DADE'S INVARIANT CONJECTURE FOR CHEVALLEY GROUPS $G_2(q)$ IN NON-DEFINING CHARACTERISTICS

JIANBEI AN

Introduction

Let G be a finite group, r a prime and B an r-block. Dade in [12] raised several conjectures on counting the number of ordinary irreducible characters in B and announced that his final conjecture can be confirmed by verifying it for all non-abelian finite simple groups. The ordinary conjecture was verified in [4] for the Chevalley groups $G_2(q)$ in non-defining characteristics. In this paper, we prove the invariant conjecture for $G_2(q)$ when r is distinct from the characteristic of the group. This implies that Dade's final conjecture holds in this case when $q \neq 3$ and 4.

We fix some notation and state the invariant conjecture in Section 1. Using the geometry and subgroup structures of the Chevalley groups of type G_2 given by Aschbacher [5], we determine the stabilizers in the automorphism group of radical chains of the $G_2(q)$-invariant subfamily $\mathcal{R}^0(G_2(q))$ given in [4]. We prove Dade's invariant conjecture for $G_2(q)$ when r is odd and even in Sections 3 and 4, respectively.

1. *Dade's invariant conjecture*

We shall follow the notation of [12]. Given an r-subgroup chain

$$C : P_0 < P_1 < \cdots < P_n \tag{1.1}$$

of a finite group G, define $|C| = n$, $C(C) = C(P_n) = C_G(P_n)$ and

$$N(C) = N_G(C) = N(P_0) \cap N(P_1) \cap \cdots \cap N(P_n), \tag{1.2}$$

where $N(P_i)$ is the normalizer $N_G(P_i)$ of P_i in G. In addition, let

$$\begin{aligned} &C_k : P_0 < P_1 < \cdots < P_k \quad \text{and} \\ &C^k : 1 < P_{k+1} < \cdots < P_n,\ 0 \leq k \leq n-1. \end{aligned}$$

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The chain C is said to be *radical* if

(a) $P_0 = O_p(G)$ and (b) $P_k = O_p(C_k)$

for $1 \leq k \leq n$. Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical r-chains of G.

Let $\mathrm{Irr}(G)$ be the set of irreducible ordinary characters of G, $\mathrm{Blk}(G)$ the set of r-blocks of G, $D(B)$ a defect group of $B \in \mathrm{Blk}(G)$, $\mathrm{Irr}(B)$ the set of irreducible ordinary characters of B and $\mathrm{Irr}^i(B)$ the characters of $\mathrm{Irr}(B)$ with height i.

If $Z(G) = 1$, we can identify G with its inner automorphism group $\mathrm{Inn}(G)$, so that $G \trianglelefteq A = \mathrm{Aut}(G)$ and $O = \mathrm{Out}(G) = A/G$. Let $N_A(C)$ be the stabilizer of C in A, and $N_A(C, \psi)$ the stabilizer of $\psi \in \mathrm{Irr}(N(C))$ in $N_A(C)$, so that $N(C) \leq N_A(C, \psi)$. Thus

$$N_A(C, \psi)/N_G(C) \simeq N_O(C, \psi) = N_A(C, \psi)G/G.$$

Given $B \in \mathrm{Blk}(G)$, $C \in \mathcal{R}(G)$, an integer $d \geq 0$ and $U \leq O$, let $\mathrm{Blk}(N(C)|B) = \{b \in \mathrm{Blk}(N(C)) : b^G = B\}$, and let $\mathrm{k}(N(C), B, d, U)$ be the number of characters in the set

$$\mathrm{Irr}(N(C), B, d, U) = \left\{\psi \in \mathrm{Irr}(N(C)) : B(\psi)^G = B,\ N_O(C, \psi) = U,\ \mathrm{d}(\psi) = d\right\} \qquad (1.3)$$

where $B(\psi)$ is the block of $N(C)$ containing ψ and $\mathrm{d}(\psi)$ is the r-defect of ψ. The following is Dade's invariant conjecture (when $Z(G) = 1$), [12, 2.5]:

Dade's Invariant Conjecture *If $O_p(G) = 1$ and B is an r-block of G with defect* $\mathrm{d}(B) > 0$, *then for any integer $d \geq 0$ and any subgroup $U \leq \mathrm{Out}(G)$,*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|}\, \mathrm{k}(N(C), B, d, U) = 0, \qquad (1.4)$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits in $\mathcal{R}$.

If $O = \mathrm{Out}(G)$ is cyclic, then U is determined uniquely by its order $|U|$. We set $\mathrm{k}(N(C), B, d, |U|) = \mathrm{k}(N(C), B, d, U)$.

2. *Stabilizers of radical chains*

The notation and terminology of Section 1 are continued in this section. Let p be a prime distinct from r, $q = p^e$ for some positive integer $e \in \mathrm{N}$ and F_q the field of q elements. Following the notation of [5], we denote V a 7-dimensional linear space over F_q with basis

$$X = \{x_0, x_i, x_i' : 1 \leq i \leq 3\}.$$

and we identify $\mathrm{GL}(V)$ with $\mathrm{GL}_7(q)$ with respect to X. Let f and B be, respectively the alternating trilinear and symmetric bilinear form on V whose monomials in the basis X are

$$f = x_0x_1x_1' + x_0x_2x_2' + x_0x_3x_3' + x_1x_2x_3 + x_1'x_2'x_3',$$
$$B = -2x_0^2 + x_1x_1' + x_2x_2' + x_3x_3'.$$

If $p \neq 2$, then denote also by B the quadratic form determined by B. If $p = 2$, then let B be the quadratic form associated to the bilinear form B such that $B(x_0) = 1$ and $B(x_i) = B(x'_i) = 0$ for $i \geq 1$. Define

$$U_3 = \langle x_0, x_3, x'_3\rangle, \quad U_4 = \langle x_1, x_2, x'_1, x'_2\rangle,$$
$$V_3 = \langle x_1, x_2, x_3\rangle, \quad V'_3 = \langle x'_1, x'_2, x'_3\rangle, \quad V_6 = V_3 + V'_3.$$

By [5, (3.4)], the group $\mathrm{O}(V, f, B)$ of all isometries of f and B is isomorphic to $G_2(q)$. In particular,

$$|\,\mathrm{O}(V, f, B)| = q^6(q-1)^2(q+1)^2(q^2+q+1)(q^2-q+1).$$

Let F_{q^2} be a quadratic extension of F_q, α the Frobinenius automorphism of F_{q^2}, so that $\alpha(t) = t^p$ for all $t \in \mathrm{F}_{q^2}$ and F_q is the fixed-point set $C_{\mathrm{F}_{q^2}}(\alpha^e)$ of α^e. Thus $|\alpha| = 2e$ in $\mathrm{Aut}(\mathrm{F}_{q^2})$ and $\alpha(\mathrm{F}_q) = \mathrm{F}_q$. In addition, α can be viewed as the Frobinenius automorphism of F_q, so that $|\alpha| = e$ in $\mathrm{Aut}(\mathrm{F}_q)$. Moreover, α induces a field automorphism, which is also denoted by α, of $\mathrm{GL}(\mathrm{F}_{q^2} \otimes V) = \mathrm{GL}_7(q^2)$ of order $2e$, and it also induces a semilinear map, denoted again by α on $\mathrm{F}_{q^2} \otimes V$, which stabilizes V. Thus if $v = a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 + a'_1x'_1 + a'_2x'_2 + a'_3x'_3 \in \mathrm{F}_{q^2} \otimes V$, then

$$\alpha(v) = \alpha(a_0)x_0 + \alpha(a_1)x_1 + \alpha(a_2)x_2 + \alpha(a_3)x_3 + \alpha(a'_1)x'_1 + \alpha(a'_2)x'_2 + \alpha(a'_3)x'_3.$$

In particular, $\alpha(V_3) = V_3$ and $\alpha(U_4) = U_4$. Let $K = \mathrm{SL}(V_3)$ be the subgroup of $\mathrm{GL}(V)$ defined by [5, (2.3)], so that $\alpha(K) = K$. Let $g(t)$ and $r(1)$ be the elements of $\mathrm{GL}(V)$ whose matrices with respect to the ordered basis $\{x_1, x'_2, x_2, x'_1\}$ are

$$g(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -t & 1 \end{pmatrix}, \quad r(1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and with respect to the ordered basis $\{x_3, x_0, x'_3\}$ are

$$g(t) = \begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix}, \quad r(1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $\alpha(g(t)) = g(\alpha(t))$ and $\alpha(r(1)) = r(1)$. Thus α stabilizes the subgroup

$$G = \langle K, g(t), r(1) : t \in \mathrm{F}_q\rangle \leq \mathrm{GL}(V),$$

which is $G_2(q) = \mathrm{O}(V, f, B)$ by [5, p.205]. So α induces an automorphism of G, which is also denoted by α. In particular, $C_{G_2(q^2)}(\alpha^e) = G = G_2(q)$. If $q \neq 3$ and 4, then $G_2(q)$ has a trivial Schur multiplier, so Dade's invariant conjecture is equivalent to his final conjecture.

We shall identify G with its inner automorphism group $\mathrm{Inn}(G)$. Let

$$A = \mathrm{Aut}(G) = G \,\mathrm{o}\, \langle \beta \rangle,$$

where β is a field automorphism of order e on G induced by the Frobenius map except when $p = 3$, in which case β is an extraordinary graph automorphism (see [11, p.xiv]), whose square is the field automorphism induced by the Frobenius map.

(2A) *In the notation above, α induces a field automorphism of order e on G.*

Proof. We may suppose $e \geq 2$. Let $C_G(\alpha) = \{y \in G : \alpha(g) = g\}$ be the fixed-point subgroup of G. Since $C_{\mathrm{GL}_7(q)}(\alpha) = \mathrm{GL}_7(p)$, it follows that $C_G(\alpha) = \mathrm{O}(V, f, B) \cap \mathrm{GL}_7(p) = G_2(p)$. If e' is a factor of e, then $C_{\mathrm{GL}_7(q)}(\alpha^{e'}) = \mathrm{GL}_7(p^{e'})$ and $C_G(\alpha^{e'}) = G_2(p^{e'})$. It follows that α has order e in $\mathrm{Aut}(G)$. Suppose $\beta_1 = \beta$ or β^2 according as $p \neq 3$ or $p = 3$. Then $|\beta_1| = e$ and $\alpha \in \mathrm{Inn}(G) \,\mathrm{o}\, \langle \beta_1 \rangle$, since $|\alpha \,\mathrm{Inn}(G)|$ is a factor of e. Thus $\alpha \in \beta_1^s \,\mathrm{Inn}(G)$ for some $1 \leq s \leq e-1$. If $|\beta_1^s| = e_1 < e$, then e_1 divides e and $\alpha^{e_1} = h \in (\beta_1^s)^{e_1} \,\mathrm{Inn}(G) = \mathrm{Inn}(G) = G$. But $C_G(\alpha^{e_1}) = G_2(p^{e_1})$, so $h^{-1}xh = x$ for each $x \in G_2(p^{e_1})$ and $h \in C_G(G_2(p^{e_1})) = 1$. It follows that $\alpha^{e_1} = 1 \in \mathrm{Aut}(G)$, which is impossible. So $|\beta_1^s| = e = |\alpha|$. By [21, Proposition 7.2], α is a field automorphism of G, in fact α and β_1^s are $\mathrm{Inn}(G)$-conjugate. This proves (2A).

We may suppose

$$\alpha = \beta \qquad \text{or} \qquad \alpha = \beta^2$$

according as $p \neq 3$ or $p = 3$.

If $\delta = \pm$, then $2_\delta^{1+2\gamma}$ denotes the extraspecial 2-group of order $2^{1+2\gamma}$ and type δ. If r is odd, then $r_\delta^{1+2\gamma}$ denotes the extraspecial r-group of order $r^{1+2\gamma}$ with exponent r or r^2 according as $\delta = +$ or $-$. Given an integer $n \geq 0$, denote by D_{2n} a dihedral group of order $2n$, by E_{r^n} an elementary abelian group of order r^n, and by Z_n or simply n a cyclic group of order n.

We may always suppose a block $B \in \mathrm{Blk}(G)$ has a non-cyclic group, since otherwise Dade's conjecture for B follows by Dade [13]. So $r|(q^2-1)$ (see [2, p.25]). Let r^a or r^{a+1} be the exact power of r dividing $q^2 - 1$ according as r is odd or even, and let the sign ϵ be chosen so that $r^a | q - \epsilon$.

Let $K(q^2) = \mathrm{SL}(\mathrm{F}_{q^2} \otimes V_3) = \mathrm{SL}_3(q^2)$ and $L_+ = K = \mathrm{SL}(V_3)$. Since α stabilizes both $\mathrm{F}_{q^2} \otimes V_3$ and V_3, it follows that $\alpha(K(q^2)) = K(q^2)$, $\alpha(L_+) = L_+$ and $C_{K(q^2)}(\alpha^e) = L_+$. By [5, Theorem 3 (3) and (2.11)], $M = N_{G_2(q^2)}(\mathrm{F}_{q^2} \otimes V_6) = \langle K(q^2), r(1) \rangle = K(q^2)\langle r(1) \rangle$, so that $\alpha(M) = M$. Since $\alpha(r(1)) = r(1)$, α stabilizes $r(1)^{-1}K(q^2)r(1) \leq M$. Since $r(1)(V_3) = V_3'$, $r(1)$ induces the inverse-transpose automorphism on $K(q^2)$ and so the stabilizer of α^e on $r(1)^{-1}K(q^2)r(1)$ is $L_- = \mathrm{SU}_3(q)$. Moreover, $r(1) \in N_G(L_-)$ and $r(1)$ induces a field automorphism of order 2 on L_-. Thus $K_\delta = N_G(L_\delta) = \langle L_\delta, r(1)\rangle$ (see [5, Corollary 11]).

Let P_1 and P_2 be the stabilizers of $\mathrm{F}_q x_1$ and $\mathrm{F}_q x_1 \oplus \mathrm{F}_q x_2'$ in G, respectively. Then P_1 and P_2 are parabolic subgroups of G, and $\alpha(P_i) = P_i$ for $i = 1, 2$. Suppose $p = 3$. By [5, (17.4)], β interchanges P_1 and P_2, so that $\beta(O_p(P_1)) = O_p(P_2)$ and β interchanges the Levi subgroups $P_1/O_p(P_1)$ and $P_2/O_p(P_2)$. By [5, (2.15) and (4.3)], $P_1 = O_p(P_1) \circ K_1$ and $P_2 = O_p(P_2) \circ K_2$, where K_1 and K_2 act as $\mathrm{GL}_2(q)$ on $\mathrm{F}_q x_2 \oplus \mathrm{F}_q x_3$ and $\mathrm{F}_q x_1 \oplus \mathrm{F}_q x_2'$, respectively and $K_1 \leq K$. Let $\{a, b\}$ be a set of fundamental roots of root system

$$\Sigma = \{\pm a, \pm(a+b), \pm(2a+b), \pm b, \pm(3a+b), \pm(3a+2b)\}$$

of type G_2 with a a short root. Since $K = \mathrm{SL}(V_3)$ is generated by three long root subgroups of G, we may suppose the root subgroup $X_{-a} = \{x_{-a}(t) = g(t) : t \in \mathrm{F}_q\}$. Thus $X_b = \{x_b(t) : t \in \mathrm{F}_q\} \leq K_1$, where $x_b(t) \in K$ such that

$$x_b(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

with respect to the ordered basis $\{x_1, x_2, x_3\}$ of V_3. Now β induces an action, denoted again by β, on the root system Σ (see, [29, p.440]) such that $\beta(\pm a) = \pm b$, $\beta(\pm(a+b)) = \pm(3a+b)$ and $\beta(\pm(2a+b)) = \pm(3a+2b)$. Thus $\beta(x_a(t)) = x_b(t^3)$ and $\beta(x_b(t)) = x_a(t)$ for all $t \in \mathrm{F}_q$.

Suppose $\overline{\mathrm{F}}$ is the algebraic closure of F_q and T is the diagonal subgroup of $\overline{G} = G_2(\overline{\mathrm{F}})$ with respect to X. Then T is a maximal torus of $\overline{G}$ (see [5, p.254]) and α stabilizes T. Let

$$N_{\overline{G}}(T) = \langle T, \rho, \sigma, \tau\rangle,$$

where $\rho = r(1)\tau\sigma$ and the actions of σ and τ on the basis X are given as [5, p.254],

$$\sigma = (x_1, x_2, x_3)(x_1', x_2', x_3') \qquad \tau = (x_2, x_3, -x_2, -x_3)(x_2', x_3', -x_2', -x_3').$$

Thus $\langle\rho\rangle$ is the center of the Weyl group $D_{12} = N_{\overline{G}}(T)/T$, $|\sigma| = 3$, $|\tau| = 2$ and $\tau\sigma\tau = \sigma^{-1}$ modulo T. In particular, $w^\alpha = w$ for all $w \in \langle\rho, \sigma, \tau\rangle$. By [5, p.254], we may suppose $T = H_1 \times H_2$, where $H_i = \{h_i(t) : t \in \overline{\mathrm{F}}^\#\}$ such that as elements of $K = \mathrm{SL}_3(q)$,

$$h_1(t) = \begin{pmatrix} t & & \\ & t^{-1} & \\ & & 1 \end{pmatrix} \qquad h_2(t) = \begin{pmatrix} 1 & & \\ & t & \\ & & t^{-1} \end{pmatrix}.$$

Set $h_3(t) = h_1(t^{-1})h_2(t^{-1})$. Then ρ inverts T,

$$\sigma = (h_1(t), h_2(t), h_3(t)) \qquad \text{and} \qquad \tau = (h_1(t), h_3(t^{-1}))(h_2(t), h_2(t^{-1})).$$

Suppose $p = 3$ and $\omega_a = r(1)$. Then $\omega_a x_{-a}(t)\omega_a^{-1} = x_a(-t)$ whose matrix with respect to the ordered basis $\{x_1, x_2', x_2, x_1'\}$ is

$$x_a(t) = \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and with respect to the ordered basis $\{x_3, x_0, x_3'\}$ is

$$x_a(t) = \begin{pmatrix} 1 & 0 & 0 \\ 2t & 1 & 0 \\ t^2 & t & 1 \end{pmatrix}.$$

So $\omega_a = x_a(1)x_{-a}(-1)x_a(1)$ and $\omega_b = \omega_a^\beta = x_b(1)x_{-b}(-1)x_b(1) = \tau$. Since $\omega_a x_b(t)\omega_a^{-1} = x_{3a+b}(t)$ and $\omega_b x_{3a+b}(t)\omega_b^{-1} = x_{3a+2b}(t)$, it follows that as elements of K

$$x_{3a+b}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -t & 0 & 1 \end{pmatrix} \quad \text{and} \quad x_{3a+2b}(t) = \begin{pmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, since $\omega_b x_a(t)\omega_b^{-1} = x_{a+b}(t)$ and $\omega_a x_{a+b}(t)\omega_a^{-1} = x_{2a+b}(t)$, it follows that $x_{a+b}(t)$ and $x_{2a+b}(t)$ are elements of $\mathrm{GL}(V)$ whose matrices with respect to $\{x_3, x_1', x_1, x_3'\}$ and $\{x_2', x_3, x_3', x_2\}$, respectively are of the form

$$\begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and with respect to $\{x_2, x_0, x_2'\}$ and $\{x_1', x_0, x_1\}$, respectively are

$$\begin{pmatrix} 1 & 0 & 0 \\ 2t & 1 & 0 \\ t^2 & t & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2 & 2t & 1 \end{pmatrix}.$$

Since $\omega_b x_{-a}(t)\omega_b^{-1} = x_{-a-b}(t)$ and $\omega_a x_{-a-b}(t)\omega_a^{-1} = x_{-2a-b}(t)$, it follows that $x_{-a-b}(t)$ and $x_{-2a-b}(t)$ are elements of $\mathrm{GL}(V)$ whose matrices with respect to $\{x_3, x_1', x_1, x_3'\}$ and $\{x_2', x_3, x_3', x_2\}$, respectively are of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -t & 1 \end{pmatrix}$$

and with respect to $\{x_2, x_0, x_2'\}$ and $\{x_1', x_0, x_1\}$, respectively are

$$\begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2t & t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

As an element of K, $\sigma = \tau\tau_1$, where $\tau_1 = (x_1, x_2, -x_1, -x_2)(x_1', x_2', -x_1', -x_2')$. Thus $\tau_1 = x_{-3a-2b}(-1)x_{3a+2b}(1)x_{-3a-2b}(-1)$ and $(\tau_1)^\beta = x_{-2a-b}(-1)x_{2a+b}(1)x_{-2a-b}(-1)$. So

$$(\tau_1)^\beta = (x_2', x_3, -x_2', -x_3)(x_2, x_3', -x_2, -x_3')(x_1', x_1)(x_0, -x_0).$$

Since $\tau^\beta = r(1)$ and $\rho^\beta = r(1)^\beta\tau^\beta\sigma^\beta = \tau r(1)\sigma^\beta$, it follows that

$$\sigma^\beta = h_2(-1)\sigma^{-1} \qquad \rho^\beta = h_3(-1)\rho \qquad \tau^\beta = \rho\tau\sigma \tag{2.1}$$

and so $(\sigma\tau)^\beta = h_2(-1)\rho\sigma\tau$. In addition, since $h_a(t) = x_{-a}(t^{-1}-1)x_a(1)x_{-a}(t-1)x_a(-t^{-1})$ (see [29, p.447]), it follows that

$$h_a(t) = h_1(t)h_2(t^2) = \begin{pmatrix} t & & \\ & t & \\ & & t^{-2} \end{pmatrix} \in K.$$

Similarly, $h_b(t) = h_2(t) \in K$. So $h_a(t)^\beta = h_b(t^3)$ and $h_b(t)^\beta = h_a(t)$ for all $t \in \overline{\mathrm{F}}$. In particular,

$$h_1(t)^\beta = h_a(t^{-2})h_b(t^3) = \begin{pmatrix} t^{-2} & & \\ & t & \\ & & t \end{pmatrix}.$$

Remark: There is no automorphism $\psi \in \mathrm{Aut}(T)$ such that $h_a(t)^\psi = h_1(t^3)$ and $h_1(t)^\psi = h_a(t)$. Indeed, otherwise $h_2(t) = h_1(s)h_a(w)$ will imply that $t = w^2 \in \overline{\mathrm{F}}^2$.

Suppose L is a maximal tori of G. As shown in the proof of [5, (15.1)], we may suppose $L = C_T(\alpha^e w)$, where $w \in \{1, \sigma, \tau, \rho, \rho\sigma, \rho\tau\}$. Since $w^\alpha = w$ and α stabilizes T, it follows that α stabilizes $L = C_T(\alpha^e w)$. If $w_+ = 1$ and $w_- = \rho$, then $T_\delta = C_T(\alpha^e w_\delta) \simeq \mathrm{Z}_{q-\delta} \times \mathrm{Z}_{q-\delta}$. If $w_2^+ = \sigma$ and $w_2^- = \rho\sigma$, then $T_2^\delta = C_T(\alpha^e w_2^\delta) \simeq \mathrm{Z}_{q^2+\delta q+1}$. If $w_3^+ = \tau$ and $w_3^- = \rho\tau$, then $T_3^\delta = C_T(\alpha^e w_3^\delta) \simeq \mathrm{Z}_{q^2-1}$. Suppose $p = 3$. Then β stabilizes each D_{12}-classes of ρT and σT, so β stabilizes each T_δ and T_2^δ. Since $\rho\tau T$ and $\rho\tau\sigma T$ are D_{12}-conjugate and $\tau^\beta = \rho\tau\sigma$, it follows that β interchanges T_3^+ and T_3^-. Thus T_δ, T_2^δ, T_3^δ are (diagonal) maximal tori of L_δ and

$$N_G(T_\delta) = N_{K_\delta}(T_\delta) = \langle T_\delta, \sigma, \tau, \rho \rangle,$$

where $\sigma, \tau, \rho \in C_G(\alpha) = G_2(p)$ are given above. In addition,

$$N_A(T_\delta) = \langle T_\delta, \sigma, \tau, \rho \rangle \text{ o } \langle \beta \rangle \tag{2.2}$$

and moreover, $|\rho| = 2$, $|\sigma| = 3$, $\tau^2 = h_2(-1)$ and

$$\tau^{-1}\rho\tau = \rho \qquad \sigma^{-1}\rho\sigma = h_1(-1)\rho \qquad \tau^{-1}\sigma\tau = h_2(-1)\sigma^{-1}.$$

Note that the actions of τ, σ on T_δ are different from that given before [2, (3B)].

Let $\iota : \overline{\mathrm{F}}^{\#} \to (\overline{\mathrm{Q}}_\ell^{\#})_{p'}$ be the fixed isomorphism between the groups of p'-th roots of unity in these fields in the Deligne-Lusztig theory. In addition, let $L = C_T(\alpha^e w)$ for some $w \in \{1, \sigma, \tau, \rho, \rho\sigma, \rho\tau\}$. Then ι gives rise to a group isomorphism (cf [9, Proposition 4.4.1]):

$$\phi : L \longrightarrow \mathrm{Irr}(L) \tag{2.3}$$

and moreover, following [9], β acts on $\mathrm{Irr}(L)$ by ${}^{\beta}(\phi(s))(t) = \phi(s)(t^{\beta})$.

(2B) *In the notation above, suppose $s \in L$ and $R_L^G(\phi(s))$ is the Deligne-Lusztig generalized character. Then $R_L^G(\phi(s))^{\beta} = R_{\beta L \beta^{-1}}^G(\phi(s^{\beta}))$. In particular, we may suppose ${}^{\beta}\phi(s) = \phi(s^{\beta})$.*

Proof. The proof is similar to that of [18, (4A)] (see [28, pp.181 and 257]. Let $\Gamma = \mathrm{Hom}(T, \overline{\mathrm{F}}^{\#})$ and $\Gamma^* = \mathrm{Hom}(\overline{\mathrm{F}}^{\#}, T)$ be the character and cocharacter groups of T. Since $G_2(\overline{\mathrm{F}})$ is self-dual, the group isomorphism $\kappa : \Gamma \to \Gamma^*$ carries the roots of $G_2(\overline{\mathrm{F}})$ to coroots of $G_2(\overline{\mathrm{F}})^* = G_2(\overline{\mathrm{F}})$. The composition $\iota^{-1}\phi(s)$ is in $\mathrm{Hom}(L, \overline{\mathrm{F}}^{\#})$ and hence the restriction of a λ_1 in Γ. If $|\phi(s)| = n$, then there exists a λ in Γ such that

$$\lambda(\alpha^e(t)w^{-1}t^{-1}w) = \lambda_1^n(t) \quad \text{for} \quad t \in T.$$

Let $\nu = \kappa(\lambda) \in \Gamma^*$. Then the $\overline{G}$-class of $\nu(\iota^{-1}(\frac{1}{n}))$ is fixed by α^e, and its intersection with G is the G-class of s.

Since β^e or β^{2e} is a standard Frobinius map α^e, it follows by [9, p.31] that β is a Frobinous map. Thus $\kappa({}^{\beta}\chi) = \kappa(\chi)^{\beta}$ for all $\chi \in \Gamma$ (cf. [9, Proposition 4.3.1]). The procedure associating $\phi(s)$, λ_1, λ and n to $R_L^G(\phi(s))$ associates ${}^{\beta}\phi(s)$, ${}^{\beta}\lambda_1$, ${}^{\beta}\lambda$ and n to $R_L^G(\phi(s))^{\beta}$. Since ${}^{\beta}\lambda$ corresponds to ν^{β} under κ, the $\overline{G}$-class of $\nu(\iota^{-1}(\frac{1}{n}))^{\beta}$ contains s^{β} and (2B) follows.

Since $h_a(s)x_b(t)h_a(s)^{-1} = x_b(s^3 t)$ and $h_b(s)x_b(t)h_b(s)^{-1} = x_b(s^2 t)$ for all $s, t \in T$, it follows by the definition (cf. [9, p.18]) that $b(h_a(s)) = s^3$ and $b(h_b(s)) = s^2$ for all $s \in T$, where b is viewed as an element of Γ. Similarly, $a(h_a(s)) = s^2$ and $a(h_b(s)) = s$. In particular, $h_a = a^{\vee}$ and $h_b = b^{\vee}$ are coroots (see [9, p.19]) corresponding to roots a and b, respectively. So $\kappa(a) = h_a$ and $\kappa(b) = h_b$. Moreover, if $p = 3$, then $({}^{\beta}a, {}^{\beta}b) = (b, 3a)$ and $(h_a^{\beta}, h_b^{\beta}) = (3h_b, h_a)$. Regard h_a and h_b as simple roots in the dual root system. Then h_a is long and h_b is short. By [9, Proposition 4.2.3], κ induces a group automorphism κ of D_{12} such that $\kappa(w_{a_1}) = w_{\kappa(a_1)}$ for each root $a_1 \in \Sigma$, where w_{a_1} is the reflection with respect to a_1. In addition, $\kappa({}^{w}\chi) = \kappa(\chi)^{\kappa(w^{-1})}$ for all $\chi \in \Gamma$ and $w \in D_{12}$. Since h_a is long and h_b is short, it follows that $\kappa(w_a) = w_b$ and $\kappa(w_b) = w_a$.

Identify wT with w for $w \in N_{\overline{G}}(T)$, so $D_{12} = \langle \rho, \sigma, \tau \rangle$. By [29, Section 2], we may suppose $w_a = \rho\sigma^{-1}\tau$ and $w_b = \tau$ (in the notation of [29], $w_a = w_{\xi_2}$, $w_b = w_{\xi_1 - \xi_2}$ and

$\sigma = w_{\xi_1-\xi_2}w_{\xi_1-\xi_3}$). Thus $\sigma = (w_aw_b)^2$ and $\kappa(\sigma) = (w_bw_a)^2 = \sigma^{-1}$, so that $\kappa({}^\sigma\chi) = \kappa(\chi)^\sigma$. Similarly, $\kappa({}^\tau\chi) = \kappa(\chi)^{\rho\tau\sigma}$ and $\kappa({}^\rho\chi) = \kappa(\chi)^\rho$, since $\kappa(\tau) = \rho\tau\sigma$ and $\kappa(\rho) = \rho$. Moreover,

$$ {}^\sigma\phi(s) = \phi(s^\sigma) \qquad {}^\tau\phi(s) = \phi(s^{\rho\tau\sigma}) \qquad {}^\rho\phi(s) = \phi(s^\rho) \tag{2.4}$$

for all $s \in L$. Indeed, we apply a similar proof to that of [18, (4A)] to the group T. In the notation above, the T-class of $\nu(\iota^{-1}(\frac{1}{n}))$ is α^e-stable containing s, so that $s = \nu(\iota^{-1}(\frac{1}{n}))$. Since $\kappa({}^\sigma\lambda) = \kappa(\lambda)^\sigma = \nu^\sigma$, it follows that ${}^\sigma\lambda$ and ν^σ corresponds under κ. Since $s^\sigma = \nu(\iota^{-1}(\frac{1}{n}))^\sigma = \nu^\sigma(\iota^{-1}(\frac{1}{n}))$, it follows that

$$ {}^\sigma\phi(s) = \pm R_T^T(\phi(s))^\sigma = \pm R_T^T(\phi(s^\sigma)) = \phi(s^\sigma).$$

Similarly, ${}^\tau\phi(s) = \phi(s^{\rho\tau\sigma})$ and ${}^\rho\phi(s) = \phi(s^\rho)$. Moreover, since ${}^\beta\lambda$ and ν^β corresponds under κ, it follows again that ${}^\beta\phi(s) = \phi(s^\beta)$.

Next G has two non-conjugate subgroups $J_\delta = \mathrm{GL}_2^\delta(q)$ and $J_\delta{}^* = (\mathrm{GL}_2^\delta(q))^*$. So $Z(J_\delta) \simeq Z(J_\delta{}^*) \simeq \mathrm{Z}_{q-\delta}$ and $J_\delta = C_G(Z(J_\delta))$ and $J_\delta{}^* = C_G(Z(J_\delta{}^*))$. In particular, $Z(J_\delta)$ is not G-conjugate to $Z(J_\delta{}^*)$. Since G has one class of maximal torus T_δ, we may suppose both $Z(J_\delta)$ and $Z(J_\delta{}^*)$ are subgroups of T_δ. But T_δ has exactly two $N_G(T_\delta)$-orbits of cyclic subgroups of order $q-\delta$, so we may suppose

$$ Z(J_\delta) = \{\mathrm{diag}\ \{s, s, s^{-2}\} \in T_\delta : s \in \mathrm{Z}_{q-\delta}\} \le L_\delta $$

and $Z(J_\delta{}^*) = \{\mathrm{diag}\ \{s, s^{-1}, 1\} \in T_\delta : s \in \mathrm{Z}_{q-\delta}\} \le L_\delta$.

We consider the action of α on J_δ and $J_\delta{}^*$. Let D be the diagonal subgroup of $K(q^2) = \mathrm{SL}(\mathrm{F}_{q^2} \otimes V_3) \le G_2(q^2)$. By [5, (4.3)] D, $r(1)$ and $g(t)$ for all $t \in \mathrm{F}_{q^2}$ generate a subgroup W of $G_2(q^2)$ such that W acts as $\mathrm{GL}_2(q^2)$ on the line $\mathrm{F}_{q^2} \otimes \ell$, where $\ell = \mathrm{F}_q x_1 \oplus \mathrm{F}_q x_2'$. So α stabilizes W. By [5, (4.1)], each element y of $Z(W)$ has the form $y = \mathrm{diag}\ \{c, c^{-1}, 1\}$ as an element of $K(q^2) = \mathrm{SL}_3(q^2)$, where $c \in \mathrm{Z}_{q^2-1}$. In particular, $Z(J_\delta^*) \le Z(W)$. Let Y be the subgroup of $K(q^2)$ stabilizes U_4. By [5, (2.7)], Y acts as $\mathrm{GL}_2(q^2)$ on $\mathrm{F}_{q^2}x_1 \oplus \mathrm{F}_{q^2}x_2$. Each element y of $Z(Y)$ has the form $\mathrm{diag}\ \{c, c, c^{-2}\}$ as an element of $K(q^2)$, so that we may suppose $Z(J_\delta) \le Z(Y)$. Moreover, W and Y stabilize $\mathrm{F}_{q^2} \otimes U_4$, and $Z(W)$, $Z(Y)$ are not $G_2(q^2)$-conjugate. Let c be a generator of Z_{q^2-1}, $y = \mathrm{diag}\ \{c, c^{-1}, 1\} \in Z(W)$ and $y' = \mathrm{diag}\ \{c, c, c^{-2}\} \in Z(Y)$. Then $Z(W) = \langle y \rangle$ and $Z(Y) = \langle y' \rangle$. If we view W and Y as subgroups of $\mathrm{SO}(\mathrm{F}_{q^2} \otimes U_4)$, then y and y' are $\mathrm{O}(\mathrm{F}_{q^2} \otimes U_4)$-conjugate and not $\mathrm{SO}(\mathrm{F}_{q^2} \otimes U_4)$-conjugate. Since $C_{\mathrm{SO}(\mathrm{F}_{q^2} \otimes U_4)}(Z(W)) = W$ and $C_{\mathrm{SO}(\mathrm{F}_{q^2} \otimes U_4)}(Z(Y)) = Y$, it follows that W and Y are $\mathrm{O}(\mathrm{F}_{q^2} \otimes U_4)$-conjugate and not $\mathrm{SO}(\mathrm{F}_{q^2} \otimes U_4)$-conjugate. Since W is a Levi subgroup of $\mathrm{O}(\mathrm{F}_{q^2} \otimes U_4)$, W has one class in $\mathrm{O}(\mathrm{F}_{q^2} \otimes U_4)$ and two classes in $\mathrm{SO}(\mathrm{F}_{q^2} \otimes U_4)$, so we may suppose $Y = h^{-1}Wh$ for some reflection h of $\mathrm{O}(U_4)$. Thus we may suppose

$$ Y = \left\{ \begin{pmatrix} A & \\ & A^{-t} \end{pmatrix} : A \in \mathrm{GL}(\mathrm{F}_{q^2} \otimes U) \right\} \le \mathrm{SO}(\mathrm{F}_{q^2} \otimes U_4),$$

where $U = \mathrm{F}_q x_1 \oplus \mathrm{F}_q x_2$. So α stabilizes Y and $N_{\mathrm{SO}(\mathrm{F}_{q^2}\otimes U_4)}(Y) = \langle Y, v\rangle$, where v induces the inverse-transpose automorphism on Y. We may choose v so that $v \in C_{\mathrm{SO}(U_4)}(\alpha) = \mathrm{SO}_4^+(p)$. So $J_+ = C_Y(\alpha^e) \simeq \mathrm{GL}_2(q)$ and $J_- = C_{v^{-1}Yv}(\alpha^e) \simeq \mathrm{U}_2(q)$. Both are subgroups of G and $N_G(J_\delta) = \langle J_\delta, v\rangle$. In particular, α induces a field automorphism of order e on J_δ and v induces a inverse-transpose or a field automorphism of order 2 on J_δ according as $\delta = +$ or $-$. Let $h \in \mathrm{O}(U_4)$ be the reflection on $x_1 + x_1'$. Then $\alpha(h) = h$ and $h \notin \mathrm{SO}(U_4)$. We may suppose $J_\delta^* = h^{-1}J_\delta h \le \mathrm{SO}(U_4)$, so that α induces a field automorphism of order e on $J_\delta{}^*$. In addition, $J_\delta{}^*$ and J_δ are not G-conjugate.

Suppose $p = 3$. Then $\beta\sigma^2$ interchanges $Z(J_\delta)$ and $Z(J_\delta{}^*)$ and $(\beta\sigma^2)^2 = h_1(-1)\beta^2 = h_1(-1)\alpha$. Thus $\beta\sigma^2$ interchanges J_δ and $J_\delta{}^*$, and $(\beta\sigma^2)^2$ stabilizes each J_δ and $J_\delta{}^*$. It follows that

$$N_A(J_\delta) = N_G(J_\delta) \circ \langle\alpha\rangle \qquad \text{and} \qquad N_A(J_\delta{}^*) = N_G(J_\delta{}^*) \circ \langle\alpha\rangle,$$

where $N_G(J_\delta) = \langle J_\delta, \rho\rangle$ and $N_G(J_\delta{}^*) = \langle J_\delta{}^*, \rho\rangle$ as ρ inverts $Z(J_\delta)$ and $Z(J_\delta{}^*)$.

By [5, (2.7)], $J_\delta J_\delta{}^* = \mathrm{SO}_4^+(q)$ on U_4, so that $\beta\sigma^2$ stabilizes $\mathrm{SO}(U_4)$ when $p = 3$. Thus

$$N_A(\mathrm{SO}(U_4)) = \langle \mathrm{SO}(U_4), \beta'\rangle,$$

where $\beta' = \beta\sigma^{-1} = h_1(-1)\sigma\beta$ or α according as $p = 3$ or $p \neq 3$. Moreover, if $p = 3$, then

$$\sigma^{\beta'} = h_1(-1)\sigma^{-1} \qquad \rho^{\beta'} = h_1(-1)\rho \qquad \tau^{\beta'} = h_3(-1)\rho\sigma\tau$$

and so $(\tau\sigma)^{\beta'} = h_3(-1)\rho\tau\sigma$.

Suppose $q \neq 2$. By [26, Proposition 2.5.7], $\Omega(U_4)$ is the only subgroup of $\mathrm{SO}(U_4)$ of index two and by [26, Proposition 2.9.1 (iv)], $\Omega(U_4)$ is the central product $M_1 \circ M_2$ of $M_1 = \mathrm{Sp}_2(q)$ and $M_2 = \mathrm{Sp}_2(q)$ over $Z(M_1) = Z(M_2)$. Let W_i be the underlying (symplectic) space of M_i. Then $M_i = \mathrm{SL}(W_i)$ and $W_1 \otimes W_2$ is an orthogonal space of plus type over F_q and we may suppose $\mathrm{SO}(U_4) = \mathrm{SO}(W_1 \otimes W_2)$. Let $g_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{GL}(W_1)$ and $g_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{GL}(W_2)$. If $g = g_1 \otimes g_2$ and q is odd, then $g \in \mathrm{SO}(U_4)\backslash\Omega(U_4)$. Thus

$$\mathrm{SO}(U_4) = \langle M_1 \circ M_2, g\rangle. \tag{2.5}$$

(2C) *Let $M = N_G(U_4) = \mathrm{SO}(U_4) = \mathrm{SO}_4^+(q)$ and $\Omega = \Omega(U_4) = \Omega_4^+(q)$. Suppose $\Omega = M_1 \circ M_2$, where $M_1 \simeq M_2 \simeq \mathrm{SL}_2(q)$.*

(a) *If $e \ge 2$, then α induces a filed automorphism α_{M_i} of order e on M_i for each i.*

(b) *If $p = 3$, then β' interchanges M_1 and M_2.*

Proof. Since Ω is the unique subgroup of index 2 in M, it follows that $\beta'(\Omega) = \Omega$. Moreover, $C_M(\alpha) = \mathrm{SO}_4^+(p)$.

Let $H = [\,C_M(\alpha), C_M(\alpha)\,]$ be the commutator subgroup of $C_M(\alpha)$ and suppose $C_M(\alpha) = \langle H_1 \circ H_2, w\rangle$, where $\Omega_4^+(p) = H_1 \circ H_2$, $H_1 \simeq H_2 \simeq \mathrm{SL}_2(p)$. If $p > 3$, then $H_i/Z(H_i) \simeq L_2(p)$ is nonabelian simple, so that $[\,H_i, H_i\,] = H_i$ and $H = \Omega_4^+(p)$. If $p = 3$, then $\mathrm{SO}_4^+(3) = 2_+^{1+4}.(\frac{1}{2}\Omega_4^+(2))$, where $2_+^{1+4} \in \mathrm{Syl}_2(H_1 \circ H_2)$ and $\frac{1}{2}\Omega_4^+(2)$ is the intersection of the three subgroups of index 2 of $\mathrm{O}_4^+(2)$ (cf. [26, Proposition 2.5.9]). Suppose $\frac{1}{2}\Omega_4^+(2) = \langle \mathrm{Z}_3 \times \mathrm{Z}_3, y\rangle$, so that $y^{-1}xy = x^y = x^{-1}$ for each $x \in \mathrm{Z}_3 \times \mathrm{Z}_3$. Thus $\Omega_4^+(3) = 2_+^{1+4}{:}\,(\mathrm{Z}_3 \times \mathrm{Z}_3)$ and $2_+^{1+4} = [\,\Omega_4^+(3), \Omega_4^+(3)\,]$. Since y inverts $\mathrm{Z}_3 \times \mathrm{Z}_3$, it follows that $(\mathrm{Z}_3 \times \mathrm{Z}_3) \leq H$ and so $H = \Omega_4^+(3)$. If $p = 2$, then $C_M(\alpha) = \mathrm{SO}_4^+(2) = \mathrm{O}_4^+(2) = S_3 \wr S_2$ and $H = \frac{1}{2}\Omega_4^+(2) = \langle \omega_1, \omega_2, y\rangle$ given above, where $\langle\omega_1\rangle \times \langle\omega_2\rangle = \mathrm{Z}_3 \times \mathrm{Z}_3$.

(a) Suppose $e \geq 2$. Then $[\,M_i, M_i\,] = M_i$, $\Omega = [\,\Omega, \Omega\,]$ and $H \leq \Omega$. Let $\mathcal{K}$ be the set of all normal subgroups N of Ω such that $|N| = |\,\mathrm{SL}_2(q)|$. Then $M_i \in \mathcal{K}$. Suppose $N \in \mathcal{K}\backslash\{M_1, M_2\}$ and $N_i = N \cap M_i$. Then $N_i \trianglelefteq M_i$ and $N_i \in \{\langle -I_2\rangle, M_i\}$, since $M_i/Z(M_i)$ is a simple group. If $N_i = M_i$, then $N = M_i$ since $|N| = |M_i|$. This is impossible. Thus $N_1 = N_2 = \{\pm I_2\}$. Suppose $x \in N\backslash\{\pm I_2\}$. Then $x = x_1x_2$ for some $x_i \in M_i$, so that $x_1 \neq \pm I_2$ and $x_2 \neq \pm I_2$ as $N_i = \{\pm I_2\}$. For each $y_1 \in M_1$, $[\,y_1, x^{-1}\,] = y_1^{-1}xy_1x^{-1} = y_1^{-1}x_1y_1x_1^{-1} \in N_1$, so $y_1^{-1}x_1y_1 = \pm x_1$ and $x_1Z(M_1)$ is commutative with $y_1Z(M_1)$ in $M_1/Z(M_1)$. Since y_1 is arbitrary, it follows that $x_1Z(M_1) = Z(M_1)$ and $x_1 \in Z(M_1) = \{\pm I_2\}$. A contradiction. Thus $\mathcal{K} = \{M_1, M_2\}$ and β' permutes $\mathcal{K}$.

Suppose $\alpha(M_1) = M_2$. For each $y_i \in M_i\backslash\{\pm I_2\}$, $\alpha(y_i) \notin M_i$, so that $\alpha(y_i) \neq y_i$. If $x \in H_1\backslash\{\pm I_2\}$, then $x = x_1x_2$ for some $x_i \in M_i$. If $x \in M_1$, then $\alpha(x) = x \in M_1 \cap M_2$, which is impossible. So $x \notin M_1$. Similarly, $x \notin M_2$ and then $x_i \neq \pm I_2$. Thus $\alpha(x_i) \notin M_i$ and $x_1x_2 = x = \alpha(x) = \alpha(x_1)\alpha(x_2)$, so that $\alpha(x_1) = \pm x_2$ and $\alpha(x_2) = \pm x_1$. In particular, $|x_1| = |x_2|$. Suppose $p \neq 2$ and let $S_1 = \{x_1 \in M_1 : x_1x_2 \in H_1 \text{ for some } x_2 \in M_2\}$. Since $H_1 \simeq \mathrm{SL}_2(p)$, it follows that S_1 is a nonabelian subgroup of M_1 containing a (generalized) quaternion group, so that $C_{M_1}(S_1) = Z(M_1)$. Similarly, let $S_2 = \{x_2 \in M_2 : x_1x_2 \in H_1 \text{ for some } x_1 \in M_1\}$. Then $C_{M_2}(S_2) = Z(M_2)$, so that $C_\Omega(H_1) = Z(\Omega)$. This is impossible, since $H_2 \leq C_\Omega(H_1)$. So $\alpha(M_1) \neq M_2$. A proof similar to above shows that if H_1 is not a subgroup of either M_1 or M_2, then both $C_{M_1}(S_1)$ and $C_{M_2}(S_2)$ are abelian, so that $C_\Omega(H_1)$ is abelian. The contradiction implies that $H_1 \leq M_1$ or $H_1 \leq M_2$. We may suppose $H_i \leq M_i$ for $i = 1, 2$, so that $H_i \leq C_{M_i}(\alpha)$. Suppose $H_i \neq C_{M_i}(\alpha)$ for some i, say $i = 1$. Since $(\mathrm{SO}_4^+(p){:}\, H_1 \circ H_2) = 2$ and $C_M(\sigma) = \mathrm{SO}_4^+(p)$, it follows that $C_{M_1}(\alpha) \circ H_2 = \mathrm{SO}_4^+(p)$ and $[\,C_{M_1}(\alpha), H_2\,] = 1$, which is impossible. Thus $C_{M_i}(\alpha) = H_i$ and $C_\Omega(\alpha) = \Omega_4^+(p)$.

Suppose $p = 2$, so that $H = \frac{1}{2}\Omega_4^+(2) \leq \Omega = M_1 \times M_2$. If $\alpha(M_1) = M_2$, let $S_1 = \{x_1 \in M_1 : x_1x_2 \in C_{M_1 \times M_2}(\sigma) \text{ for some } x_2 \in M_2\}$. Since $\alpha(x_1) = x_2$, it follows that

$C_{M_1\times M_2}(\alpha) \simeq S_1 \le M_1$. But $\mathrm{Z}_3 \times \mathrm{Z}_3 \le H \le C_\Omega(\alpha) = C_{M_1\times M_2}(\alpha)$, so $M_1 = \mathrm{SL}_2(q)$ has an elementary abelian group of order 3^2, which is impossible. Thus $\alpha(M_i) = M_i$.

Suppose e' is a proper factor of e. Then $C_H(\sigma^{e'}) = \mathrm{SO}_4^+(p^{e'})$, $C_\Omega(\alpha^{e'}) = \Omega_4^+(p^{e'})$ and $C_{M_i}(\alpha^{e'}) \simeq \mathrm{SL}_2(p^{e'})$. In particular, the induced automorphism of $\alpha^{e'}$ on M_i is non-trivial and so $|\alpha_{L_i}| = e$. Now $\mathrm{Aut}(M_i)$ consists of only field, diagonal and inner automorphisms of M_i (see [26, Proposition 2.2.3]), so $\alpha_{M_i} = \psi^s h$, where ψ is a field automorphism of order e on M_i and h is an element of the subgroup W of $\mathrm{Aut}(M_i)$ generated by inner and diagonal automorphisms. Suppose $|\psi^s| = e_1 < e$, then $(\alpha_{M_i})^{e_1} = y$ is an element of W and $y^{-1}xy = x$ for each $x \in C_{M_i}(\alpha^{e_1})$, where W is identified with the group $\mathrm{GL}_2(q)/Z(\mathrm{GL}_2(q))$. If $p \ge 3$ or $p = 2$ and $e_1 \ge 2$, then $C_{M_i}(\alpha^{e_1}) = \mathrm{SL}_2(p^{e_1})$ and $y = 1$, so that $(\alpha_{M_i})^{e_1} = 1$, which is impossible. Suppose $e_1 = 1$, $p = 2$ and suppose $(\alpha_{M_i})^{e_1} = y_i$, then $y_i^{-1}x_iy_i = x_i$ for each $x_1x_2 \in C_{M_1\times M_2}(\alpha)$. Thus y_1y_2 centralizes $\frac{1}{2}\Omega_4^+(2)$. It follows that $y_1y_2 = 1$ and $\alpha_{M_i} = 1$. A contradiction. So the order of ψ^s is e. By [21, Proposition 7.2], α_{M_i} is a field automorphism of M_i. In articular, α induces a field automorphism of Ω of order e. Thus $C_{M_i}(\alpha) = \mathrm{SL}_2(p)$ for all primes p.

(b) Suppose $p = 3$. In the notation of (2.5), let $s_i \in M_i$ be an element of order 2^a, so that $C_{M_i}(s_i) = \mathrm{Z}_{q-\epsilon}$. Thus $J_i = C(s_i) \simeq \mathrm{GL}_2^\epsilon(q)$, $J_1 \ge C_\Omega(s_1) = C_{M_1}(s_1) \circ M_2$ and $J_2 \ge C_\Omega(s_2) = M_1 \circ C_{M_2}(s_2)$. In addition, J_1 and J_2 are not G-conjugate. Indeed, if $J_1^w = J_2$ for some $w \in G$, then $\langle s_1\rangle^w = O_2(Z(J_1))^w = O_2(Z(J_2)) = \langle s_2\rangle$ and $\Omega_1(\langle s_1\rangle)^w = \Omega_1(\langle s_2\rangle)$, and so $w \in \mathrm{SO}(U_4)$, since $\Omega_1(\langle s_1\rangle) = \Omega_1(\langle s_2\rangle) = Z(\mathrm{SO}(U_4))$. Thus $w = t$ or tg for some $t \in M_1 \circ M_2$, so $s_1^w \in M_1$. But $\langle s_1\rangle^w = \langle s_2\rangle \le M_2$, so $s_1^w \in M_1 \cap M_2 = Z(M)$, which is impossible.

Since J_1 and J_2 are not G-conjugate, it follows that J_i^α and J_i are G-conjugate, so that $J_i^{c_i\alpha} = J_i$ for some $c_i \in G$. Thus $\Omega_1(\langle s_i\rangle)^{c_i\alpha} = \Omega_1(\langle s_i\rangle)$ and $c_i \in \mathrm{SO}(U_4)$, since α centralizes $Z(M) = \Omega_1(\langle s_i\rangle)$. But $M_i = [\,J_i, J_i\,]$ and by (2.5), $c_i = t$ or tg for some $t \in \Omega$, so $M_i^{c_i\alpha} = M_i^\alpha = M_i$. Since A has only one class of subgroups J_ϵ, it follows that J_1 and J_2 are A-conjugate and $s_1^\omega = s_2$ for some $\omega \in A\backslash G$. So $Z(M)^\omega = Z(M)$, $J_1^\omega = J_2$ and $\omega \in \langle \mathrm{SO}(U_4), \beta'\rangle \backslash \mathrm{SO}(U_4) \,\mathrm{o}\, \langle\alpha\rangle$. Since $\Omega(U_4)$ is the only subgroup of index two in $\mathrm{SO}(U_4)$, it follows that $C_\Omega(s_1)^\omega = C_\Omega(s_2)$. Let $Z_j = [\,C_\Omega(s_j), C_\Omega(s_j)\,]$ for $j = 1, 2$. Then $Z_1^\omega = Z_2$ and $Z_j \le M_i$, where $i \ne j$ and $i, j \in \{1, 2\}$. If $q \ne 3$, then $Z_j = M_i$. Suppose $q = 3$, so that $Z_j = O_2(M_i)$ is quaternion. If m_2 is an element of order 3 in $\mathrm{SL}(W_2) \le C_\Omega(s_1)$, then $m_2^\omega \in C_\Omega(s_2) = \mathrm{SL}(W_1) \circ \langle s_2\rangle$, so that $m_2^\omega \in \mathrm{SL}(W_1)$. But $\langle Z_1, m_2\rangle = \mathrm{SL}(W_2)$, so $\mathrm{SL}(W_2)^\omega = \mathrm{SL}(W_1)$. In all cases, $M_2^\omega = M_1$. Since $\omega \notin \mathrm{SO}(U_4) \,\mathrm{o}\, \langle\alpha\rangle$, it follows that $\omega = t\beta'$ for some $t \in \mathrm{SO}(U_4) \,\mathrm{o}\, \langle\alpha\rangle$, so that $\mathrm{SL}(W_1) = \mathrm{SL}(W_2)^\omega = \mathrm{SL}(W_2)^{t\beta'} = \mathrm{SL}(W_2)^{\beta'}$. Since ${\beta'}^2 \in \mathrm{SO}(U_4) \,\mathrm{o}\, \langle\alpha\rangle$, it follows that $\mathrm{SL}(W_1)^{\beta'} = \mathrm{SL}(W_2)$. This completes the proof.

Let $\Phi(G, r)$ be a set of representatives for conjugacy G-classes of radical r-subgroups of G, where $R \leq G$ is radical if $R = O_r(N(R))$. In addition, let $\mathrm{Z}_{r^a} = Z(J_\epsilon)$ and $\mathrm{Z}^*_{r^a} = Z(J_\epsilon{}^*)$. Suppose $r \geq 5$. Then we may take

$$\Phi(G, r) = \{1, \mathrm{Z}_{r^a}, \mathrm{Z}^*_{r^a}, O_r(T_\epsilon)\}.$$

Suppose $r = 3$. Then we may take

$$\Phi(G,3) = \begin{cases} \{1, Z(L_\epsilon) = \mathrm{Z}_3, \mathrm{Z}^*_3, O_3(T_\epsilon), E = S\} & \text{if } a = 1 \text{ and } 3 \neq q - \epsilon, \\ \{1, Z(L_\epsilon) = \mathrm{Z}_3, \mathrm{Z}^*_3, E = S\} & \text{if } a = 1 \text{ and } 3 = q - \epsilon, \\ \{1, Z(L_\epsilon), \mathrm{Z}_{3^a}, \mathrm{Z}^*_{3^a}, O_3(T_\epsilon), E, E^*, S\} & \text{if } a \geq 2 \text{ and } 3^a \neq q - \epsilon, \\ \{1, Z(L_\epsilon), \mathrm{Z}_{3^a}, \mathrm{Z}^*_{3^a}, E, E^*, S\} & \text{if } a \geq 2 \text{ and } 3^a = q - \epsilon, \end{cases}$$

where $E \simeq E^* \simeq 3^{1+2}_+ \leq L_\epsilon$ and $S \in \mathrm{Syl}_3(L_\epsilon)$ is a Sylow 3-subgroup of L_ϵ.

Define radical r-chains $C(1)$ and $C(2)$ as follows:

$$\begin{aligned} &C(1) : \begin{cases} 1 < \mathrm{Z}_{r^a} < O_r(T_\epsilon) & \text{if } r \geq 5, \\ 1 < Z(L_\epsilon) & \text{if } r = 3, \end{cases} \\ &C(2) : 1 < \mathrm{Z}^*_{r^a} < O_r(T_\epsilon) \qquad \text{if } r \geq 3. \end{aligned} \tag{2.6}$$

If $r \geq 5$, then we may take

$$\mathcal{R}(G)/G = \{C(1)_0, C(1)_1, C(2)_1, C(1)^1, C(1), C(2)\}$$

and set $\mathcal{R}^0(G) = \mathcal{R}(G)$. If $r = 3$, then let $\mathcal{R}^0(G)$ be a G-invariant subfamily of $\mathcal{R}(G)$ such that

$$\mathcal{R}^0(G)/G = \{C(1)_0, C(1), C(2)_1, C(2)\}. \tag{2.7}$$

(2D) *Let $G = G_2(q)$. In the notation above, suppose $r \geq 3$. Let $B \in \mathrm{Blk}(G)$ with $\mathrm{d}(B) > 0$ and $d, u \in \mathrm{Z}$. Then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|}\, \mathrm{k}(N(C), B, d, u) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|}\, \mathrm{k}(N(C), B, d, u).$$

Proof. Suppose $r = 3$, so that $p \neq 3$. If R is nonabelian or $R = \mathrm{Z}_{3^a}$, then $Z(L_\epsilon) = \Omega_1(Z(R))$ and so $N_A(R) \leq N_A(Z(L_\epsilon))$. If $R = O_3(T_\epsilon)$, then $N_A(R) = N_G(T_\epsilon) \circ \langle \alpha \rangle \leq N_A(L_\epsilon)$, since $N_G(T_\epsilon) = N_{K_\epsilon}(T_\epsilon)$. Thus the proof of [4, (2A)] can be apply here with some obvious modifications.

Suppose $r = 2$, so that p is odd and $\Phi(G, 2)$ is given in Section 2 of [2]. A 2-group R is called of *symplectic type* if R is a central product EP over $Z(E) = \Omega_1(Z(P))$ of an extraspecial 2-group E and a 2-group P such that P is either cyclic or isomorphic to a semidihedral group SD_{2^λ}, a dihedral group D_{2^λ} or a generalized quaternion group

Q_{2^λ} of order 2^λ, where $\lambda \geq 4$. By [25, Lemma 2.4], $G_2(p) = C_G(\alpha)$ has a class of subgroups E_8 such that $C_G(E_8) = E_8$ and $N_G(E_8) = N_{G_2(p)}(E_8) = E_8.\,\mathrm{GL}_3(2)$. Suppose $p = 3$. By the uniqueness of E_8 in G, we may suppose $E_8 = \langle \rho, \Omega_1(O_2(T_\epsilon))\rangle$. Then $\Omega_1(O_2(T_\epsilon)) = \langle h_1(-1), h_2(-1)\rangle$ and β interchanges $h_1(-1)$ and $h_2(-1)$. Since β induces a non-trivial automorphism of $G_2(3)$, it follows by [11, p.61] that

$$N_A(G_2(3)) = G_2(3) : 2 \times \langle \alpha \rangle$$

and $N_A(E_8) = E_8.L_3(2){:}\,2 \times \langle \alpha \rangle$. Let $\Omega_2(O_2(T_\epsilon)) = \langle u, w\rangle$, where $u = h_a(s)$ and $w = h_b(s)$ for some $s \in \mathrm{Z}_{q-\epsilon}$ of order 4. Then $s^2 = -1$, $u^2 = h_1(-1)$, $w^2 = h_2(-1)$ and $u^{\beta\sigma^2} = h_3(-1)u$. In addition, $u^{\tau\sigma} = u$ and $w^{\tau\sigma} = uw^{-1}$, so $(\tau\sigma u)^2 = h_1(-1)$ and $\rho^{\tau\sigma u} = \rho$. Let $S_1 = \langle h_2(-1), \tau\sigma u\rangle$ and $S_2 = \langle \rho, u\rangle$. Then $[\,S_1, S_2\,] = 1$, $S_1 \simeq S_2 \simeq D_8$ and S_1S_2 is a central product over $Z(S_1) = Z(S_2) = \langle h_1(-1)\rangle$. Thus

$$S_1S_2 = 2_+^{1+4} = \langle h_2(-1), \tau\sigma u\rangle \circ \langle \rho, u\rangle$$

and moreover, $\beta\sigma^2 \in N_{N_A(E_8)}(2_+^{1+4})$.

If $2^a \neq q - \epsilon$, then $O_2(T_\epsilon) \neq T_\epsilon$ and $O_2(T_{-\epsilon}) \neq T_{-\epsilon}$. From Section 2 of [2], we may take

$$\Phi(G,2) = \{1, \mathrm{Z}_2, \mathrm{Z}_{2^a}, \mathrm{Z}_{2^{a+1}}, E_8, O_2(T_{-\epsilon}), O_2(T_\epsilon), \mathrm{Z}_{2^a} \wr \mathrm{Z}_2, \langle O_2(T_\epsilon), \rho\rangle, EP, S\},$$

where $\mathrm{Z}_2 = O_2(\mathrm{SO}(U_4)) = O_2(\mathrm{SO}_4^+(q))$, $\mathrm{Z}_{2^a} = O_2(C_G(\mathrm{Z}_{2^a})) = O_2(\mathrm{GL}_2^\epsilon(q))$, $\mathrm{Z}_{2^{a+1}} = O_2(T_3^\delta)$, EP is a non-abelian 2-groups of symplectic type, and $S \in \mathrm{Syl}_2(G)$. In addition, G has two classes of Z_{2^a}, $\mathrm{Z}_{2^{a+1}}$, $\mathrm{Z}_{2^a} \wr \mathrm{Z}_2$, and EP except when $EP = 2_+^{1+4}$, $q \equiv \pm 3 (\mathrm{mod}\, 8)$, and $EP = 2_+^{1+2}\mathrm{Z}_{2^a}$, in which cases G has one class of EP. If $2^a = q - \epsilon$ and $q \neq 3$, then $O_2(T_\epsilon) = T_\epsilon$ and $O_2(T_{-\epsilon}) \neq T_{-\epsilon}$, $O_2(T_\epsilon)$ is not a radical 2-subgroup of G since $O_2(N_G(T_\epsilon)/T_\epsilon) \simeq \mathrm{Z}_2$, so that by Section 2 of [2], we may take

$$\Phi(G,2) = \{1, \mathrm{Z}_2, \mathrm{Z}_{2^a}, \mathrm{Z}_{2^{a+1}}, E_8, O_2(T_{-\epsilon}), \langle O_2(T_\epsilon), \rho\rangle, EP, S\},$$

and similarly, if $q = 3$, then $O_2(T_\epsilon)$ and $O_2(T_{-\epsilon})$ are non-radical 2-subgroups in G, and moreover, $\mathrm{SO}(U_4) = \mathrm{SO}_4^+(3) = 2_+^{1+4}{:}\,3^2.2$. Suppose $R \in \Phi(G,2)$ and $Z(R)$ is cyclic. We may suppose $\Omega_1(Z(R)) = Z(\mathrm{SO}(U_4))$, so that $N_G(R) \leq \mathrm{SO}(U_4)$ and R is radical in $\mathrm{SO}(U_4)$. In particular, $2_+^{1+4} = O_2(\mathrm{SO}(U_4)) \leq R$ and $R = 2_+^{1+4}$ or $R \in \mathrm{Syl}_2(G)$ is Sylow. By Section 2 of [2], we may take

$$\Phi(G,2) = \{1, E_8, \langle O_2(T_\epsilon), \rho\rangle, 2_+^{1+4}, S\},$$

where the radical 2-subgroups are defined as above.

We define chains $C(i)$ for $1 \leq i \leq 4$ as follows.

$$\begin{aligned}
&C(1) : 1 < \mathrm{Z}_2 < O_2(T_{-\epsilon}),\\
&C(2) : \begin{cases} 1 < \mathrm{Z}_2 < O_2(T_\epsilon) & \text{if } 2^a \neq q-\epsilon,\\ 1 < \langle O_2(T_\epsilon), \rho\rangle < S' & \text{if } 2^a = q-\epsilon,\end{cases}\\
&C(3) : 1 < E_8 < 2_+^{1+4},\\
&C(4) : 1 < E_8 < W < S'',
\end{aligned} \tag{2.8}$$

where $2_+^{1+4} \leq N_G(E_8)$, $W = \langle \rho, \Omega_2(O_2(T_\epsilon))\rangle \leq N_G(E_8)$ containing $E_8 = \langle \rho, \Omega_1(O_2(T_\epsilon))\rangle$, $W/E_8 \simeq \mathrm{Z}_2 \times \mathrm{Z}_2$, $S'' = \langle \Omega_2(O_2(T_\epsilon)), \rho, \tau\rangle \in \mathrm{Syl}_2(N_G(E_8))$ and $S' = \langle O_2(T_\epsilon), \rho, \tau\sigma\rangle \in \mathrm{Syl}_2(N_G(T_\epsilon))$. Note that if $q = 3$, then $S', S'' \in \mathrm{Syl}_2(G)$. Let $\mathcal{R}^0(G)$ be a G-invariant subfamily of $\mathcal{R}(G)$ such that if $q \neq 3$, then

$$\mathcal{R}^0(G)/G = \{C(1)_0, C(1)_1, C(1)^1, C(1), C^1(2), C(2), C(3)_1, C(3), C(4)_2, C(4)\}, \tag{2.9}$$

where $C^1(2) = C(2)^1$ or $C(2)_1$ according to whether $2^a \neq q-\epsilon$ or $2^a = q-\epsilon$; if $q = 3$, then

$$\mathcal{R}^0(G)/G = \{C(1)_0, C(3)_1, C(3)^1, C(3)\}. \tag{2.10}$$

We have the following proposition.

(2E) *Suppose B is a 2-block of $G = G_2(q)$ with $\mathrm{d}(B) > 0$ and $d, u \in \mathrm{Z}$. Then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|}\, \mathrm{k}(N_G(C), B, d, u) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|}\, \mathrm{k}(N_G(C), B, d, u).$$

Proof. Suppose $C \in \mathcal{R}(G)$ is given by (1.1) with $|C| \geq 1$ and $P_1 \in \Phi(G, 2)$. Let $N(C) = N_G(C)$ and $C(C) = C_G(C)$.

(1) Suppose $P_1 = \mathrm{Z}_2$ and $|C| \geq 2$, so that we may take $P_2 \in \Phi(\mathrm{SO}(U_4), 2)$. As shown in the proof of [2, (2B) and (2C)], we may suppose $P_2 \in \Phi(G, 2)$ and both E_8 and $\langle O_2(T_\epsilon), \rho\rangle$ are non-radical 2-subgroups of $\mathrm{SO}(U_4)$. Let

$$\Omega^* = \{1, \mathrm{Z}_2, O_2(T_\epsilon), E_8, O_2(T_{-\epsilon}), \langle O_2(T_\epsilon), \rho\rangle\}$$

and take $R \in \Phi(G, 2)\backslash\Omega^*$. Then $\mathrm{Z}_2 = \Omega_1(Z(R))$, so that $N_A(R) \leq N_A(\mathrm{Z}_2) = \langle \mathrm{SO}(U_4), \beta'\rangle$. Replace $Z(L_\epsilon)$ by Z_2 in the definition of $\mathcal{M}^0(R)$ in [4, (2.3)]. Then $N(C') = N(g(C'))$ and $N_A(C') = N_A(g(C'))$, and so

$$\mathrm{k}(N(C'), B, d, u) = \mathrm{k}(N(g(C')), B, d, u) \tag{2.11}$$

for all $C' \in \mathcal{M}^0(R)$, where $g(C')$ is defined after [4, (2.3)]. We may suppose

$$C \notin \bigcup_R (\mathcal{M}^+(R) \cup \mathcal{M}^0(R)),$$

where R runs over the set $\Phi(G,2)\backslash\Omega^*$. Thus we may take $P_1 \in \Omega^*$ and if $P_1 = \mathrm{Z}_2$ with $|C| \geq 2$, then $q \neq 3$ and we may suppose $P_2 \in \{O_2(T_{-\epsilon}), O_2(T_\epsilon)\}$.

(2) If $P_1 = O_2(T_\epsilon)$, then $2^a \neq q - \epsilon$, $N_A(P_1) = N_A(T_\epsilon) = \langle T_\epsilon, \sigma, \rho, \tau\rangle \circ \langle\beta\rangle$ and $\langle\rho T_\epsilon, \tau\sigma T_\epsilon\rangle \in \mathrm{Syl}_2(D_{12})$. Thus we may take

$$\Phi(N(O_2(T_\epsilon)), 2) = \{O_2(T_\epsilon), \langle O_2(T_\epsilon), \rho\rangle, \mathrm{Z}_{2^a} \wr \mathrm{Z}_2, (\mathrm{Z}_{2^a} \wr \mathrm{Z}_2)^*, S'\},$$

where $\mathrm{Z}_{2^a} \wr \mathrm{Z}_2 = \langle O_2(T_\epsilon), \tau\rangle$, $(\mathrm{Z}_{2^a} \wr \mathrm{Z}_2)^* = \langle O_2(T_\epsilon), \rho\tau\sigma\rangle$ and $S' = \langle O_2(T_\epsilon), \rho, \tau\sigma\rangle$. If $p = 3$, then $(\rho\tau\sigma)^\beta = \tau^\alpha = \tau$ and $\sigma^{-2}\tau\sigma^2 = \sigma\tau\sigma^{-1} = h_2(-1)\tau\sigma$, so

$$N_A(S') = \langle N_G(S'), \beta'\rangle \qquad N_A(R) = N_G(R) \circ \langle\beta\rangle$$

for $R \in \{O_2(T_\epsilon), \langle O_2(T_\epsilon), \rho\rangle\}$ and $N_A(R) = N_G(R) \circ \langle\alpha\rangle$ for $R \in \{\mathrm{Z}_{2^a} \wr \mathrm{Z}_2, (\mathrm{Z}_{2^a} \wr \mathrm{Z}_2)^*\}$. Replace $Z(L_\epsilon)$ by $O_2(T_\epsilon)$ and R by $\langle O_2(T_\epsilon), \rho\rangle$ in [4, (2.3)]. Then (2.11) holds. Thus we may suppose $P_2 \in \{\mathrm{Z}_{2^a} \wr \mathrm{Z}_2, (\mathrm{Z}_{2^a} \wr \mathrm{Z}_2)^*, S'\}$. Moreover, we may suppose P_1 and $\langle O_2(T_\epsilon), \rho\rangle$ are not conjugate in G.

Suppose $2^a = q - \epsilon$. If $P_1 = \langle O_2(T_\epsilon), \rho\rangle$, then $N(P_1) = N(T_\epsilon)$ and $P_2 = S' \in \mathrm{Syl}_2(N(P_1))$. Thus $C(2)$ and $C(2)_1$ are radical chains.

(3) Suppose $C' \in \mathcal{R}(G)$ such that

$$C' : 1 < O_2(T_\epsilon) < P_2 < \ldots < P_n,$$

where $P_2 \in \{\mathrm{Z}_{2^a} \wr \mathrm{Z}_2, (\mathrm{Z}_{2^a} \wr \mathrm{Z}_2)^*, S'\}$, so that $q \neq 3$. Define radical 2-chain

$$g(C') : 1 < \mathrm{Z}_2 < O_2(T_\epsilon) < P_2 < \ldots < P_n.$$

Then $N(C') = N(g(C'))$ since $N(T_\epsilon) \cap N(P_2) \leq \langle T_\epsilon, \rho, \tau\sigma\rangle \leq N(\mathrm{Z}_2)$. Moreover, $N_A(C') = N_A(g(C'))$ since $N_A(T_\epsilon) \cap N_A(P_2) \leq \langle T_\epsilon, \rho, \tau\sigma, \beta'\rangle \leq N_A(\mathrm{Z}_2)$. Thus (2.11) holds. By [4], we may take

$$\Phi(N_{N(\mathrm{Z}_2)}(O_2(T_\epsilon)), 2) = \{O_2(T_\epsilon), \mathrm{Z}_{2^a} \wr \mathrm{Z}_2, (\mathrm{Z}_{2^a} \wr \mathrm{Z}_2)^*, S'\}.$$

It follows that the remaining chains $C \in \mathcal{R}(G)$ such that P_1 is conjugate to Z_2 or $O_2(T_\epsilon)$ and $|P_2| \neq 4$ have representatives, $C(1)_1$, $C(2)$ and $C(2)^1$.

(4) Suppose $P_1 = E_8$, so that $N_G(E_8) = N_{C_G(\alpha)}(E_8)$. If $p \neq 3$, then $N_A(E_8) = N(E_8) \times \langle\alpha\rangle$. By the proof of [4, (2B) (4)] we may suppose the remaining chains in $\mathcal{R}(G)$ with $P_1 \simeq E_8$ have representatives $C(3)$, $C(3)_1$, $C(4)$ and $C(4)_2$.

Suppose $p = 3$. As shown in the proof of [4, (2B) (4)] we may suppose $P_2 \in \{2_+^{1+4}, W, S''\}$, where $W = \langle\rho, \Omega_2(O_2(T_\epsilon))\rangle$ and $S'' \in \mathrm{Syl}_2(N(E_8))$. Thus $N_{N(E_8)}(W) = \langle\Omega_2(O_2(T_\epsilon)), \sigma, \tau, \rho\rangle$ and $N_{N_A(E_8)}(W) = N_{N(E_8)}(W) \circ \langle\beta\rangle$. We may suppose

$$S'' = \langle\Omega_2(O_2(T_\epsilon)), \tau\sigma, \rho\rangle,$$

so that $N_{N_A(E_8)}(S'') = \langle S'', \beta' \rangle$. Let $M = C_{\mathrm{SO}(U_4)}(\alpha)$. Then $M = \mathrm{SO}_4^+(3)$ is a maximal subgroup of $G_2(3) = C_G(\alpha)$ and $M = 2_+^{1+4} : \frac{1}{2}\Omega_4^+(2)$, where $\frac{1}{2}\Omega_4^+(2)$ is the intersection of the three subgroups of index 2 in $\mathrm{O}_4^+(2)$.

In the notatio of (2.5) (with $q = 3$), let $v_i = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $y_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $m_i = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ be elements of $M_i = \mathrm{SL}_2(3)$. Then $y_i^{v_i} = y_i^{-1} = zy_i$, $|v_i| = |y_i| = 4$ and $O_2(M_i) = \langle v_i, y_i \rangle \simeq Q_8$, where $Z(M) = \langle z \rangle$. Thus $2_+^{1+4} = O_2(M_1) \circ O_2(M_2) = O_2(M)$ and we may suppose $E_8 = \langle z, v_1v_2, y_1y_2 \rangle$, so that $E_8 \leq O_2(M) \leq N(E_8)$. Now $v_i^{m_i} = y_i$, $y_i^{m_i} = zv_iy_i$, $v_i^{g_i} = zv_iy_i$ and $y_i^{g_i} = zy_i$. Both $m = m_1 \otimes m_2$ and g normalize E_8, and $N_{\mathrm{SO}_4^+(3)}(E_8) = \langle O_2(M), m, g \rangle$, since $|\,\mathrm{SO}_4^+(3) : \langle O_2(M), m, g \rangle| = 3$. Since $N(E_8) \leq C_G(\alpha)$, it follows that

$$N_{N(E_8)}(2_+^{1+4}) = N(E_8) \cap N(2_+^{1+4}) = N_{\mathrm{SO}_4^+(3)}(E_8) = \langle 2_+^{1+4}, m, g \rangle. \tag{2.12}$$

In particular, we may suppose $\Phi(N(E_8), 2) = \{E_8, O_2(C_G(\alpha)), \langle \Omega_2(O_2(T_\epsilon)), \rho \rangle, S''\}$. Since β' stabilizes M, it stabilizes $2_+^{1+4} = O_2(M)$ and so

$$N_{N_A(E_8)}(2_+^{1+4}) = \langle N_{N(E_8)}(2_+^{1+4}), \beta' \rangle.$$

Apply (2.11) to

$$C' : 1 < E_8 < 2_+^{1+4} < S'' \qquad \text{and} \qquad g(C') : 1 < E_8 < S''.$$

Then the remaining chains in $\mathcal{R}(G)$ with $P_1 \simeq E_8$ have representatives $C(3)$, $C(3)_1$, $C(4)$ and $C(4)_2$.

If $q = p = 3$, then we can suppose $S'' = S' = S \in \mathrm{Syl}_2(G)$. If $C' = C(4)$ and $g(C') = C(2)$, then $N(C') = N(g(C')) = S$ and $N_A(C') = N_A(g(C')) = \langle S, \beta' \rangle$, so that (2.11) still holds. Similarly, we can apply (2.11) to $C' = C(2)_1$ and $g(C') = C(4)_2$, since $\langle O_2(T_\epsilon), \rho \rangle = W = \langle \Omega_2(T_\epsilon), \rho \rangle$. This implies (2E).

(5) If $P_1 = O_2(T_{-\epsilon})$, then $q \neq 3$ and $N(P_1) = N(T_{-\epsilon}) = \langle T_{-\epsilon}, \sigma, \tau, \rho \rangle$, where σ, τ, ρ are given as before. Thus we may suppose $P_2 \in \{E_8, D_8, D_8^*, Q\}$, where $E_8 = \langle P_1, \rho \rangle$, $D_8 = \langle P_1, \tau \rangle$, $D_8^* = \langle P_1, \rho\tau\sigma \rangle$, and $Q = \langle E_8, \tau\sigma \rangle \in \mathrm{Syl}_2(N(P_1))$. In particular, $N_A(E_8) = \langle N(E_8), \beta' \rangle$, $N_A(R) = N(R) \times \langle \alpha \rangle$ for $R \in \{D_8, D_8^*\}$ and $N_A(Q) = \langle N(Q), \beta' \rangle$. Let

$$C' : 1 < O_2(T_{-\epsilon}) < E_8 \qquad \text{and} \qquad g(C') : 1 < O_2(T_{-\epsilon}) < E_8 < Q.$$

Since $C(E_8) = E_8 \trianglelefteq N(C')$, it follows by [16, Corollary V.3.11] that $\mathrm{Irr}(b(C')) = \mathrm{Irr}(N(C'))$, where $b(C')$ is the principal block $B_0(N(C'))$ of $N(C')$. Similarly,

$$\mathrm{Irr}(b(g(C'))) = \mathrm{Irr}(N(g(C')))$$

with $b(g(C')) = B_0(N(g(C')))$. Let $h_1 = h_1(-1), h_2 = h_2(-1)$ and $h_3 = h_3(-1)$, so that $O_2(T_{-\epsilon}) = \langle h_1, h_2\rangle$. Since $\rho^\sigma = h_1\rho = \rho^{\tau\sigma}$ and $h_2^\sigma = h_2^{\tau\sigma} = h_3$, it follows that $N(g(C')) = Q = \mathrm{Z}_2 \times D_8$ and $N(C') = \langle Q, \sigma\rangle = \mathrm{Z}_2 \times A_4{:}\,2$, where $\mathrm{Z}_2 = \langle h_2\rho\rangle$, $D_8 = \langle h_1, h_2, \tau\sigma\rangle$ and $A_4{:}\,2 = \langle h_1, h_2, \sigma, \tau\sigma\rangle$. In addition, $N_A(g(C')) = \langle Q, \beta'\rangle$ and $N_A(C') = \langle Q, \sigma, \beta'\rangle$. Thus $\mathrm{Irr}(Q)$ has 8 linear characters and 2 characters of degree 2, and $\mathrm{Irr}(\langle Q, \sigma\rangle)$ has 4 linear characters, 2 characters of degree 2 and 4 of degree 3. If $p \neq 3$, then $N_A(g(C')) = Q \times \langle\alpha\rangle$ and $N_A(C') = \langle Q, \sigma\rangle \times \langle\alpha\rangle$, so that (2.11) holds.

Suppose $p = 3$, so that ${\beta'}^2 = h_1\alpha$ and α acts trivially on $\langle Q, \sigma\rangle$. Let $\theta = \beta'\langle\alpha\rangle \in \langle\beta'\rangle/\langle\alpha\rangle$. Then $\theta^2 = h_1$ and

$$\rho^\theta = h_1\rho, \quad h_2^\theta = h_3, \quad (\tau\sigma)^\theta = h_3\rho\tau\sigma, \quad \sigma^\theta = h_1\sigma^{-1}.$$

Let $H = \langle h_1, h_2, \rho, \tau, \sigma, \theta\rangle = \langle Q, \sigma, \theta\rangle$ and $h = \theta\tau\sigma h_2$. Then $h^2 = (\theta\tau\sigma h_2)(\theta\tau\sigma h_2) = \theta^2(\tau\sigma)^\theta h_2^\theta \tau\sigma h_2 = h_1 h_3 \rho h_3^{\tau\sigma} h_2 = h_2\rho$ and $C_H(h) = \langle h, h_1, h_2, \sigma\rangle = \mathrm{Z}_4 \times A_4$, where $\mathrm{Z}_4 = \langle h\rangle$ and $A_4 = \langle h_1, h_2, \sigma\rangle$. Moreover, $h^\theta = \theta\tau\sigma\rho = h^{-1}$, $A_4^\theta = A_4$ and $H = \langle C_H(h), \theta\rangle$ with $(H{:}\, C_H(h)) = 2$. But the commutator subgroup $[\,H, H\,] = \langle h_1, \sigma, h_2, \rho\rangle$, so H has exactly 4 linear characters. So θ stabilizes exactly the trivial character and the non-linear character of $\mathrm{Irr}(A_4)$. Since $h^\theta = h^{-1}$, it follows that θ stabilizes exactly two linear characters and two characters of degree 3 in $\mathrm{Irr}(C_H(h))$. Since $\mathrm{Irr}(C_H(h))$ has 12 linear chacaters and 4 characters of degree 3, it follows that $\mathrm{Irr}(H)$ has 4 linear characters, 4 characters of degree 3, 1 of degree 6 and 5 of degree 2. Thus θ stabilizes exactly 2 linear characters, 2 characters of degree 3 and 2 of degree 2 in $\mathrm{Irr}(N(C'))$.

Let $M = \langle h_1, h_2, \tau\sigma, \rho, \theta\rangle$, so that $h \in M$. Thus $C_M(h) = \langle h, h_1, h_2\rangle = \mathrm{Z}_4 \times O_2(T_{-\epsilon})$ is a subgroup of index 2 in M and $M = \langle C_M(h), \theta\rangle$. Since $[\,M, M\,] = \langle h_1, h_3\rho\rangle$, it follows that M has exactly 8 linear characters. But $C_H(h)$ has 16 linear characters, so $\mathrm{Irr}(M)$ has exactly 8 linear characters and 6 of degree 2. It follows that θ stabilizes exactly 4 linear characters and 2 characters of degree 2 in $\mathrm{Irr}(N(g(C')))$. Thus

$$\mathrm{k}(N(C'), B_0, d, u) = \mathrm{k}(N(g(C')), B_0, d, u) = \begin{cases} 4 & \text{if } d = 4 \text{ and } u = 2e, \\ 4 & \text{if } d = 4 \text{ and } u = e, \\ 2 & \text{if } d = 3 \text{ and } u = 2e, \\ 0 & \text{otherwise.} \end{cases}$$

In all cases we may take $P_2 \in \{D_8, D_8^*, Q\}$. Let

$$C' : 1 < O_2(T_{-\epsilon}) < P_2 < \ldots < P_n \in \mathcal{R}(G)$$

$$g(C') : 1 < \mathrm{Z}_2 < O_2(T_{-\epsilon}) < P_2 < \ldots < P_n.$$

Then $g(C') \in \mathcal{R}(G)$, $N(C') = N(g(C'))$ and $N_A(C') = N_A(g(C'))$, since $N(T_{-\epsilon}) \cap N(P_2) \leq T_{-\epsilon}Q \leq N(\mathrm{Z}_2)$ and since $N_A(T_{-\epsilon}) \cap N_A(P_2) \leq \langle T_{-\epsilon}, Q, \beta'\rangle \leq N_A(\mathrm{Z}_2)$. So (2.11) holds and a proof similar to that of (3) above shows that we may take

$$\Phi(N_{N(\mathrm{Z}_2)}(O_2(T_{-\epsilon})), 2) = \{O_2(T_{-\epsilon}), D_8, D_8^*, Q\}.$$

Thus the remaining radical chains $C \in \mathcal{R}(G)$ such that P_1 is conjugate to Z_2 or $O_2(T_{-\epsilon})$ and P_2 is not conjugate to $O_2(T_\epsilon)$ have representatives, $C(1)_1$, $C(1)$ and $C(1)^1$. Thus (2E) follows.

(2F) *Let $C \in \mathcal{R}^0(G)$ with $|C| \geq 1$. If $r \geq 3$, then*

C	$N_A(C)$	Conditions
$C(1)_1 : 1 < \mathrm{Z}_{r^a}$	$\langle \mathrm{GL}_2^\epsilon(q), \rho, \alpha \rangle$	$r \geq 5$
$C(1)^1 : 1 < O_r(T_\epsilon)$	$N_A(T_\epsilon)$	$r \geq 5$
$C(2)_1 : 1 < \mathrm{Z}^*_{r^a}$	$\langle \mathrm{GL}_2^\epsilon(q), \rho, \alpha \rangle$	$r \geq 3$
$C(1) : 1 < \mathrm{Z}_{r^a} < O_r(T_\epsilon)$	$\langle T_\epsilon, \rho, \tau, \alpha \rangle$	$r \geq 5$
$C(2) : 1 < \mathrm{Z}^*_{r^a} < O_r(T_\epsilon)$	$\langle T_\epsilon, \rho, \tau, \alpha \rangle$	$r \geq 3$
$C(1) : 1 < Z(L_\epsilon)$	$\langle L_\epsilon, \rho, \alpha \rangle$	$r = 3$

where $\rho, \sigma, \tau, \alpha$ are given by (2.2).

Proof. The proof is given by that of (2D).

(2G) *Let $C \in \mathcal{R}^0(G)$ with $|C| \geq 1$. If $r = 2$, then*

C	$N_A(C)$	Conditions
$C(1)_1 : 1 < \mathrm{Z}_2$	$\langle \mathrm{SO}_4^+(q), \beta' \rangle$	$q \neq 3$
$C(1)^1 : 1 < O_2(T_{-\epsilon})$	$N_A(T_{-\epsilon})$	$q \neq 3$
$C(2)^1 : 1 < O_2(T_\epsilon)$	$N_A(T_\epsilon)$	$q - \epsilon \neq 2^a$
$C(2)_1 : 1 < \langle O_2(T_\epsilon), \rho \rangle$	$N_A(T_\epsilon)$	$q - \epsilon = 2^a$
$C(3)_1 : 1 < E_8$	$\langle E_8.\, \mathrm{GL}_3(2), \beta' \rangle$	none
$C(3)^1 : 1 < 2_+^{1+4}$	$\langle \mathrm{SO}_4^+(3), \beta' \rangle$	$q = 3$
$C(1) : 1 < \mathrm{Z}_2 < O_2(T_{-\epsilon})$	$\langle T_{-\epsilon}, \rho, \tau, \beta' \rangle$	$q \neq 3$
$C(2) : 1 < \mathrm{Z}_2 < O_2(T_\epsilon)$	$\langle T_\epsilon, \tau, \rho, \beta' \rangle$	$q - \epsilon \neq 2^a$
$C(2) : 1 < \langle O_2(T_\epsilon), \rho \rangle < S'$	$\langle S', \beta' \rangle$	$q - \epsilon = 2^a$
$C(3) : 1 < E_8 < 2_+^{1+4}$	$\langle 2_+^{1+4}.S_3, \beta' \rangle$	none
$C(4)_2 : 1 < E_8 < W$	$\langle W, \sigma, \tau, \beta \rangle$	$q \neq 3$
$C(4) : 1 < E_8 < W < S''$	$\langle S'', \beta' \rangle$	$q \neq 3$

where ρ, σ, τ are given by (2.2), *$\beta' = \alpha$ or $\beta\sigma^2$ according as $p \neq 3$ or $p = 3$, $E_8 = \langle \rho, \Omega_1(O_2(T_\epsilon)) \rangle$, $W = \langle \rho, \Omega_2(O_2(T_\epsilon)) \rangle \leq N(E_8)$, and $S'' \in \mathrm{Syl}_2(N(E_8))$ containing W.*

Proof. The proof is given by that of (2E).

Remark: We note that if $q = 3$, the classification of radical 2-chains given by [4, (2.6)] is incorrect, since some of the 2-chains there are not radical. The family defined by

[4, (2.8)] should be replaced by the family $\mathcal{R}^0(G)$ defined by (2.10). However, the proof of the main theorem [4, (4C)] still works if we replace the chain $C : 1 < \mathrm{Z}_2$ by $C : 1 < 2_+^{1+4}$, since $N(1 < \mathrm{Z}_2) = N(1 < 2_+^{1+4}) = \mathrm{SO}_4^+(3)$ (see also the proofs in Section 4).

3. *The proof of the conjecture for odd primes*

The notation and terminology of Sections 1 and 2 are continued in this section. We shall identify a dual group of $G = G_2(q)$ with G. Let $\mathcal{E}\big(G, (s)\big)$ be the set of the irreducible constituents of Deligne-Lusztig generalized characters associated with the conjugacy class (s) of a semisimple element $s \in G$, and let

$$\mathcal{E}_r(G, (s)) = \bigcup_y \mathcal{E}(G, (sy)),$$

where $s \in G$ is a semisimple r'-element and y runs over all the r-elements of $C_G(s)$. In this section, we first prove two lemmas and then verify Dade's invariant conjecture for odd primes r.

Let $\mathrm{Blk}^0(G, r)$ be the set of r-blocks of G with non-cyclic defect groups. By [22], [23] and [24],

$$\mathrm{Blk}^0(G, r) = \begin{cases} \{B_1, B_2, B_3, B_a, B_b, B_{X_\gamma}\} & \text{if } r \geq 3, \\ \{B_1, B_3, B_{1a}, B_{1b}, B_{2a}, B_{2b}, B_{X_1}, B_{X_2}\} & \text{if } r = 2, \end{cases} \tag{3.1}$$

where $B_1 = B_0(G)$ is the principal block and $\gamma = 1$ or 2 according as $\epsilon = +$ or $-$. In addition, if q is even, then there is no block B_2. If $\gcd(q, 3) = 3$, then there is no block B_3.

Suppose $B \in \mathrm{Blk}^0(G, r)$ with $D = D(B)$ and $B \subseteq \mathcal{E}_r(G, (s))$ for some semisimple r'-element s of G. If $r \geq 3$, then by [22] and [23], D is abelian unless $r = 3$ and $B = B_1$. If $r = 2$, then by [2, (3D)], D is isomorphic to a Sylow 2-subgroup of $C_G(s)$. If $D \simeq O_2(T_\epsilon)$, then we may suppose $D = O_2(T_\epsilon)$ and so $C_G(D) = T_\epsilon$. By [7, Theorem 3.2], we may suppose $s \in C_G(D)$, so that $sy \in T_\epsilon$ for all $y \in D$. Since $D \leq C_G(sy)$, it follows that $C_G(sy) \in \{G, \mathrm{SO}(U_4), L_\delta, J_\epsilon, T_\epsilon\}$.

We denote $\chi_{t,\mu}$ the character of $\mathcal{E}(G, (t))$ labelled by (t, μ), where μ is a unipotent irreducible character of $C_G(t)$. It follows by (2B) that $\chi_{t,\mu}^\beta \in \mathcal{E}(G, (t^\beta))$.

(3A) *Let $G = G_2(q)$ and $B \in \mathrm{Blk}^0(G, r)$. In the notation of* [22], *suppose $t \in T_\epsilon$.*

(a) *If $p = 3$ and $t = 1$, then $X_{13}^\beta = X_{14}$ and β stabilizes the other 4 or 6 characters of $\mathrm{Irr}(B_1) \cap \mathcal{E}(G, (1))$ according as $r \geq 5$ or $r = 2$.*

(b) *If $p = 3$ and $|t| = 2$, then $X_{23}^\beta = X_{24}$ and β stabilizes the other 2 characters of $\mathcal{E}(G, (t))$.*

(c) *If $p \neq 3$ or $p = 3$ and $t^2 \neq 1$, then*

$$\chi_{t,\mu}^\beta = \chi_{t^\beta,\mu}$$

for all $\chi_{t,\mu} \in \mathcal{E}(G,(t)) \cap \mathrm{Irr}(B) \backslash \{X_{19}, \overline{X}_{19}\}$.

Proof. Suppose $p = 3$, so that $r \neq 3$. If $t^2 = 1$, then $t = 1$ or $|t| = 2$ and $\phi(t)^2 = 1$. Since G has only one class of involutions, it follows that $\mathcal{E}(G,(t))^\beta = \mathcal{E}(G,(t))$. In the notation of [14], the classes A_2 and A_{31} contains $x_{3a+2b}(1)$ and $x_{2a+b}(1)$, respectively. Thus β fuses A_2 and A_{31}. In addition, $X_{13} = \theta_3$, $X_{14} = \theta_4$, $X_{23} = \theta_8$ and $X_{24} = \theta_9$ (see [22, Appendix B]). By [14, Table VII-2], $\theta_3^\beta = \theta_4$ and $\theta_8^\beta = \theta_9$. By the degrees of characters given in [14, Table VII-2], $\theta_i^\beta = \theta_i$ for $0 \leq i \leq 11$ with $i \notin \{3,4,8,9\}$. This implies (b). By [22] and [24],

$$\mathrm{Irr}(B_1) \cap \mathcal{E}(G,(t)) \subseteq \{\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_{10}, \theta_{11}\},$$

which implies (a).

Suppose $p = 2$. In the notation of [15], $\theta_4(A_2) = 0$ and $\theta_3^\alpha(A_2) = \theta_3(A_2) = q$. Thus $\theta_3^\alpha \neq \theta_4$. The rest of (c) can be proved similarly using (2B), [14, Table IV-2] and [10].

Let $M_\epsilon = N_G(T_\epsilon) = \langle T_\epsilon, \rho, \tau, \sigma \rangle$ and $N(\phi(t))$ the stabilizer of $\phi(t) \in \mathrm{Irr}(T_\epsilon)$ in M_ϵ, where ϕ is given by (2.3). A similar proof to that of [4, (4.2)] shows that we may suppose

$$N(\phi(t)) \in \{M_\epsilon, \langle T_\epsilon, \rho, \tau\sigma \rangle, \langle T_\epsilon, \tau, \sigma \rangle, \langle T_\epsilon, \rho^j \tau \sigma^i \rangle,\ j, i = 0, 1\},$$

since we only consider the characters in $\mathrm{Irr}(M_\epsilon)$. Let $\mathcal{S}(t)$ be the subset of all characters of $\mathrm{Irr}(N(\phi(t)))$ consisting of characters which cover $\phi(t)$.

(3B) *Let $G = G_2(q)$ and $t \in T_\epsilon$. If $p = 3$ and $\phi(t)^\beta = \phi(t)$, then $t^2 = 1$.*

(a) *Suppose $t = 1$. Then*

$$\mathcal{S}(t) = \{\xi_{1,i} : 1 \leq i \leq 6\},$$

where $\xi_{1,i}(1) = 1$ for $1 \leq i \leq 4$ and $\xi_{1,5}(1) = \xi_{1,6}(1) = 2$. If $p = 3$, then we may suppose $\xi_{1,3}^\beta = \xi_{1,4}$ and $\xi_{1,i}^\beta = \xi_{1,i}$ for all $i \neq 3, 4$. If $p \neq 3$, then α stabilizes each character of $\mathcal{S}(1)$.

(b) *Suppose $|t| = 2$ and $N(\phi(t)) = \langle T_\epsilon, \rho, \tau\sigma \rangle$. Then*

$$\mathcal{S}(t) = \{\xi_{t,i} : 1 \leq i \leq 4\},$$

where $\xi_{t,i}(1) = 1$ for all i. If $p = 3$, then we may suppose $\xi_{t,3}^{\beta'} = \xi_{t,4}$ and $\xi_{t,i}^{\beta'} = \xi_{t,i}$ for all $i \neq 3, 4$. If $p \neq 3$, then α stabilizes each character of $\mathcal{S}(t)$.

(c) *Suppose $|t| = 3$ and $C_G(t) \simeq L_\epsilon$. Then $p \neq 3$ and*

$$\mathcal{S}(t) = \{\xi_{t,i} : 1 \leq i \leq 3\},$$

where $\xi_{t,1}(1) = \xi_{t,2}(1) = 1$ and $\xi_{t,3}(1) = 2$. In addition, $\xi_{t,i}^\alpha = \xi_{t,i}$ for all i.

(d) *Suppose $C_G(t) \simeq \mathrm{GL}_2^\epsilon(q)$. Then*

$$\mathcal{S}(t) = \{\xi_{t,1}, \xi_{t,2}\},$$

where both $\xi_{t,1}$ and $\xi_{t,2}$ are linear. In addition, $\xi_{t,i}^\alpha = \xi_{t^\alpha,i}$ for $i = 1, 2$.

(e) *Suppose $C_G(t) = T_\epsilon$. Then*

$$\mathcal{S}(t) = \{\xi_{t,1}\},$$

where $\xi_{t,1}$ is linear. In addition, $\xi_{t,1}^\beta = \xi_{t^\beta,1}$.

Proof. Suppose $p = 3$ and $\phi(t)^\beta = \phi(t)$. Then $\phi(t)^{\beta^2} = \phi(t)^\alpha = \phi(t^\alpha) = \phi(t)$, so that $t^\alpha = t^3 = t$ and $t^2 = 1$.

(a) Since $N(\phi(1)) = M_\epsilon$ and $M_\epsilon/T_\epsilon \simeq \mathrm{Z}_2 \times S_3$, $\phi(1)$ has 6 extensions to M_ϵ. Suppose $p = 3$. As shown in the proof of (2E) (5), $\langle M_\epsilon, \theta\rangle/T_\epsilon \simeq (\mathrm{Z}_4 \times 3).2$ which has 4 linear characters and 5 irreducible characters of degree 2, where $\theta = \beta'\langle\alpha\rangle$. Thus β stabilizes exactly 2 linear characters and the two characters of degree 2 in $\mathcal{S}(1)$. If $p \neq 3$, then α stabilizes each character of $\mathcal{S}(1)$, since $w^\alpha = w$ for all $w \in \langle\rho, \tau, \sigma\rangle$.

(b) Since $N(\phi(t)) = \langle T_\epsilon, \rho, \tau\sigma\rangle$, $\phi(t)$ has four linear extensions to $N(\phi(t))$. If $p = 3$, then β' stabilizes $N(\phi(t))$ and $\langle N(\phi(t)), \theta\rangle/T_\epsilon \simeq D_8$, so that β' stabilizes exactly two linear characters of $\mathcal{S}(t)$. If $p \neq 3$, then the proof is similar to that of (a).

(c) If $|t| = 3$, then $C_G(t) = L_\delta$, $N(\phi(t)) = \langle T_\epsilon, \sigma, \tau\rangle$, and $\phi(t)$ has three extensions to $N(\phi(t))$. Since $w^\alpha = w$ for $w \in \langle\sigma, \tau\rangle$, it follows that $\xi_{t,i}^\alpha = \xi_{t^\alpha,i}$.

(d) If $C_G(t) \simeq J_\epsilon$, then $N(\phi(t)) = \langle T_\epsilon, \rho^j\tau\sigma^i\rangle$, and $\phi(t)$ has two extensions to $N(\phi(t))$, non is stabilized by β. A proof similar to above shows that $\xi_{t,i}^\alpha = \xi_{t^\alpha,i}$.

(e) If $C_G(t) = T_\epsilon$, then $N(\phi(t)) = T_\epsilon$ and $\mathcal{S}(t) = \{\xi_{t,1} = \phi(t)\}$. A proof similar to above shows that $\xi_{t,1}^\beta = \xi_{t^\beta,1}$.

(3C) *Let $G = G_2(q)$ and let r be a prime such that $\gcd(r, q) = 1$. If $r \geq 3$, then Dade's invariant conjecture holds for G.*

Proof. (1) Suppose $B \in \mathrm{Blk}^0(G, r)$. Let $C = C(1)$ or $C(1)^1$ according as $r = 3$ or $r \geq 5$. In the former case $N(C) = K_\epsilon$ and in the latter case $N(C) = N(T_\epsilon)$. If D is abelian, then $N(D) \leq N(C)$ and $\mathrm{Blk}(B|N(C)) = \{b(C)\}$ by Brauer's First Main Theorem. If D is non-abelian, then $r = 3$ and B is principal, so that $\mathrm{Blk}(B|N(C)) = \{b(C)\}$ by Brauer Third Main Theorem. We shall define a bijection

$$\Psi : \mathrm{Irr}(b(C)) \longrightarrow \mathrm{Irr}(B)$$

such that w stabilizes $\Psi(\varphi)$ if and only if w stabilizes $\varphi \in \mathrm{Irr}(b(C))$ for $w \in \langle\beta\rangle$.

Suppose $B \subseteq \mathcal{E}_r(G,(s))$ and b is a block of $C_G(D)$ such that $b^{N(C)} = b(C)$. If $r \neq 3$ or $r = 3$ and $B \neq B_1$, then $C_G(D) = T_\epsilon$ and we may suppose $b = \mathcal{E}_r(T_\epsilon,(s))$.

Let $t \in T_\epsilon$ and $\mathrm{I}(\xi) = \mathrm{Ind}_{N(\phi(t))}^{N(C)}(\xi)$ for $\xi \in \mathrm{Irr}(N(\phi(t))$. Suppose $r \geq 5$. We define the map $\Psi : \cup_{b(C)} \mathrm{Irr}(b(C)) \longrightarrow \cup_B \mathrm{Irr}(B)$ as follows:

If $C_G(t) = T_\epsilon$, then $\mathcal{E}(G,(t)) = \{\chi_{t,1}\}$ and $\mathcal{S}(t) = \{\xi_{t,1}\}$. Define $\Psi(I(\xi_{t,1})) = \chi_{t,1}$.

If $C_G(t) \simeq J_\epsilon$, then $\mathcal{E}(G,(t)) = \{\chi_{t,1}, \chi_{t,\mathrm{St}}\}$ and $\mathcal{S}(t) = \{\xi_{t,1}, \xi_{t,2}\}$, where St is the Stainberg character of $C_G(t)$. Define $\Psi(I(\xi_{t,1})) = \chi_{t,1}$ and $\Psi(I(\xi_{t,2})) = \chi_{t,\mathrm{St}}$.

If $C_G(t) = L_\epsilon$, then $\mathcal{E}(G,(t)) = \{X_{31}, X_{32}, X_{33}\}$ (in the notation of [22, Appendix B]), and we define $\Psi(I(\xi_{t,i})) = X_{3i}$ for $\xi_{t,i} \in \mathcal{S}(t)$

If $|t| = 2$, then $\mathcal{E}(G,(t)) = \{X_{21}, X_{22}, X_{23}, X_{24}\}$, and we define $\Psi(I(\xi_{t,i})) = X_{2i}$ for $\xi_{t,i} \in \mathcal{S}(t)$.

If $t = 1$ and $r \geq 5$, then $\mathcal{E}(G,(t)) \cap \mathrm{Irr}(B_1) = \{X_{11}, X_{12}, X_{13}, X_{14}, X_{1\gamma_1}, X_{1\gamma_2}\}$, where $(\gamma_1, \gamma_2) = (5,6)$ or $(7,8)$ according as $\epsilon = +$ or $-$. Define $\Psi(I(\xi_{1,i})) = X_{1i}$ for $\xi_{1,i} \in \mathcal{S}(1)$ except when $\epsilon = -$ and $i = 5, 6$, in which case $\Psi(I(\xi_{1,i})) = X_{1i+2}$.

Then Ψ induces a bijection between $\mathrm{Irr}(b(C))$ and $\mathrm{Irr}(B)$ (see [22] and the proof of [3, (1A)]). By (3A) and (3B), w stabilizes $\Psi(I(\xi))$ if and only if w stabilizes $I(\xi)$ for $\xi \in \mathcal{S}(t)$, since

$$\mathrm{Ind}_{N(\phi(t))}^{N(C)}(\xi)^\beta = \mathrm{Ind}_{N(\phi(t))^\beta}^{N(C)}(\xi^\beta).$$

It follows that

$$\mathrm{k}(G,B,d,u) = \mathrm{k}(N(C),B,d,u) \tag{3.2}$$

for all $d, u \in \mathrm{Z}$.

(2) Suppose $r = 3$ and $B = B_1$. Then $p \neq 3$, $N(C) = K_\epsilon = \langle L_\epsilon, \rho \rangle$ and $b(C) = B_0(K_\epsilon)$. Let $b_0 = B_0(L_\epsilon)$ and $B_H = B_0(H)$, where $L_\epsilon \leq H = \mathrm{GL}_3^\epsilon(q)$. Then B_H covers b_0, $B_H = \mathcal{E}_3(H,(1))$ by [17] and $b_0 = \mathcal{E}_3(L_\epsilon,(1))$. If t is a 3-element of L_ϵ, then $C_{L_\epsilon}(t) \in \{L_\epsilon, T_2^\epsilon, T_\epsilon, J_\epsilon\}$. Let $\zeta_{t,\mu}$ be an irreducible character of H labelled by (t,μ) and $\zeta_{t,\mu}|_{L_\epsilon}$ the restriction of $\zeta_{t,\mu}$ to L_ϵ, where $t \in L_\epsilon$ is semisimple and μ is a unipotent character of $C_H(t)$.

Let y be a semisimple element of order 3^{a+1} in H. Then $C_H(y)$ is a Coxeter torus $T_1 = \mathrm{Z}_{q^3-\epsilon}$. The irreducible character $\zeta_y = \pm R_{T_1}^H(y)$ of H labelled by $(y,1)$ has degree $|H\colon T_1|_{p'} = (q-\epsilon)^2(q+\epsilon)$, where 1 is the trivial character of T_1. Since $\zeta_y^{h\rho} = \zeta_y^\rho$ for $h \in H$, we may suppose the inverse-transpose map ρ of H stabilizes T_1, so that $\zeta_y^\rho = \pm R_{T_1}^H(y^\rho)$. If Δ and Δ' are elementary divisors of y and y^ρ, respectively, then $\Delta(c) = 0$ if and only if $\Delta'(c^{-\epsilon q}) = 0$ for some c in some extension field of F_q. If $\Delta(c) = 0$, then Δ has exactly three roots $\{c, c^{\epsilon q}, c^{q^2}\}$, so that $\Delta(c^{-\epsilon q}) \neq 0$ as $|c| = 3^{a+1}$. Thus y and y^ρ are not conjugate in H, so that $\zeta_y \neq \zeta_{y^\rho}$. Since $\zeta_y \in \mathrm{Irr}(B_H)$, it follows that $\zeta_y|_{L_\epsilon}$ and $\zeta_{y^\rho}|_{L_\epsilon}$ are characters of b_0. By [30, Table 1b], L_ϵ has exactly $\frac{1}{3}(q^2 - \epsilon q + 1 - 3)$ irreducible characters of degree $(q-\epsilon)^2(q+\epsilon)$, which are given by the restrictions of $\pm R^H_{T_1}(t)$ to L_ϵ, where $t \in O_{3'}(T_1)$. It follows that $\zeta_y|_{L_\epsilon}$ and $\zeta_{y^\rho}|_{L_\epsilon}$ are reducible, so that each has three irreducible constituents in $\mathrm{Irr}(b_0)$ by [30, Corollary 4.8]. In particular, by Clifford theory $\zeta_y|_{L_\epsilon} \neq \zeta_{y^\rho}|_{L_\epsilon}$. Since $\zeta_y^\rho = \zeta_{y^\rho}$, it follows that ρ induces a bijection between the three irreducible constituents of $\zeta_y|_{L_\epsilon}$ and of $\zeta_{y^\rho}|_{L_\epsilon}$. By [30] or [20], $\mathrm{Irr}(L_\epsilon)$ has exactly 6 characters of degree $(q-\epsilon)^2(q+\epsilon)/3$, so that $\mathrm{Irr}^{2a-1}(b_0)$ has exactly 6 characters and $\mathrm{Irr}(b(C))$ has 3 characters of height $2a-1$. Note that if y' is a 3-element of the maximal torus T_2^ϵ of L_ϵ, then $C_H(y') = T_1$ and $R^H_{T_1}(y)|_{L_\epsilon} = R^{L_\epsilon}_{T_2^\epsilon}(y')$ by [27].

Suppose $\Omega = \{\xi_0, \xi_1, \xi_2\}$ are three irreducible constituents of $\zeta_s|_{L_\epsilon}$. Let x be a diagonal element of $H\backslash L_\epsilon$ such that x induces an automorphism of order 3 on L_ϵ. Then x permutes characters of Ω transitively. Since $x^\alpha \in \langle x\rangle$, it follows that α also permutes Ω. In the notation of [30] we may suppose $\xi_i = \chi^{(i)}_{(q-\epsilon)^2(q+\epsilon)/3}$. (In the notation of [20], $\xi_i = \chi^{(i,1)}_{(q-\epsilon)^2(q+\epsilon)/3}$.) Since α stabilizes the class $C_3^{0,0}$, it follows from the character table that α stabilizes ξ_0, and $\xi_1^\alpha = \xi_2$ if and only if $(C_3^{0,1})^\alpha = C_3^{0,2}$.

By [23, Tables I and II], $\mathrm{Irr}^{2a-1}(B_1) = \{X_{19}, \overline{X}_{19}, X_{18}\}$ or

$$\mathrm{Irr}^{2a-1}(B_1) = \{X_{19}, \overline{X}_{19}, X_{16}\}$$

according as $\epsilon = +$ or $-$. Suppose $p = 2$. In the notation of [15], α stabilizes each class B_0, B_1 and $B_2(0)$, and moreover, $B_2(1)^\alpha = B_2(2)$ if and only if $(C_3^{0,1})^\alpha = C_3^{0,2}$. By the character table [15, Table IV-2], $B_2(1)^\alpha = B_2(2)$ if and only if $X_{19}^\alpha = \overline{X}_{19}$. Suppose $p \geq 5$. In the notation of [10], α stabilizes each class k_3, $k_{3,1}$ and $k_{3,2}$, and moreover, $k_{3,3,1}^\alpha = k_{3,3,2}$ if and only if $(C_3^{0,1})^\alpha = C_3^{0,2}$. Similarly, $k_{3,3,1}^\alpha = k_{3,3,2}$ if and only if $X_{19}^\alpha = \overline{X}_{19}$. It follows that $\xi_1^\alpha = \xi_2$ if and only if $X_{19}^\alpha = \overline{X}_{19}$.

Define $\Psi : \mathrm{Irr}^{2a-1}(b(C)) \longrightarrow \mathrm{Irr}^{2a-1}(B_1)$ such that $\Psi(\xi_0 + \xi_0^\rho) = X_{18}$ or X_{16} according as $\epsilon = +$ or $-$, $\Psi(\xi_1 + \xi_1^\rho) = X_{19}$ and $\Psi(\xi_2 + \xi_2^\rho) = \overline{X}_{19}$. Then w stabilizes $\Psi(\varphi)$ if and only if w stabilizes φ for each $\varphi \in \mathrm{Irr}^{2a-1}(b(C))$.

Suppose $\xi \in \mathrm{Irr}(L_\epsilon)$. Then ξ has defect $2a+1$ if and only if ξ is in a block with full defect and ξ has height 0. Since b_0 is the only block of L_ϵ with full defect, it follows that $\mathrm{Irr}^0(b)$ consists of all characters of $\mathrm{Irr}(L_\epsilon)$ with defect $2a+1$. In the notation of [30] and [20],

$$\mathrm{Irr}^0(b_0) = \{\chi_1, \chi_{q^2+\epsilon q}, \chi_{q^3}, \chi^{(u)}_{(q+\epsilon)(q^2+\epsilon q+1)/3} : 0 \leq u \leq 2\}.$$

Suppose t is an element of L_ϵ such that $C_H(t)$ is the torus $T_1 = \mathrm{Z}_{q-\epsilon} \times \mathrm{Z}_{q-\epsilon} \times \mathrm{Z}_{q-\epsilon}$. Let ζ_t and $\zeta_{t^{-1}}$ be characters of $\mathrm{Irr}(H)$ labelled by $(t,1)$ and $(t^{-1},1)$, respectively. Then $\zeta_t^\rho = \zeta_{t^{-1}}$ and both have degree $(q+\epsilon)(q^2+\epsilon q+1)$. Since L_ϵ has three irreducible characters of degree $(q+\epsilon)(q^2+\epsilon q+1)/3$ in $\mathrm{Irr}(b_0)$, there is some 3-element, say y such that $C_H(y) = T_1$ and $\zeta_y|_{L_\epsilon}$ is reducible. Since L_ϵ only has three such irreducible characters, it follows that

$$\zeta_y|_{L_\epsilon} = \zeta_{y^{-1}}|_{L_\epsilon}$$

and ρ permutes the three constituents. In addition, we may suppose $y = \operatorname{diag}\{x, x^{-1}, 1\}$ for some non-trivial 3-element $x \in O_3(\mathrm{Z}_{q-\epsilon})$. Since $\rho^2 = 1$ and since $\langle H, \rho\rangle/Z(H)L_\epsilon \simeq S_3$, it follows that ρ stabilizes exactly one constituent $\chi^{(0)}_{(q+\epsilon)(q^2+\epsilon q+1)/3}$. Thus ρ stabilizes exactly 4 characters of $\mathrm{Irr}^0(b_0)$, so that $\mathrm{Irr}^0(b(C))$ has 9 characters. Since α stabilizes the class $C_3^{(0,0)}$, it follows that α stabilizes $\chi^{(0)}_{(q+\epsilon)(q^2+\epsilon q+1)/3}$. By the degrees, α stabilizes χ_1, $\chi_{q^2+\epsilon q}$ and χ_{q^3}. Since $\rho^\alpha = \rho$, it follows that α stabilizes each character of $\mathrm{Irr}^0(b(C))$.

Let $t \in O_3(T_\epsilon)$ such that $C_{L_\epsilon}(t) = T_\epsilon$, so that $T_1 = C_H(t) = \mathrm{Z}_{q-\epsilon} \times \mathrm{Z}_{q-\epsilon} \times \mathrm{Z}_{q-\epsilon}$. The irreducible character $\zeta_t = \pm R^H_{T_1}(t)$ of H has degree $(q+\epsilon)(q^2+\epsilon q+1)$. By [30, Table 1b] and [20, Theorem 4.5], there are $\frac{1}{6}3^a(3^a-3)$ characters in $\mathrm{Irr}(b_0)$ such that $\xi_t = \zeta_t|_{L_\epsilon}$ is irreducible and $C_{L_\epsilon}(t) = T_\epsilon$. Let $\mathcal{X}(b_0)$ be the subset consisting of these characters of $\mathrm{Irr}(b_0)$,

$$\mathcal{X}_0(b_0) = \{\xi_t \in \mathcal{X}(b_0) : \xi_t^\rho = \xi_t\}$$

and $\mathcal{X}_1(b_0) = \mathcal{X}(b_0)\backslash\mathcal{X}_0(b_0)$. If $\xi_t \in \mathcal{X}_0(b_0)$, then we may suppose $t = \operatorname{diag}\{t_1, t_1^{-1}, 1\}$ for some $t_1 \in O_3(\mathrm{Z}_{q-\epsilon})$, since $\xi_t^\rho = \xi_{t^{-1}}$. As shown above L_ϵ has a unique class of non-trivial 3-elements containing $y = \operatorname{diag}\{x, x^{-1}, 1\}$ such that $\zeta_y|_{L_\epsilon}$ is reducible. Thus L_ϵ has exactly $\frac{1}{2}(3^a-3)$ classes of element $t = \operatorname{diag}\{t_1, t_1^{-1}, 1\}$ with $t_1 \in O_3(\mathrm{Z}_{q-\epsilon})\backslash\{1, x, x^{-1}\}$, and for each class $(t)_{L_\epsilon}$, ξ_t is stabilized by ρ. So $\mathcal{X}_0(b_0)$ has $\frac{1}{2}(3^a-3)$ characters and $\mathcal{X}_1(b_0)$ has $\frac{1}{6}3^a(3^a-3) - \frac{1}{2}(3^a-3) = \frac{1}{6}(3^a-3)^2$ characters. Let $\varphi_{t,1}$ and $\varphi_{t,2}$ be characters of $\mathrm{Irr}(b(C))$ which cover $\xi_t \in \mathcal{X}_0(b_0)$, and let $\varphi_t = \mathrm{Ind}^{K_\epsilon}_{L_\epsilon}(\xi_t)$ for $\xi_t \in \mathcal{X}_1(b_0)$. In addition, denote $\mathcal{X}_1(C) = \{\varphi_t : \xi_t \in \mathcal{X}_1(b_0)\}$ and

$$\mathcal{X}_0(C) = \{\varphi_{t,1}, \varphi_{t,2} : \xi_t \in \mathcal{X}_0(b_0)\}.$$

Then $\mathcal{X}_0(C)$ and $\mathcal{X}_1(C)$ are the characters of $\mathrm{Irr}(b(C))$ covering characters of $\mathcal{X}_0(b_0)$ and $\mathcal{X}_1(b_0)$, respectively. Moreover, $|\mathcal{X}_0(C)| = 3^a - 3$ and $|\mathcal{X}_1(C)| = \frac{1}{12}(3^a-3)^2$.

Let $\mathcal{X}_0(B_1) = \{\{X_\gamma\}, \{X'_\gamma\}\} \subseteq \mathrm{Irr}^1(B_1)$ and $\mathcal{X}_1(B_1) = \{\{X_\lambda\}\} \subseteq \mathrm{Irr}^1(B_1)$, where $(\gamma, \lambda) = (1a, 1)$ or $(2b, 2)$ according as $\epsilon = +$ or $-$. Then $|\mathcal{X}_i(B_1)| = |\mathcal{X}_i(C)|$ for $i = 0, 1$. In the notation of Lusztig's Jordan decomposition $\mathcal{X}_0(B_1) = \{\chi_{t,1}, \chi_{t,\mathrm{St}}\}$, where $t = \operatorname{diag}\{t_1, t_1^{-1}, 1\} \in O_3(T_\epsilon)$ and St is the Steinberg character of $C_G(t) = J_\epsilon$. Similarly, $\mathcal{X}_1(B_1) = \{\chi_{t,1}\}$, where $t \in O_3(T_\epsilon)$ such that $C_G(t) = T_\epsilon$. Define $\Psi : \mathcal{X}_i(C) \longrightarrow \mathcal{X}_i(B_1)$ by $\Psi(\varphi_{t,1}) = \chi_{t,1} \in \mathcal{X}_0(B_1)$, $\Psi(\varphi_{t,2}) = \chi_{t,\mathrm{St}}$, and $\Psi(\varphi_t) = \chi_{t,1} \in \mathcal{X}_1(B_1)$. Then Ψ is a bijection between $\mathcal{X}_i(C)$ and $\mathcal{X}_i(B_1)$, and w stabilizes $\Psi(\varphi)$ if and only if w stabilize $\varphi \in \mathcal{X}_i(C)$ for all $w \in \langle\alpha\rangle$.

Let $t \in O_3(T_\epsilon)$ be such that $C_{L_\epsilon}(t) = J_\epsilon$, so that $C_H(t) = J_\epsilon \times \mathrm{Z}_{q-\epsilon}$. In addition, let ζ_t and $\zeta_{t,\mathrm{St}}$ be the irreducible characters of H labelled by $(t, 1)$ and (t, St), respectively, where St is the Steinberg character of degree q of $C_H(t)$. Then $\zeta_t(1) = q^2 + \epsilon q + 1$ and $\zeta_{t,\mathrm{St}}(1) = q(q^2+\epsilon q+1)$. By the degrees of characters of $\mathrm{Irr}(L_\epsilon)$, both $\zeta_t|_{L_\epsilon}$ and

$\zeta_{t,\mathrm{St}}|_{L_\epsilon}$ are irreducible. Let $\xi_{t,1} = \zeta_t|_{L_\epsilon}$, $\xi_{t,\mathrm{St}} = \zeta_{t,\mathrm{St}}|_{L_\epsilon}$, and $\mathcal{X}_2(b_0) = \{\xi_{t,1}, \xi_{t,\mathrm{St}}\}$. Then $|\mathcal{X}_2(b_0)| = 2(3^a-1)$, since $\mathrm{Irr}(b_0)$ has 3^a-1 characters of degree $q^2+\epsilon q+1$ and $q(q^2+\epsilon q+1)$. Since $\zeta_t^\rho = \zeta_{t^{-1}}$ and $\zeta_{t,\mathrm{St}}^\rho = \zeta_{t^{-1},\mathrm{St}}$, it follows that $\zeta_t|_{L_\epsilon} \neq \zeta_{t^{-1}}|_{L_\epsilon}$ and $\zeta_{t,\mathrm{St}}|_{L_\epsilon} \neq \zeta_{t^{-1},\mathrm{St}}|_{L_\epsilon}$. Thus ρ does not stabilizes any character of $\mathcal{X}_2(b_0)$. Let

$$\mathcal{X}_2(C) = \{\varphi_{t,\mu} = \mathrm{Ind}_{L_\epsilon}^{K_\epsilon}(\xi_{t,\mu}) : \xi_{t,\mu} \in \mathcal{X}_2(b_0)\}$$

and $\mathcal{X}_2(B_1) = \{\chi_{t,1}, \chi_{t,\mathrm{St}}\}$, where $t \in \mathrm{O}_3(T_\epsilon)$ such that $C_{L_\epsilon}(t) = J_\epsilon$. Then $|\mathcal{X}_2(C)| = |\mathcal{X}_2(B_1)| = 3^a - 1$. Define $\Psi : \mathcal{X}_2(C) \longrightarrow \mathcal{X}_2(B_1)$ by $\Psi(\varphi_{t,1}) = \chi_{t,1}$. Then Ψ is a bijection between $\mathcal{X}_2(C)$ and $\mathcal{X}_2(B_1)$, and $w \in \langle\alpha\rangle$ stabilizes $\Psi(\varphi)$ if and only if w stabilize $\varphi \in \mathcal{X}_2(C)$.

Since each character of $\mathrm{Irr}^1(b_0)$ has degree $q^2+\epsilon q+1$, $q(q^2+\epsilon q+1)$ or $(q+\epsilon)(q^2+\epsilon q+1)$, it follows that $\mathrm{Irr}^1(b_0) = \cup_{i=0}^2 \mathcal{X}_i(b_0)$, so

$$\mathrm{Irr}^1(b(C)) = \cup_{i=0}^2 \mathcal{X}_i(C).$$

By [22, Tables 1 and 2], Ψ is a bijection between $\mathrm{Irr}^1(b(C))$ and $\mathrm{Irr}^1(B_1)$. Since $\mathrm{Irr}(b_0) = \mathrm{Irr}^{2a-1}(b_0) \cup \mathrm{Irr}^1(b_0) \cup \mathrm{Irr}^0(b_0)$, it follows that

$$\mathrm{Irr}(b(C)) = \mathrm{Irr}^{2a-1}(b(C)) \cup \mathrm{Irr}^1(b(C)) \cup \mathrm{Irr}^0(b(C)),$$

so that Ψ is a bijection between $\mathrm{Irr}(b(C))$ and $\mathrm{Irr}(B_1)$.

(3) Suppose $r = 3$ and $B \neq B_1$. Then $p \neq 3$ and $b(C)$ is not a principal block of K_ϵ. If $D = D(B)$, then $C_G(D) = T_\epsilon$ and $b_D^{K_\epsilon} = b(C)$, where b_D is a root block of B. If $B \subseteq \mathcal{E}_r(G,(s))$, then we may suppose $s \in T_\epsilon$ and $b_D = \mathcal{E}_r(T_\epsilon,(s))$. Let $b = b_D^{L_\epsilon}$, so that $b = \mathcal{E}_r(L_\epsilon,(s))$.

If $B \neq B_2$, then $|s| \neq 2$ and $b^\rho \neq b$, so that $\xi^\rho \neq \xi$ for each $\xi \in \mathrm{Irr}(b)$. Moreover, if $B = B_a$ or B_b, then $C_{L_\epsilon}(s) = J_\epsilon$ and

$$\mathcal{E}_r(L_\epsilon,(s)) = \{\xi_{sy,\mu} : y \in O_3(T_\epsilon),\ \mu \in \{1, \mathrm{St}\}\}.$$

Let $\mathcal{X}_1(b) = \{\xi_{sy,\mu} : C_{L_\epsilon}(sy) = J_\epsilon, \mu = 1, \mathrm{St}\}$, $\mathcal{X}_2(b) = \mathrm{Irr}(b)\backslash\mathcal{X}_1(b)$ and

$$\mathcal{X}_i(C) = \{\varphi_{t,\mu} = \mathrm{Ind}_{L_\epsilon}^{K_\epsilon}(\xi_{t,\mu}) : \xi_{t,\mu} \in \mathcal{X}_i(b)\}.$$

Then $|\mathcal{X}_1(b)| = |\mathcal{X}_1(C)| = 2 \cdot 3^a$ and $|\mathcal{X}_2(b)| = |\mathcal{X}_2(C)| = \frac{1}{2}(3^{2a} - 3^a)$. In addition, $\varphi_{t,\mu}^\alpha = \varphi_{t^\alpha,\mu}$.

Suppose $\mathcal{X}_1(B) = \{\chi_{sy,\mu} : y \in O_3(T_\epsilon), C_{L_\epsilon}(sy) = J_\epsilon, \mu = 1, \mathrm{St}\}$ and $\mathcal{X}_2(B) = \mathrm{Irr}(B)\backslash\mathcal{X}_1(B)$. By [21, pp.374, 377], $|\mathcal{X}_1(B)| = 2 \cdot 3^a$ and $|\mathcal{X}_2(B)| = \frac{1}{2}(3^{2a} - 3^a)$. Define $\Psi : \mathcal{X}_i(C) \longrightarrow \mathcal{X}_i(B)$ by $\Psi(\varphi_{sy,\mu}) = \chi_{sy,\mu}$. Then Ψ is a bijection between $\mathcal{X}_i(C)$

and $\mathcal{X}_i(B)$ for each $i = 1, 2$. So Ψ is a bijection between $\mathrm{Irr}(b(C))$ and $\mathrm{Irr}(B)$. Moreover, $w \in \langle \alpha \rangle$ stabilizes $\Psi(\varphi)$ if and only if w stabilizes $\varphi \in \mathrm{Irr}(b(C))$.

Suppose $B = B_{X_\gamma}$, where $\gamma = 1$ or 2 according as $\epsilon = +$ or $-$. Then $C_{L_\epsilon}(s) = T_\epsilon$ and $\mathrm{Irr}(b) = \{\xi_{sy,1} : y \in O_3(T_\epsilon)\}$. Thus $|\,\mathrm{Irr}(b)| = 3^{2a}$ and

$$\mathrm{Irr}(b(C)) = \{\varphi_{sy,1} = \mathrm{Ind}_{L_\epsilon}^{K_\epsilon}(\xi_{sy,1}) : y \in O_3(T_\epsilon)\}.$$

Define $\Psi : \mathrm{Irr}(b(C)) \longrightarrow \mathrm{Irr}(B)$ by $\Psi(\varphi_{sy,1}) = \chi_{sy,1}$. Then Ψ is a bijection and w stabilizes $\Psi(\varphi)$ if and only is it stabilizes $\varphi \in \mathrm{Irr}(b(C))$ for $w \in \langle \alpha \rangle$.

Finally, suppose $B = B_2$, so that $|s| = 2$ and $C_{L_\epsilon}(s) = J_\epsilon$. Thus

$$\mathrm{Irr}(b) = \{\xi_{sy,\mu} : y \in O_3(T_\epsilon),\ \mu \in \{1, \mathrm{St}\}\}.$$

We may suppose $s = h_1(-1) = \mathrm{diag}\,\{-1, -1, 1\} \in L_\epsilon$. Let $\mathcal{X}_0(b)$ be the character of $\mathrm{Irr}(b)$ stabilized by ρ. If $\xi_{sy,\mu}^\rho = \xi_{sy,\mu}$, then y is conjugate to y^{-1}. We may suppose $y = \mathrm{diag}\,\{c, c^{-1}, 1\}$ for $c \in O_3(\mathrm{Z}_{q-\epsilon})$, so that $sy = \mathrm{diag}\,\{-c, -c^{-1}, 1\}$ and $C_{L_\epsilon}(sy) = J_\epsilon$ or T_ϵ according as $c = 1$ or $c \neq 1$. Conversely, each character $\xi_{sy,\mu}$ with $sy = \mathrm{diag}\,\{-c, -c^{-1}, 1\}$ is stabilized by ρ. Since sy and sy^{-1} are conjugate in L_ϵ, it follows that $|\mathcal{X}_0(b)| = \frac{1}{2}(3^a - 1) + 2$. Let $\mathcal{X}_1(b) = \{\xi_{sy,\mu} : C_{L_\epsilon}(sy) = J_\epsilon, y \neq 1, \mu = 1, \mathrm{St}\}$ and

$$\mathcal{X}_2(b) = \mathrm{Irr}(b) \backslash (\mathcal{X}_0(b) \cup \mathcal{X}_1(b)).$$

If $\xi_{sy,\mu} \in \mathcal{X}_1(b)$, then we may suppose $sy = \mathrm{diag}\,\{-c, -c, c^{-2}\}$ for some non-trivial element $c \in O_3(\mathrm{Z}_{q-\epsilon})$, so that $|\mathcal{X}_1(b)| = 2(3^a - 1)$. If $\xi_{sy,\mu} \in \mathcal{X}_2(b)$, then we may suppose $sy = \mathrm{diag}\,\{-z_1, -z_2, z_1^{-1}z_2^{-1}\}$ for some non-trivial elements $z_1, z_2 \in O_3(\mathrm{Z}_{q-\epsilon})$ with $z_1 \neq z_2$. Since $\mathrm{diag}\,\{-z_1, -z_2, z_1^{-1}z_2^{-1}\}$ and $\mathrm{diag}\,\{-z_2, -z_1, z_1^{-1}z_2^{-1}\}$ are conjugate in L_ϵ, it follows that $|\mathcal{X}_2(b)| = \frac{1}{2}(3^{2a} - 3^a - (3^a - 1)) = \frac{1}{2}(3^a - 1)^2$. Let

$$\mathcal{X}_i(C) = \{\varphi_{t,\mu} = \mathrm{Ind}_{L_\epsilon}^{K_\epsilon}(\xi_{t,\mu}) : \xi_{t,\mu} \in \mathcal{X}_i(b)\}$$

for $i = 1, 2$. Let $\varphi_{sy,1}$ and $\varphi_{sy,\mathrm{St}}$ be the two extensions of $\xi_{sy,1} \in \mathcal{X}_0(b)$ to K_ϵ except when $y = 1$, in which cases, let $\varphi_{s,j}$ for $1 \leq j \leq 4$ be the four extensions of $\xi_{s,1}$ and $\xi_{s,\mathrm{St}}$ to K_ϵ. In addition, let

$$\mathcal{X}_0(C) = \{\varphi_{sy,\mu}, \varphi_{s,j} : \xi_{sy,1} \in \mathcal{X}_0(b), y \neq 1, 1 \leq j \leq 4\}.$$

Since ρ permutes $\mathcal{X}_i(b)$, it follows that

$$|\mathcal{X}_i(C)| = \begin{cases} 3 + 3^a & \text{if } i = 0, \\ 3^a - 1 & \text{if } i = 1, \\ \frac{1}{4}(3^a - 1)^2 & \text{if } i = 2. \end{cases}$$

Let $\mathcal{X}_1(B) = \{\chi_{sy,\mu} : 1 \neq y \in O_3(T_\epsilon), C_{L_\epsilon}(sy) = J_\epsilon\}$,

$$\mathcal{X}_0(B) = \{\chi_{sy,\mu} : y = \mathrm{diag}\,\{-c, -c^{-1}, 1\} \in O_3(T_\epsilon)\},$$

$\mathcal{X}_2(B) = \mathrm{Irr}(B)\backslash(\mathcal{X}_0(B) \cup \mathcal{X}_1(B))$, where μ runs over the unipotent characters of $C_{L_\epsilon}(sy)$. By [23], $|\mathcal{X}_i(B)| = |\mathcal{X}_i(C)|$ for $i = 0, 1, 2$. Given $i = 1, 2$, define $\Psi : \mathcal{X}_i(C) \longrightarrow \mathcal{X}_i(B)$ by $\Psi(\varphi_{sy,\mu}) = \chi_{sy,\mu}$. In addition, define $\Psi : \mathcal{X}_0(C) \longrightarrow \mathcal{X}_0(B)$ by $\Psi(\varphi_{sy,\mu}) = \chi_{sy,\mu}$ except when $y = 1$, in which case $\Psi(\varphi_{s,j}) = X_{2j}$. Then Ψ is a bijection between $\mathcal{X}_i(C)$ and $\mathcal{X}_i(B)$, so a bijection between $\mathrm{Irr}(b(C))$ and $\mathrm{Irr}(B)$. Moreover, w stabilizes $\Psi(\varphi)$ if and only is it stabilizes $\varphi \in \mathrm{Irr}(b(C))$ for $w \in \langle\alpha\rangle$. Thus (3.2) holds.

(4) We may suppose $B \in \mathrm{Blk}^0(G, r)$. Let $C = C(2)_1$ and $C' = C(2)$. Then $N(C) = N_G({J_\epsilon}^*)$ and $N(C') = N_{N(C)}(T_\epsilon)$. Since $N_G(J_\epsilon) \simeq N_G({J_\epsilon}^*)$, for simplicity of notation, we identify J_ϵ with ${J_\epsilon}^*$ and suppose $Z(J_\epsilon)$ is generated by an element $\mathrm{diag}\,\{c^{-2}, c, c\} \in L_\epsilon$ for some $c \in \mathrm{Z}_{q-\epsilon}$. Thus $\tau \in J_\epsilon = C_G(Z(J_\epsilon))$ and $\rho \in N_G(Z(J_\epsilon)) = N_G(J_\epsilon)$. So $N(C) = \langle J_\epsilon, \rho\rangle$ and $N(C') = \langle T_\epsilon, \tau, \rho\rangle$, where ρ acts as a inverse-transpose or a field automorphism of order 2 on $J_\epsilon = \mathrm{GL}_2^\epsilon(q)$ according as $\epsilon = +$ or $-$, $T_\epsilon \leq J_\epsilon$ and the actions of τ, ρ on T_ϵ are given by (2.2). Moreover, $N_A(C) = N(C)\mathrm{o}\langle\alpha\rangle$ and $N_A(C') = \langle T_\epsilon, \tau, \rho, \alpha\rangle$, where α acts as a field automorphism of order e on J_ϵ.

If $B = B_0(G)$, then $\mathrm{Blk}(N(C)|B) = \{b(C)\}$ and $\mathrm{Blk}(N(C')|B) = \{b(C')\}$, where $b(C) = B_0(N(C))$ and $b(C') = B_0(N(C'))$. If $B \neq B_0(G)$, then we may suppose $D = O_r(T_\epsilon)$, so that $C(D) = T_\epsilon$ and $N(D) = N(T_\epsilon)$, where $D = D(B)$. Let (D, b_D) be a maximal B-pair. Then $b_D^{N(C)}$ is a block $b(C)$ of $\mathrm{Blk}(N(C)|B)$ with a defect group D and $b(C') = b_D^{N(C')} \in \mathrm{Blk}(N(C')|B)$ with a defect group D. Since $N(C') = N_{N(C)}(D)$, it follows by Brauer First Main Theorem that $b(C) = b(C')^{N(C)}$. Moreover, the Brauer correspondence induces a bijection between $\mathrm{Blk}(N(C')|B)$ and $\mathrm{Blk}(N(C)|B)$. Let $b(C') = b_D^{N(C')} \in \mathrm{Blk}(N(C')|B)$ and $b(C) = b_D^{N(C)} \in \mathrm{Blk}(N(C)|B)$.

Let $H_\epsilon = \langle T_\epsilon, \tau\rangle$, $B' = b_D^{J_\epsilon}$ and $b' = b_D^{H_\epsilon}$, and let $N(B')$ and $N(b')$ be stabilizers of B' and b' in $N(C)$ and $N(C')$, respectively. Then $H_\epsilon = N_{J_\epsilon}(D)$, ${b'}^{J_\epsilon} = B'$, ${b'}^{N(C')} = b(C')$ and ${B'}^{N(C)} = b(C)$. If $N(B') = J_\epsilon$, then $N(b') = H_\epsilon$, since ${b'}^{J_\epsilon} = B'$. By [16, Theorem V.2.5], the induction map $\mathrm{Ind}_{J_\epsilon}^{N(C)}$ induces a bijection between $\mathrm{Irr}(B')$ and $\mathrm{Irr}(b(C))$. Since $N(C) = \langle J_\epsilon, \rho\rangle$ and $[\,\alpha, \rho\,] = 1$, it follows that

$$\left(\mathrm{Ind}_{J_\epsilon}^{N(C)}(\varphi)\right)^\alpha = \mathrm{Ind}_{J_\epsilon}^{N(C)}(\varphi^\alpha), \tag{3.3}$$

so that w stabilizes $\mathrm{Ind}_{J_\epsilon}^{N(C)}(\varphi)$ if and only if w stabilizes $\varphi \in \mathrm{Irr}(B')$ for any $w \in \langle\alpha\rangle$. Similarly, $\mathrm{Ind}_{H_\epsilon}^{N(C')}$ induces a bijection between $\mathrm{Irr}(b')$ and $\mathrm{Irr}(b(C'))$, and $w \in \langle\alpha\rangle$ stabilizes $\mathrm{Ind}_{H_\epsilon}^{N(C')}(\varphi)$ if and only if w stabilizes $\varphi \in \mathrm{Irr}(b')$. Since $\mathrm{k}(B') = \mathrm{k}(B', 0) = \mathrm{k}(b') = \mathrm{k}(b', 0)$, it follows that

$$\mathrm{k}(b(C), d, u) = \mathrm{k}(B', d, u) = \mathrm{k}(b', d, u) = \mathrm{k}(b(C'), d, u), \tag{3.4}$$

for all integers $d, u \in \mathbf{Z}$, where $\mathrm{k}(b(C), d, u) = \mathrm{k}(N(C), B, d, u) \cap \mathrm{Irr}(b(C))$.

Suppose $N(B') = N(C)$, so that $N(b') = N(C')$. Let ϕ be the isomorphism from T_ϵ to $\mathrm{Irr}(T_\epsilon)$ given by (2.3). If W is a subgroup of G containing T_ϵ, we denote by $[T_\epsilon, \phi(t)]_W$ the W-orbit containing the pair $(T_\epsilon, \phi(t))$. In particular, $N(C)$ and $N(C')$ permute $\{[T_\epsilon, \phi(t)]_{J_\epsilon}\}$ and $\{[T_\epsilon, \phi(t)]_{H_\epsilon}\}$, respectively. Since ϕ is a group isomorphism, it follows that

$$[T_\epsilon, \phi(t)]_W^v = [T_\epsilon, \phi(t^v)]_W, \qquad \text{for } v \in \langle \alpha, \rho \rangle, \tag{3.5}$$

where $W \in \{J_\epsilon, H_\epsilon\}$. By [7, Theorem 3.2], $\mathrm{Irr}(B') = \mathcal{E}_r(J_\epsilon, (s))$ for some semisimple r'-element $s \in J_\epsilon$. Moreover, $s = 1$ if and only if B' and b' are principal. We may suppose $D \leq J_\epsilon$, so that $C_{J_\epsilon}(sy) = J_\epsilon$ or T_ϵ for $y \in D$.

If $C_{J_\epsilon}(sy) = T_\epsilon$, then

$$\chi = \pm R_{T_\epsilon}^{J_\epsilon}(\phi(sy)) \in \mathrm{Irr}(B') \tag{3.6}$$

and $R_{T_\epsilon}^{J_\epsilon}(\phi(sy)) = R_{T_\epsilon}^{J_\epsilon}(\phi(sy'))$ for some $y' \in D$ if and only if $[T_\epsilon, \phi(sy)]_{J_\epsilon} = [T_\epsilon, \phi(sy')]_{J_\epsilon}$. Let $Q \leq J_\epsilon$ and

$$\mathcal{Y}(B', Q) = \mathrm{Irr}(B') \cap \left(\bigcup_t \mathcal{E}(J_\epsilon, (t)) \right), \tag{3.7}$$

where t runs over elements of T_ϵ such that $C_{J_\epsilon}(t) \simeq Q$. Thus each element of $\mathcal{Y}(B', T_\epsilon)$ is labelled by $(sy, 1)$ with $C_{J_\epsilon}(sy) = T_\epsilon$. So (3.6) induces a bijection between $\mathcal{Y}(B', T_\epsilon)$ and the set

$$\mathcal{X}(J_\epsilon) = \{[T_\epsilon, \phi(sy)]_{J_\epsilon} : y \in D,\ C_{J_\epsilon}(sy) = T_\epsilon\}.$$

Suppose $[T_\epsilon, \phi(sy')]_{H_\epsilon} \subseteq [T_\epsilon, \phi(sy)]_{J_\epsilon} \in \mathcal{X}(J_\epsilon)$. Then $[T_\epsilon, \phi(sy')]_{H_\epsilon}^h = [T_\epsilon, \phi(sy)]_{H_\epsilon}$ for some $h \in J_\epsilon$, and $(T_\epsilon, \phi(sy'))^h = (T_\epsilon, \phi(sy))^v = (T_\epsilon, \phi(sy)^v)$ for some $v \in H_\epsilon$. In particular, ${T_\epsilon}^h = T_\epsilon$, $h \in H_\epsilon$ and $[T_\epsilon, \phi(sy')]_{H_\epsilon} = [T_\epsilon, \phi(sy)]_{H_\epsilon}$. So the inclusion map $[T_\epsilon, \phi(sy)]_{H_\epsilon} \subseteq [T_\epsilon, \phi(sy)]_{J_\epsilon}$ is a bijection between

$$\mathcal{X}(H_\epsilon) = \{[T_\epsilon, \phi(sy)]_{H_\epsilon} : y \in D, C_{H_\epsilon}(sy) = T_\epsilon\}$$

and $\mathcal{X}(J_\epsilon)$. Let $\mathcal{Y}(b', T_\epsilon)$ be the subset of $\mathrm{Irr}(b')$ consisting of irreducible characters $\zeta(sy)$ of H_ϵ covering $\phi(sy)$ with $[T_\epsilon, \phi(sy)]_{H_\epsilon} \in \mathcal{X}(H_\epsilon)$. If $\zeta(sy) \in \mathcal{Y}(b', T_\epsilon)$, then $\phi(sy)^\tau \neq \phi(sy)$, since $C_{J_\epsilon}(sy) = T_\epsilon$. Thus $\zeta(sy)$ covers exactly $\phi(sy)$ and $\phi(sy)^\tau$ in $\mathrm{Irr}(b_D)$. Since $[T_\epsilon, \phi(sy)]_{H_\epsilon} = \{(T_\epsilon, \phi(sy)), (T_\epsilon, \phi(sy)^\tau)\}$, it follows that $\zeta(sy) \mapsto [T_\epsilon, \phi(sy)]_{H_\epsilon}$ is a bijection between $\mathcal{Y}(b', T_\epsilon)$ and $\mathcal{X}(H_\epsilon)$. Moreover, $\zeta(sy)^\alpha = \zeta((sy)^\alpha)$, since $\phi(sy)^\alpha = \phi((sy)^\alpha)$. It follows that the mapping $\Psi_1 : \mathcal{Y}(b', T_\epsilon) \longrightarrow \mathcal{Y}(B', T_\epsilon)$ given by $\Psi_1(\zeta(sy)) = \pm R_{T_\epsilon}^{J_\epsilon}(\phi(sy)))$ for suitable sign $\pm$ is a bijection and moreover,

$$\Psi_1(\zeta(sy))^\alpha = \pm R_{T_\epsilon}^{J_\epsilon}(\phi(sy))^\alpha = \pm R_{T_\epsilon}^{J_\epsilon}(\phi((sy)^\alpha)) = \Psi_1(\zeta((sy)^\alpha)). \tag{3.8}$$

It follows that $w \in \langle\alpha\rangle$ stabilizes $\Psi_1(\zeta(sy))$ if and only if it stabilizes $\zeta(sy)$.

Next suppose $\varphi \in \mathrm{Irr}(b(C))$ and $\psi \in \mathrm{Irr}(b(C'))$ covering $\xi = \Psi_1(\zeta(sy))$ and $\zeta(sy)$, respectively. If ρ stabilizes ξ, then $[T_\epsilon, \phi(sy)]_{J_\epsilon} = [T_\epsilon, \phi(sy)^{-1}]_{J_\epsilon}$ and $\xi = \varphi|_{J_\epsilon}$. In addition, $\varphi^w = \varphi$ if and only if $\xi^w = \xi$ for $w \in \langle\alpha\rangle$, since $[\alpha, \rho] = 1$. Moreover, since $[T_\epsilon, \phi(sy)]_{H_\epsilon} = [T_\epsilon, \phi(sy)^{-1}]_{H_\epsilon}$, it follows that ρ stabilizes $\zeta(sy)$, so that w stabilizes ψ if and only if w stabilizes $\zeta(sy)$.

If $\xi^\rho \neq \xi$, then $[T_\epsilon, \phi(sy)]_{J_\epsilon} \neq [T_\epsilon, \phi(sy)^{-1}]_{J_\epsilon}$, $[T_\epsilon, \phi(sy)]^\rho_{J_\epsilon} = [T_\epsilon, \phi(sy)^{-1}]_{J_\epsilon}$ and $\varphi|_{J_\epsilon} = \xi + \xi^\rho$. Thus $[T_\epsilon, \phi(sy)]_{H_\epsilon} \neq [T_\epsilon, \phi(sy)^{-1}]_{H_\epsilon}$ $[T_\epsilon, \phi(sy)]^\rho_{H_\epsilon} = [T_\epsilon, \phi(sy)^{-1}]_{H_\epsilon}$ and $\psi = \zeta(sy) + \zeta(sy)^\rho$. In addition, $w \in \langle\alpha\rangle$ stabilizes φ if and only if it stabilizes the set $\{[T_\epsilon, \beta(sy)]_{J_\epsilon}, [T_\epsilon, \beta(sy)^{-1}]_{J_\epsilon}\}$, if and only if it stabilizes the set

$$\{[T_\epsilon, \phi(sy)]_{H_\epsilon}, [T_\epsilon, \phi(sy)^{-1}]_{H_\epsilon}\},$$

if and only if w stabilizes ψ.

Let $\mathcal{Y}(b(C), T_\epsilon)$ and $\mathcal{Y}(b(C'), T_\epsilon)$ be the subsets of characters of $\mathrm{Irr}(b(C))$ and of $\mathrm{Irr}(b(C'))$ covering $\mathcal{Y}(B', T_\epsilon)$ and $\mathcal{Y}(b', T_\epsilon)$, respectively. Then there is a bijection Ψ between $\mathcal{Y}(b(C), T_\epsilon)$ and $\mathcal{Y}(b(C'), T_\epsilon)$ such that $\Psi(\varphi)^w = \Psi(\varphi)$ if and only if $\varphi^w = \varphi$.

Suppose $C_{J_\epsilon}(sy) = J_\epsilon$. Then $\mathrm{Irr}(B')$ contains characters ξ_1 and ξ_2 labelled by $(sy, 1)$ and (sy, St) respectively, where St is the Steinberg character of J_ϵ. Since $sy \in Z(J_\epsilon)$ and $\tau \in J_\epsilon$, it follows that $\phi(sy)^\tau = \phi(sy)$ and $\mathrm{Irr}(b')$ contains two characters ζ_1 and ζ_2 covering $\phi(sy)$. Define $\Psi_1(\xi_i) = \zeta_i$. By [17, (2H)], $\xi_i^\rho = \xi_i$ if and only $\phi(sy)^\rho = \phi(sy)$, if and only if $\zeta_i^\rho = \zeta_i$, since $[\tau, \rho] = 1$. If ρ stabilizes ξ_i, then $\mathrm{Irr}(b(C))$ has two characters φ_i and φ_i' covering ξ_i, and $\mathrm{Irr}(b(C'))$ has two characters ψ_i and ψ_i' covering ζ_i. Since α stabilizes both trivial and the Steinberg characters of J_ϵ, it follows that $w \in \langle\alpha\rangle$ stabilizes φ_i (respectively, φ_i') if and only if w permutes $\{\phi(sy), \phi(sy)^{-1}\}$, if and only if w stabilizes ζ_i (respectively, ζ_i'). Define $\Psi(\varphi_i) = \psi_i$ and $\Psi(\varphi_i') = \psi_i'$. If $\xi_i^\rho \neq \xi_i$, then $\zeta_i^\rho \neq \zeta_i$, and so $\mathrm{Irr}(b(C))$ and $\mathrm{Irr}(b(C'))$ each has exactly one character φ_i and ψ_i covering ξ_i and ζ_i, respectively. We define $\Psi(\varphi_i) = \psi_i$. Thus Ψ is a bijection between the characters φ of $\mathrm{Irr}(b(C)) \backslash \mathcal{Y}(b(C), T_\epsilon)$ and $\mathrm{Irr}(b(C')) \backslash \mathcal{Y}(b(C'), T_\epsilon)$ such that $w \in \langle\alpha\rangle$ stabilizes φ if and only if w stabilizes $\Psi(\varphi)$. It follows that (3.4) holds and

$$\mathrm{k}(N(C), B, d, u) = \mathrm{k}(N(C'), B, d, u) \tag{3.9}$$

for all $d, u \in \mathrm{Z}$.

If $r \geq 5$, let $C = C(1)_1$ and $C' = C(1)$. The same proof as above shows that (3.9) holds. This completes the proof.

4. *The proof of the conjecture for the even prime*

The notation and terminology of Sections 1 and 2 are continued in this section. Suppose $r = 2$ and $B \in \mathrm{Blk}^0(G, 2)$ such that $B \subseteq \mathcal{E}_2(G, (s))$ for some semisimple $2'$-element s of G, where $\mathrm{Blk}^0(G, 2)$ is given by (3.1). Then $D(B)$ is either abelian or $D(B) \in \{SD_{2^{a+2}}, \mathrm{Z}_{2^a} \wr \mathrm{Z}_2, S\}$, where $S \in \mathrm{Syl}_2(G)$. In addition, let $\mathcal{R}^0(B)$ the subfamily of $\mathcal{R}^0(G)$ consisting of the chain C such that $\mathrm{Blk}(N(C)|B) \neq \emptyset$. By [4, (4A)],

$$\mathcal{R}^0(B)/G = \begin{cases} \{1, 1 < \mathrm{Z}_2, 1 < O_2(T_\delta), 1 < \mathrm{Z}_2 < O_2(T_\delta)\} & \text{if } D = O_2(T_\delta), \\ \{1, 1 < \mathrm{Z}_2, 1 < O_2(T_{-\epsilon}), 1 < \mathrm{Z}_2 < O_2(T_{-\epsilon})\} & \text{if } D \simeq SD_{2^{a+2}}, \\ \{1, 1 < \mathrm{Z}_2, 1 < O_2(T_\epsilon), 1 < \mathrm{Z}_2 < O_2(T_\epsilon)\} & \text{if } D \simeq \mathrm{Z}_{2^a} \wr \mathrm{Z}_2, \\ \mathcal{R}^0(G)/G & \text{if } B = B_1. \end{cases}$$

If $q = 3$, then we may suppose $1 \neq D \in \Phi(G, 2)$, so that $C_G(D) = Z(D)$ and $B = B_0(G)$. In the following propositions (4A)-(4C) we suppose $q \neq 3$.

(4A) *Let q be a power of an odd prime p, and let B a 2-block of $G = G_2(q)$ with a non-cyclic abelian defect group D. Then B satisfies Dade's invariant conjecture.*

Proof. Since D is non-cyclic abelian, we may suppose $D = O_2(T_\delta)$, so that $O_2(T_\delta) \neq T_\delta$. Let

$$C : 1 < \mathrm{Z}_2 \qquad \text{and} \qquad C' : 1 < \mathrm{Z}_2 < O_2(T_\delta).$$

Then $N(C) = \mathrm{SO}(U_4)$, $N(C') = \langle T_\delta, \rho, \tau\sigma \rangle$ and $C(C')/T_\delta$ is a 2-group. Moreover, $N(C') = N_{N(C)}(T_\delta) = N_{N(C)}(D)$. If $b(C) \in \mathrm{Blk}(N(C)|B)$ and $b(C') \in \mathrm{Blk}(N(C')|B)$, then both $b(C)$ and $b(C')$ have a defect group D. In addition, $b^{N(C)} = b(C)$ for some block $b \in \mathrm{Blk}(N(C')|B)$ by Brauer's First Main Theorem. Thus the Brauer correspondence induces a bijection between $\mathrm{Blk}(N(C)|B)$ and $\mathrm{Blk}(N(C')|B)$, we may suppose $b(C')^{N(C)} = b(C)$.

If $B \subseteq \mathcal{E}_2(G, (s))$, then $C_G(s) = T_\delta$ and by [7, Theorem 3.2], we may suppose $b(C) \subseteq \mathcal{E}_2(N(C), (s))$. Since T_δ is a regular subgroup of $N(C)$, it follows by [6, Theorem 2.3] that there exists a perfect isometry between $\mathcal{E}_2(N(C), (s))$ and $\mathcal{E}_2(T_\delta, (s))$. Since $\mathcal{E}_2(T_\delta, (s))$ is a single block, it follows that $\mathcal{E}_2(N(C), (s))$ is a block, so that $b(C) = \mathcal{E}_2(N(C), (s))$. Let b_D be a block of T_δ covered by $b(C')$, and ϑ the canonical character of b_D. Then we may suppose $b_D = \mathcal{E}_2(T_\delta, (s))$ and $\vartheta = \phi(s)$. By (2.4), the stabilizer $N'(\vartheta)$ of ϑ in $N(C')$ is T_δ, so that $N'(b_D) = T_\delta$ and $N'(\xi) = T_\delta$ for all $\xi \in \mathrm{Irr}(b_D)$, since $N'(\xi) \leq N'(b_D)$. Since $\mathrm{Irr}(b_D) = \{\phi(sy) : y \in O_2(T_\delta)\}$, it follows that

$$\mathrm{Irr}(b(C')) = \{\mathrm{Ind}_{T_\delta}^{N(C')}(\phi(sy)) : y \in O_2(T_\delta)\}.$$

On the other hand, since $\phi(sy)$ is in the general position, $\pm R_{T_\delta}^{N(C)}(\phi(sy))$ is irreducible for some sign $\pm$. Thus

$$\mathrm{Irr}(b(C)) = \{\pm R_{T_\delta}^{N(C)}(\phi(sy)) : y \in O_2(T_\delta)\}.$$

Since $\beta'(N(C)) = N(C)$ and $\beta'(T_\delta) = T_\delta$, it follows that

$$\mathrm{Ind}_{T_\delta}^{N(C')}(\phi(sy))^{\beta'} = \mathrm{Ind}_{T_\delta}^{N(C')}(\phi(sy)^{\beta'}) \qquad R_{T_\delta}^{N(C)}(\phi(sy))^{\beta'} = R_{T_\delta}^{N(C)}(\phi(sy)^{\beta'}).$$

Define $\Psi : \mathrm{Irr}(b(C')) \longrightarrow \mathrm{Irr}(b(C))$ such that $\Psi(\mathrm{Ind}_{T_\delta}^{N(C')}(\phi(sy))) = \pm R_{T_\delta}^{N(C)}(\phi(sy))$. Thus w stabilizes $\Psi(\psi)$ if and only if w stabilizes $\psi \in \mathrm{Irr}(b(C'))$ for $w \in \langle \beta' \rangle$. It follows that

$$\mathrm{k}(N(C'), B, d, u) = \mathrm{k}(N(C), B, d, u) \tag{4.1}$$

for all $d, u \in \mathrm{Z}$.

If $C : 1 < O_2(T_\delta)$ and $b(C) \in \mathrm{Blk}(N(C)|B)$, then $N(C) = N_G(T_\delta) = N(D)$, so that D is a defect group of $b(C)$ and $\mathrm{Blk}(N(C)|B) = \{b(C)\}$. If b_D is a block of T_δ covered by $b(C)$, then we may suppose $b_D = \mathcal{E}_2(T_\delta, (s))$ and $N(b_D) = T_\delta$, where $N(b_D)$ is the stabilizer of b_D in $N(C)$. So

$$\mathrm{Irr}(b(C)) = \{\mathrm{Ind}_{T_\delta}^{N(C)}(\phi(sy)) : y \in O_2(T_\delta)\}.$$

Since $\mathrm{Irr}(B) = \{\pm R_{T_\delta}^G(\phi(sy)) : y \in O_2(T_\delta)\}$, it follows that

$$\Psi(\mathrm{Ind}_{T_\delta}^{N(C)}(\phi(sy))) = \pm R_{T_\delta}^G(\phi(sy))$$

is a bijection between $\mathrm{Irr}(b(C))$ and $\mathrm{Irr}(B)$. Since $\beta(N(C)) = N(C)$ and $\beta(T_\delta) = T_\delta$, it follows that w stabilizes $\Psi(\varphi)$ if and only if w stabilizes $\varphi \in \mathrm{Irr}(b(C))$ for $w \in \langle \beta \rangle$. Thus (4A) holds.

(4B) *Let q be a power of an odd prime p and $G = G_2(q)$. In the notation of* [24], *suppose $B \in \{B_{1a}, B_{1b}, B_{2a}, B_{2b}\}$. Then B satisfies Dade's invariant conjecture.*

Proof. If $B \subseteq \mathcal{E}_2(G, (s))$, then $C_G(s) = \mathrm{GL}_2(q)$ or $\mathrm{GL}_2^-(q)$ according to whether $B \in \{B_{1a}, B_{1b}\}$ or $B \in \{B_{2a}, B_{2b}\}$. Thus $D \simeq \mathrm{Z}_{2^a} \wr \mathrm{Z}_2$ or $SD_{2^{a+2}}$ according as $B \in \{B_{\gamma a}, B_{\gamma b}\}$ or $B \in \{B_{\gamma' a}, B_{\gamma' b}\}$, where $D = D(B)$, $\gamma = 1$ or 2 according as $\epsilon = +$ or $-$ and $\gamma + \gamma' = 3$.

Suppose $D \simeq \mathrm{Z}_{2^a} \wr \mathrm{Z}_2$. So $2^a \neq q - \epsilon$. Let $C = C(2)^1$ and $C' = C(2)$. Then $N(C) = N(T_\epsilon)$ and $N(C') = \langle T_\epsilon, \rho, \tau\sigma \rangle$. Since $2^a \geq 4$, it follows by [1, (1A)] that each normal abelian subgroup of D is a subgroup of the base subgroup of D. Thus $O_2(T_\epsilon)$ is the unique maximal normal abelian subgroup of D, so that $N(D) \leq N(O_2(T_\epsilon)) = N(C)$. Similarly, $N(D) \leq N(C')$ since $\Omega_1(Z(D)) = \mathrm{Z}_2$.

Suppose $b(C) \in \mathrm{Blk}(N(C)|B)$ and $b(C') \in \mathrm{Blk}(N(C')|B)$. Let b be a block of $C(C) = T_\epsilon$ such that b induces $b(C)$ or $b(C')$. Then $b^G = B$ and we may suppose $b = \mathcal{E}_2(T_\epsilon, (s))$. Thus $O_2(T_\epsilon)$ is a defect group of b. Since D is a defect group of B and $(O_2(T_\epsilon), b)$ is a B-pair, it follows that $(O_2(T_\epsilon), b)$ is not a Sylow B-pair in $N(T_\epsilon)$, so that $O_2(T_\epsilon)$ is a proper subgroup in a defect group Q of $b^{N(C)} = b(C)$. Since $b(C)^G = B$, we may suppose $Q \leq D$. But $(D\colon O_2(T_\epsilon)) = 2$, so $Q = D$ and D is a defect group of $b(C)$.

Since $T_\epsilon \leq C_G(z)$ for some involution z of G, $b^{C_G(z)}$ is a block B' of $C_G(z) = \mathrm{SO}_4^+(q)$. We may suppose $B' \subseteq \mathcal{E}_2(C_G(z), (s))$. Since $C_{C_G(z)}(T_\epsilon) = T_\epsilon$ is a regular subgroup of $C_G(z)$, it follows by the proof of (4A) that $B' = \mathcal{E}_2(C_G(z), (s))$. So a Sylow 2-subgroup of $C_{C_G(z)}(s) = C_G(sz)$ is a defect group of B'. But $C_G(sz) = J_\epsilon$ and ${B'}^G = B$, so D is a defect group of B'. Replace B by B', $b(C)$ by $b(C')$, and $N(C)$ by $N(C')$ in the proof above. Then $(O_2(T_\epsilon), b)$ is not a Sylow B'-pair of $N(C')$ as $N_{C_G(z)}(O_2(T_\epsilon)) = N(C')$. Thus D is a defect group of $b(C')$. By Brauer's First Main Theorem,

$$\mathrm{Blk}(N(C)|B) = \{b(C)\} \qquad \mathrm{Blk}(N(C')|B) = \{b(C')\}.$$

Since $\phi(s)$ is the canonical character of b, it follows by (2.4) that $\phi(s)^\sigma \neq \phi(s)$, and so $\sigma \notin N(b)$, where $N(b)$ is the stabilizer of b in $N(C)$. By a result of Fong, [16, Theorem V.3.14], we may suppose $D \leq N(b)$ and $(N(b)\colon DT_\epsilon)_2 = 1$. Since $\langle \sigma T_\epsilon\rangle \trianglelefteq N(T_\epsilon)/T_\epsilon$, it follows that $N(b) = DT_\epsilon \leq N(C')$. But $N(\xi) \leq N(b)$ for each $\xi \in \mathrm{Irr}(b)$, so $N(\xi) = N'(\xi)$, where $N'(\xi) = N(\xi) \cap N(C')$. In particular,

$$(N(C) : N(\xi))_2 = (N(C') : N'(\xi)).$$

Thus the mappings $\mathrm{Ind}_{N(\xi)}^{N(C)}$ and $\mathrm{Ind}_{N'(\xi)}^{N(C')}$ induce a height preserving bijection Ψ between $\mathrm{Irr}(b(C'))$ and $\mathrm{Irr}(b(C))$. If $p = 3$, then $B^\beta \neq B$, so that $b(C)^\beta \neq b(C)$ and $b(C')^\beta \neq b(C')$. If $\xi = \phi(t)$ for some $t \in T_\epsilon$, then $\mathrm{Ind}_{N(\xi)}^{N(C)}(\zeta_t)^\alpha = \mathrm{Ind}_{N(\xi)}^{N(C)}(\zeta_{t^\alpha})$, since $y^\alpha = y$ for $y \in \langle \rho, \tau, \sigma\rangle$, where ζ_t is an extension of $\phi(t)$ to $N(\phi(t))$. Similarly, $\mathrm{Ind}_{N'(\xi)}^{N(C')}(\zeta_t)^\alpha = \mathrm{Ind}_{N'(\xi)}^{N(C')}(\zeta_{t^\alpha})$. So $w \in \langle\alpha\rangle$ stabilizse $\Psi(\psi)$ if and only if w stabilizes $\psi \in \mathrm{Irr}(b(C'))$. Thus (4.1) holds.

Suppose $D \simeq SD_{2^{a+2}}$. Let $C = C(1)^1 : 1 < O_2(T)$ and $C' = C(1) : 1 < \mathrm{Z}_2 < O_2(T)$, where $T = T_{-\epsilon}$. Then $N(C) = N(T)$, $N(C') = \langle T, \rho, \tau\sigma\rangle$. Let $b(C) \in \mathrm{Blk}(N(C)|B)$, $b(C') \in \mathrm{Blk}(N(C')|B)$, and b a block of $C(C) = \langle T, \rho\rangle$ indices $b(C)$ or $b(C')$. Since the 2-rank of D is 2, an elementary abelian group E_8 of order 8 is not a subgroup of D. Thus a Sylow 2-subgroup $E_8 = \langle O_2(T), \rho\rangle$ of $\langle T, \rho\rangle$ is not a defect group of b, since $b^G = B$. Thus the B-pair $(O_2(T), b)$ is not a Sylow B-pair, so that it is not maximal in $N(C)$. Since $(1, b(C)) \leq (O_2(T), b)$ holds in $N(C)$, it follows that $O_2(T)$ is a proper subgroup of a defect group Q of $b(C)$. In particular, Q is dihedral of order 8 conjugate to $\langle O_2(T), \tau\sigma\rangle$ or $\langle O_2(T), \rho\tau\sigma\rangle$. A proof similar to that of the case $D \simeq \mathrm{Z}_{2^a} \wr \mathrm{Z}_2$ shows that $B' = b^{\mathrm{SO}(U_4)}$ has a defect group D, $(O_2(T), b)$ is not maximal in $N(C')$, and a defect group W of $b(C') = b^{N(C')}$ is dihedral of order 8. Since $b(C')^{N(C)} \in \mathrm{Blk}(N(C)|B)$, it follows that both blocks $b(C')$ and $b(C')^{N(C)}$ have a same defect group Q. Thus the Brauer correspondence induces a bijection between $\mathrm{Blk}(N(C)|B)$ and $\mathrm{Blk}(N(C')|B)$. We may suppose $b(C')^{N(C)} = b(C)$.

Let b_D be a block of T covered by b. Then b is the unique block of $C(C)$ covering b_D, so that $N(b_D) \le N(b)$ as $x \in N(b_D)$ implies that $x \in N(T)$ and b^x is a block of $C(C)$ covering b_D. In addition, b covers exactly two or one conjugate of b_D, since $(C(C){:}\,T) = 2$. Suppose $\sigma \in N(b)$. Then σ permutes the conjugates of b_D covered by b, so that $\sigma \in N(b_D)$ since $|\sigma| = 3$. By (2.4), $\sigma \notin N(b_D)$ and so $\sigma \notin N(b)$. By a result of Fong, [16, Theorem V.3.14] $(N(b){:}\,QC(C))_2 = 1$, so that $N(b) = QC(C) = N(C')$. A similar proof to above shows that the two mappings $\mathrm{Ind}_{N(\xi)}^{N(C)}$ and $\mathrm{Ind}_{N'(\xi)}^{N(C')}$ induce a bijection Ψ between $\mathrm{Irr}(b(C'))$ and $\mathrm{Irr}(b(C))$. Moreover, w stabilizes $\Psi(\psi)$ if and only if w stabilizes ψ for $w \in \langle\alpha\rangle$. Thus (4.1) holds.

Finally, suppose $C = C(1)_1 : 1 < \mathrm{Z}_2$ and $b(C) \in \mathrm{Blk}(N(C)|B)$. Then $N(C) = \mathrm{SO}(U_4)$ and we may suppose $b(C) \subseteq \mathcal{E}_2(N(C),(s))$. We may also suppose $D \le C_{N(C)}(s)$. Thus $L = C_{N(C)}(s)$ is a regular subgroup of $N(C)$, where $L \simeq \mathrm{GL}_2^{\epsilon}(q)$ or $\mathrm{GL}_2^{-\epsilon}(q)$ according as $D \simeq \mathrm{Z}_{2^a} \wr \mathrm{Z}_2$ or $SD_{2^{a+2}}$. By a result of Broué, [6, Theorem 2.3], there exists a perfect isometry $R_L^{N(C)}$ between $\mathcal{E}_2(N(C),(s))$ and $\mathcal{E}_2(L,(1))$. Since $\mathcal{E}_2(L,(1))$ is the principal block b_L of L, it follows that $b(C) = \mathcal{E}_2(N(C),(s))$. If $\xi_{y,\mu} \in \mathrm{Irr}(b_L)$, then $\pm R_L^{N(C)}(\xi_{y,\mu}) \in \mathrm{Irr}(b(C))$ for some sign $\pm$. Since $\alpha(L) = L$ and $\alpha(N(C)) = N(C)$, it follows that $R_L^{N(C)}(\xi_{y,\mu})^\alpha = R_L^{N(C)}(\xi_{y^\alpha,\mu})$.

If $B = B_{1a}$ or B_{1b}, then $C_G(s) = L \simeq \mathrm{GL}_2(q)$. If $B = B_{2a}$ or B_{2b}, then $C_G(s) = L \simeq \mathrm{GL}_2^{-\epsilon}(q)$. So L is also a regular subgroup of G. Again by [6, Theorem 2.3], there exists a perfect isometry R_L^G between $B = \mathcal{E}_2(G,(s))$ and b_L. If $\xi_{y,\mu} \in \mathrm{Irr}(b_L)$, then $\pm R_L^G(\xi_{y,\mu}) \in \mathrm{Irr}(B)$ for some sign $\pm$. Thus $R_L^G(\xi_{y,\mu})^\alpha = R_L^G(\xi_{y^\alpha,\mu})$. It follows that there exists a bijection Ψ from $\mathrm{Irr}(b(C))$ to $\mathrm{Irr}(B)$ such that $\Psi(\pm R_L^{N(C)}(\xi_{y,\mu})) = \pm R_L^G(\xi_{y,\mu})$. Since both maps $R_L^{N(C)}$ and R_L^G are perfect isometry, it follows that Ψ preserves height. Moreover, $w \in \langle\beta\rangle$ stabilizes $\Psi(\varphi)$ is and only if w stabilizes $\varphi \in \mathrm{Irr}(b(C))$. This proves (4B).

(4C) *Let q be a power of an odd prime p and $G = G_2(q)$. In the notation of* [24], *suppose $B = B_3$. Then B satisfies Dade's invariant conjecture.*

Proof. Since $B = B_3$, it follows that $p \ne 3$, so $\beta = \alpha$. Thus $D = D(B) \simeq \mathrm{Z}_{2^a} \wr \mathrm{Z}_2$ or $SD_{2^{a+2}}$ according as $\eta = \epsilon$ or $-\epsilon$, where η is the sign $\pm$ such that $q \equiv \eta (\mathrm{mod}\, 3)$.

Suppose $D \simeq SD_{2^{a+2}}$, so that $\eta = -\epsilon$. Let $C = C(1)^1 : 1 < O_2(T)$ and $C' = C(1) : 1 < \mathrm{Z}_2 < O_2(T)$, where $T = T_{-\epsilon}$. Then $N(C) = N(T)$, $N(C') = \langle T, \rho, \tau\sigma\rangle$ and $C(C) = \langle T, \rho\rangle$. Let $b(C) \in \mathrm{Blk}(N(C)|B)$ and $b(C') \in \mathrm{Blk}(N(C')|B)$. A proof similar to that of (4B) shows that $b(C)$ and $b(C')$ both have a dihedral defect group of order 8, and the Brauer correspondence induces a bijection between $\mathrm{Blk}(N(C)|B)$ and $\mathrm{Blk}(N(C')|B)$. We may suppose $b(C')^{N(C)} = b(C)$ and Q is a defect group of $b(C')$.

Let $b \in \mathrm{Blk}(C(C)|b(C'))$, and b_D a block of T covered by b. Then b is a unique block of $C(C)$ covering b_D, so that $N(b_D) \leq N(b)$. We may suppose $b_D = \mathcal{E}_2(T, (s))$. By (2.4), $\sigma \in N(b_D) \leq N(b)$. Since $(N(b){:}\, QT)_2 = 1$, it follows that $N(b) = N(C)$. Since $\mathrm{Irr}(b_D) = \{\phi(sy) : y \in O_2(T)\}$ and $\rho \notin N(b_D)$, it follows that $\mathrm{Irr}(b) = \{\mathrm{Ind}_T^{C(C)}(\phi(sy)) : y \in O_2(T)\}$, so that $\mathrm{Irr}(b)$ has four characters of degree 2. Let $\vartheta_y = \mathrm{Ind}_T^{C(C)}(\phi(sy))$. Since $\rho \in Z(N(T)/T)$, it follows that

$$N(\vartheta_1) = N(C) \qquad (N(\vartheta_y) : C(C)) = 2,$$

where $1 \neq y \in O_2(T)$. Thus ϑ_1 has three extensions $\varphi_{s,i}$ in $N(C)$ such that $\varphi_{s,1}(1) = 4$ and $\varphi_{s,i}(1) = 2$ for $i = 2, 3$. In addition, for $y \neq 1$, ϑ_y has two extensions $\xi_{sy,i}$ to $N(\vartheta_y)$ of degree 2, and we let $\varphi_{sy,i} = \mathrm{Ind}_{N(\vartheta_y)}^{N(C)}(\xi_{sy,i})$. It follows that

$$\mathrm{Irr}(b(C)) = \{\varphi_{s,j}, \varphi_{sy,i} : 1 \leq j \leq 3, 1 \leq i \leq 2, 1 \neq y \in O_2(T)\}$$

and in particular, $\mathrm{Irr}(b(C))$ has one character of degree 4, 2 of degree 2 and degree 6.

Since $N(b) = N(C)$, the stabilizer $N'(b)$ of b in $N(C')$ is $N(C')$. Since $\rho^{\tau\sigma} = h_1\rho$ and $h_1^{\tau\sigma} = h_1$, it follows that $N'(\vartheta_1) = N'(\vartheta_{h_1}) = N(C')$ and $N'(\vartheta_{h_2}) = N'(\vartheta_{h_3}) = C(C)$, where $h_i = h_i(-1)$. Let $\psi_{sy,1}$ and $\psi_{sy,2}$ be the two extensions of ϑ_y to $N(C')$ and let $\psi_{sh_2,1} = \mathrm{Ind}_{C(C)}^{N(C')}(\vartheta_{h_2})$, where $y \in \{1, h_1\}$. Then

$$\mathrm{Irr}(b(C')) = \{\psi_{sy,i}, \psi_{sh_2,1} : 1 \leq i \leq 2, y = 1, h_1\}.$$

Define $\Psi : \mathrm{Irr}(b(C')) \longrightarrow \mathrm{Irr}(b(C))$ such that $\Psi(\psi_{sh_2,1}) = \varphi_{s,1}$, $\Psi(\psi_{s,i}) = \varphi_{s,1+i}$ and $\Psi(\psi_{sh_1,i}) = \varphi_{sy,i}$. Since $t^\alpha = t$ for all $t \in O_2(T)$, it follows that $w \in \langle\alpha\rangle$ stabilizes $\Psi(\psi)$ if and only if w stabilizes $\psi \in \mathrm{Irr}(b(C'))$. So (4.1) holds.

Let $M = N(1 < \mathrm{Z}_2) = \mathrm{SO}(U_4)$ and $b_M \in \mathrm{Blk}(M|B)$. Then $L = C_M(s)$ is a regular subgroup of M, where $L \simeq \mathrm{GL}_2^\epsilon(q)$ or $\mathrm{GL}_2^{-\epsilon}(q)$ according as $D \simeq \mathrm{Z}_{2^a} \wr \mathrm{Z}_2$ or $SD_{2^{a+2}}$. Let $b_L = \mathcal{E}_2(L, (1))$. The same proof as that of (4D) shows that $\mathrm{Blk}(M|B) = \{b_M\}$. Since R_L^M is a prefect isometry between b_L and b_M and since $\alpha(L) = L$ and $\alpha(M) = M$, it follows that w stabilizes $\pm R_L^M(\xi)$ if and only if w stabilizes $\xi \in \mathrm{Irr}(b_L)$ for $w \in \langle\alpha\rangle$. Let $B_\eta = B_{1a}$ or B_{2a} according as $\eta = \epsilon = +$ or $-$, and $B_\eta = B_{2a}$ or B_{1a} according as $\eta = -\epsilon = +$ or $-$. If $\eta = \epsilon$, then $D(B_\eta) \simeq \mathrm{Z}_{2^a} \wr \mathrm{Z}_2$, and if $\eta = -\epsilon$, then $D(B_\eta) \simeq SD_{2^{a+2}}$, so that $D(B_\eta) \simeq D$. As shown in the proof of (4B), R_L^G is a perfect isometry between b_L and B_η, so that w stabilizes $R_L^G(\xi)$ if and only it w stabilizes $\xi \in \mathrm{Irr}(b_L)$. It follows that we may identify $\mathrm{Irr}(b_M)$ with $\mathrm{Irr}(B_\eta)$. In particular, if $D \simeq SD_{2^{a+2}}$, then by [24],

$$\mathrm{k}(G, B, d, u) = \mathrm{k}(M, B, d, u) = \mathrm{k}(G, B_\eta, d, u)$$

for all $d, u \in \mathrm{Z}$, which implies (4C).

Finally, suppose $D \simeq \mathrm{Z}_{2^a} \wr \mathrm{Z}_2$, so that $\eta = \epsilon$ and $2^a \neq q - \epsilon$. Thus $B_\eta = B_{1a}$ or B_{2a} according as $\epsilon = +$ or $-$. So

$$\operatorname{Irr}^j(B_\eta) = \begin{cases} \{\varphi_{sy,\mu} : C(sy) = J_\epsilon, \mu = 1, \mathrm{St}\} & \text{if } j = 0, \\ \{\varphi_{sy,1} : C(sy) \simeq \mathrm{Z}_{q^2-1}\} & \text{if } j = a, \\ \{\varphi_{sy,1} : C(sy) = T_\epsilon\} & \text{if } j = 1, \\ \emptyset & \text{otherwise,} \end{cases}$$

and moreover, $|\operatorname{Irr}^0(B_\eta)| = 2^{a+1}$, $|\operatorname{Irr}^a(B_\eta)| = 2^{a-1}$ and $|\operatorname{Irr}^1(B_\eta)| = 2^{a-1}(2^a - 1)$. By [24],

$$\operatorname{Irr}^j(B_3) = \begin{cases} \{\chi_{sy,\mu}, \chi_{s,\lambda_i} : C(sy) = J_\epsilon, \mu = 1, \mathrm{St}\} & \text{if } j = 0, \\ \{\chi_{sy,1} : C(sy) \simeq \mathrm{Z}_{q^2-1}\} & \text{if } j = a, \\ \{\chi_{sy,1}, \chi_{s,\lambda_3} : C(sy) = T_\epsilon\} & \text{if } j = 1, \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\lambda_i \in \{\lambda_1, \lambda_2\}$ and λ_3 are a unipotent character of $C_G(s) = L_\epsilon$ such that $\lambda_1(1) = 1$, $\lambda_2(1) = q^3$ and $\lambda_3(1) = q(q + \epsilon)$. Thus $|\operatorname{Irr}^0(B_3)| = 2^{a+1}$, $|\operatorname{Irr}^a(B_3)| = 2^{a-1}$ and $|\operatorname{Irr}^1(B_3)| = 1 + \frac{1}{6}(2^a - 1)(2^a - 2)$. Define

$$\Psi : \operatorname{Irr}^a(B_\eta) \longrightarrow \operatorname{Irr}^a(B_3)$$

such that $\Psi(\varphi_{sy,1}) = \chi_{sy,1} \in \operatorname{Irr}^a(B_3)$ for $\varphi_{sy,1} \in \operatorname{Irr}(B_\eta)$. Then w stabilizes $\Psi(\varphi)$ if and only if w stabilizes φ.

Let $C = C(2)^1$ and $C' = C(2)$. Then $N(C) = N(T_\epsilon)$ and $N(C') = \langle T_\epsilon, \rho, \tau\sigma\rangle$. A proof similar to that of (4B) shows that $N(D) \leq N(C)$, $N(D) \leq N(C')$ and D is a defect group of $b(C) \in \operatorname{Blk}(N(C)|B)$ and $b(C') \in \operatorname{Blk}(N(C')|B)$. By Brauer's First Main Theorem,

$$\operatorname{Blk}(N(C)|B) = \{b(C)\} \qquad \operatorname{Blk}(N(C')|B) = \{b(C')\}.$$

If $b \in \operatorname{Blk}(T_\epsilon|b(C'))$, then $\phi(s)$ is the canonical character of b and $b = \mathcal{E}_2(T_\epsilon, (s)) = \{\phi(sy) : y \in O_3(T_\epsilon)\}$. Suppose $y = \operatorname{diag}\{y_1, y_2, y_3\} \in T_\epsilon$. By (2.4), $N(\phi(s))/T_\epsilon \simeq S_3$, and let $\xi_{s,i}$ be the three extensions of $\phi(s)$ in $N(\phi(s))$ such that $\xi_{s,3}(1) = 2$ and $\xi_{s,i}(1) = 1$ for $i = 1, 2$. If $y_i = y_j$ for some $i \neq j \in \{1, 2, 3\}$ and $y \neq 1$, then $C_G(sy) \simeq J_\epsilon$, $N(\phi(sy))/T_\epsilon \simeq \mathrm{Z}_2$ and $\phi(sy)$ has two extensions $\xi_{sy,1}$ and $\xi_{sy,2}$ in $N(\phi(sy))$ and T_ϵ contains $3(2^a - 1)$ such classes of sy. If $N(\phi(sy)) = T_\epsilon$, then let $\xi_{sy,1} = \phi(sy)$ and T_ϵ has $2^{2a} - 1 - 3(2^a - 1)$ such classes. Let $\varphi_{sy,i} = \operatorname{Ind}_{N(\phi(sy))}^{N(C)}(\xi_{sy,i})$. By Clifford theory,

$$\operatorname{Irr}^j(b(C)) = \begin{cases} \{\varphi_{sy,i}, \varphi_{s,i} : C_G(sy) = J_\epsilon, i = 1, 2\} & \text{if } j = 0, \\ \{\varphi_{sy,1}, \varphi_{s,3} : C_G(sy) = T_\epsilon\} & \text{if } j = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Since σ and τ act on $\operatorname{Irr}(b)$, it follows that $|\operatorname{Irr}^0(b(C))| = 2(2^a - 1) + 2 = 2^{a+1}$ and $|\operatorname{Irr}^1(b(C))| = 1 + \frac{1}{6}(2^a - 1)(2^a - 2)$. Define

$$\Psi : \operatorname{Irr}^1(b(C)) \longrightarrow \operatorname{Irr}^1(B_3)$$

such that $\Psi(\varphi_{s,3}) = \chi_{s,\lambda_3}$, $\Psi(\varphi_{sy,1}) = \chi_{sy,1}$. Similarly, define

$$\Psi : \mathrm{Irr}^0(b(C)) \longrightarrow \mathrm{Irr}^0(B_3)$$

such that $\Psi(\varphi_{s,i}) = \chi_{s,\lambda_i}$, $\Psi(\varphi_{sy,1}) = \chi_{sy,1}$ and $\Psi(\varphi_{sy,2}) = \chi_{s,\mathrm{St}}$. Then Ψ is a bijection between $\mathrm{Irr}^j(b(C))$ and $\mathrm{Irr}^j(B_3)$ for $j = 0, 1$, and $w \in \langle \alpha \rangle$ stabilizes $\Psi(\varphi)$ if and only if w stabilizes $\varphi \in \mathrm{Irr}^j(b(C))$.

Let $N'(b) = N(b) \cap N(C')$ and $N'(\phi(sy)) = N(\phi(sy)) \cap N(C')$ for $\phi(sy) \in \mathrm{Irr}(b)$. Then $N'(b) = \langle T_\epsilon, \tau\sigma \rangle$ and $N'(\phi(sy)) = N'(b)$ or T_ϵ. In the former case $y = \mathrm{diag}\,\{y_1, y_1, y_1^{-2}\}$, so that $C_M(sy) = J_\epsilon$ and $\mathrm{Irr}(b)$ has 2^a such characters. We denote by $\zeta_{sy,1}$ and $\zeta_{sy,2}$ the two extensions of $\phi(sy)$ to $N(\phi(sy))$. Similarly, $\mathrm{Irr}(b)$ has $2^{2a} - 2^a$ characters $\phi(sy)$ such that $N'(\phi(sy)) = T_\epsilon$ and we set $\zeta_{sy,1} = \phi(sy)$. Let $\psi_{sy,i} = \mathrm{Ind}_{N(\phi(sy))}^{N(C')}(\zeta_{sy,i})$. By Clifford theory,

$$\mathrm{Irr}^j(b(C')) = \begin{cases} \{\psi_{sy,i} : C_M(sy) = J_\epsilon, i = 1, 2\} & \text{if } j = 0, \\ \{\psi_{sy,1} : C_M(sy) = T_\epsilon\} & \text{if } j = 1, \\ \emptyset & \text{otherwise,} \end{cases}$$

and moreover, $|\mathrm{Irr}^0(b(C'))| = 2^{a+1}$ and $|\mathrm{Irr}^1(b(C'))| = 2^{a-1}(2^a - 1)$. For $j = 0, 1$, define

$$\Psi^{-1} : \mathrm{Irr}^j(b(C')) \longrightarrow \mathrm{Irr}^j(B_\eta)$$

such that $\Psi^{-1}(\psi_{sy,1}) = \varphi_{sy,1}$, $\Psi^{-1}(\psi_{sy,2}) = \varphi_{sy,\mathrm{St}}$ for $\psi_{sy,1} \in \mathrm{Irr}^0(b(C'))$ and $\Psi^{-1}(\psi_{sy,1}) = \varphi_{sy,1}$ for $\psi_{sy,1} \in \mathrm{Irr}^1(b(C'))$.

It follows that Ψ is a bijection between $\mathrm{Irr}(b(C)) \cup \mathrm{Irr}(B_\eta)$ and $\mathrm{Irr}(B_3) \cup \mathrm{Irr}(b(C'))$ such that w stabilizes $\Psi(\varphi)$ if and only if w stabilizes $\varphi \in (\mathrm{Irr}(b(C)) \cup \mathrm{Irr}(B_\eta))$ for $w \in \langle \alpha \rangle$. This implies (4C).

(4D) *Let q be a power of an odd prime p and $B = B_1$ the principal 2-block of $G = G_2(q)$. Then B satisfies Dade's invariant conjecture.*

Proof. Suppose $q \neq 3$ and let $C \in \mathcal{R}^0(G)$ be given by (2.8). By Brauer's Third Main Theorem, $\mathrm{Blk}(N(C)|B_1) = \{b(C)\}$, where $b(C) = B_0(N(C))$.

(1) Set $C = C(1)^1$ and $C' = C(1)$. Let $T = T_{-\epsilon}$ and $N(T) = N_G(T) = \langle T, \rho, \sigma, \tau \rangle$. Then $C(C) = C(C') = \langle T, \rho \rangle$, $N(C) = N(T)$, and $N(C') = \langle C(C'), \tau\sigma \rangle$. If $b = B_0(C(C))$ then $E_8 = \langle O_2(T), \rho \rangle$ is a defect group of b, $b^{N(C)} = b(C)$ and $b^{N(C')} = b(C')$. Let $b(T)$ be the principal block of T. Since $\phi(T) = \phi(O_2(T)) \times \phi(O_{2'}(T))$, it follows that $\mathrm{Irr}(b(T)) = \phi(O_2(T))$ and we can view b as the principal block of $E_8 = \langle O_2(T), \rho \rangle$. Thus $b(C)$ is the principal block of $\langle O_2(T), \rho, \sigma, \tau \rangle$ and $b(C')$ is the principal block of $\langle O_2(T), \rho, \tau\sigma \rangle$. As shown in the proof of (2E) (5), (4.1) holds.

(2) Let $C = C(2)^1$ or $C(2)_1$ according to whether $2^a \neq q - \epsilon$ or $2^a = q - \epsilon$, and let $C' = C(2)$. If $T = T_\epsilon$, then $N(C) = N(T)$ and $N(C') = \langle T, \rho, \tau\sigma \rangle$. If $b = B_0(T)$

with a defect group $O_2(T)$, then $|\operatorname{Irr}(b)| = 2^{2a}$. Now $\phi(y) \in \Omega_1(\operatorname{Irr}(b))$ if and only if $y \in \Omega_1(O_2(T))$, so $D_{12} = N(T)/T$ and β permutes $\Omega_1(\operatorname{Irr}(b))$. Let $\mathcal{X}_1(C)$ be the subset of $\operatorname{Irr}(b(C))$ consisting of characters covering $\Omega_1(\operatorname{Irr}(b))$. A proof similar to that of (2E) (5) shows that $\mathcal{X}_1(C)$ has 10 characters, 4 linear characters, 4 characters of degree 3, and 2 of degree 2. Moreover, α stabilizes each character of $\mathcal{X}_1(C)$ and if $p = 3$, then β stabilizes exactly 2 linear characters, 2 of degree 3 and 2 of degree 2.

Let $\mathcal{Z}(\tau\sigma)$ be characters of $\operatorname{Irr}(b)\backslash\Omega_1(\operatorname{Irr}(b))$ stabilized by $\tau\sigma$. By (2.4), $|\mathcal{Z}(\tau\sigma)| = 2^a - 2$ and ρ permutes $\mathcal{Z}(\tau\sigma)$. Similarly, $|\mathcal{Z}(\rho\tau\sigma)| = 2^a - 2$, ρ permutes $\mathcal{Z}(\rho\tau\sigma)$. Moreover, if $p = 3$, then β' interchanges $\mathcal{Z}(\tau\sigma)$ and $\mathcal{Z}(\rho\tau\sigma)$, since $(\tau\sigma)^{\beta'} = \rho\tau\sigma$ modulo T. Thus $\operatorname{Irr}(b)$ has $2^{2a} - 3(2^a - 2) - 3(2^a - 2) - 4$ characters $\phi(y)$ such that $N(\phi(y)) = T$, so that $\operatorname{Irr}(b(C))$ has $\frac{1}{12}(2^{2a} - 3 \times 2^{a+1} + 8)$ characters of degree 12. Let $\xi_{y,1}$ and $\xi_{y,2}$ be the two extensions of $\phi(y) \in \mathcal{Z}(\tau\sigma) \cup \mathcal{Z}(\rho\tau\sigma)$ to $N(\phi(y))$, and

$$\mathcal{X}_2(C) = \{\varphi_{y,i} = \operatorname{Ind}_{N(\phi(y))}^{N(C)}(\xi_{y,i}) : \phi(y) \in \mathcal{Z}(\tau\sigma) \cup \mathcal{Z}(\rho\tau\sigma), i = 1, 2\}.$$

Thus $|\mathcal{X}_2(C)| = 2(2^a - 2)$. In addition,

$$\operatorname{Irr}^2(b(C)) = \{\varphi_{y,1} = \operatorname{Ind}_T^{N(C)}(\phi(y))\},$$

where $\phi(y) \in \operatorname{Irr}(b)\backslash(\Omega_1(\operatorname{Irr}(b)) \cup \mathcal{Z}(\tau\sigma) \cup \mathcal{Z}(\rho\tau\sigma))$. So $|\operatorname{Irr}^2(b(C))| = \frac{1}{12}(2^{2a} - 3 \times 2^{a+1} + 8)$ and if $\varphi_{y,1} \in \operatorname{Irr}^2(b(C))$, then $C(y) = T$.

Similarly, if $\mathcal{X}_1(C')$ is the characters of $\operatorname{Irr}(b(C'))$ covering characters of $\Omega_1(\operatorname{Irr}(b))$, then $|\mathcal{X}_1(C')| = 10$ and moreover, α stabilizes each character of $\mathcal{X}_1(C')$ and if $p = 3$, then β stabilizes exactly 4 linear characters and 2 of degree 2 in $\mathcal{X}_1(C')$. Let $\zeta_{y,1}$ and $\zeta_{y,2}$ be the two extensions of $\phi(y) \in \mathcal{Z}(\tau\sigma) \cup \mathcal{Z}(\rho\tau\sigma)$ in $N(\phi(y))$, and

$$\mathcal{X}_2(C') = \{\psi_{y,i} = \operatorname{Ind}_{N(\phi(y)}^{N(C')}(\zeta_{y,i}) : \phi(y) \in \mathcal{Z}(\tau\sigma) \cup \mathcal{Z}(\rho\tau\sigma), i = 1, 2\}.$$

Then $|\mathcal{X}_2(C')| = 2(2^a - 2)$ and each character of $\mathcal{X}_2(C')$ has degree 2. Denote by

$$\operatorname{Irr}^2(b(C')) = \{\psi_{y,1} = \operatorname{Ind}_T^{N(C')}(\phi(y))\},$$

where $\phi(y) \in \operatorname{Irr}(b)\backslash(\Omega_1(\operatorname{Irr}(b)) \cup \mathcal{X}(\tau\sigma) \cup \mathcal{X}(\rho\tau\sigma))$. So $|\operatorname{Irr}^2(b(C'))| = \frac{1}{4}(2^{2a} - 2^{a+1})$ and if $\psi_{y,1} \in \operatorname{Irr}^2(b(C'))$, then we may suppose $y = \operatorname{diag}\{y_1, y_2, y_3\}$ such that $y_2 \notin \{y_1, y_1^{-1}\}$ and $y_1 y_2 y_3 = 1$.

As shown in the proof of (2E) (5) there is a bijection Ψ from $\mathcal{X}_1(C')$ to $\mathcal{X}_1(C)$ such that w stabilizes $\Psi(\psi)$ if and only if w stabilizes $\psi \in \mathcal{X}_1(C')$ for $w \in \langle\beta\rangle$. Define

$$\Psi : \mathcal{X}_2(C') \longrightarrow \mathcal{X}_2(C)$$

such that $\Psi(\psi_{y,i}) = \varphi_{y,i}$. Then w stabilizes $\Psi(\psi)$ if and only if w stabilizes $\psi \in \mathcal{X}_2(C')$ for $w \in \langle\beta\rangle$. Thus Ψ is a bijection between $\mathrm{Irr}^0(C') \cup \mathrm{Irr}^1(C')$ and $\mathrm{Irr}^0(C) \cup \mathrm{Irr}^1(C)$. Moreover, if $\psi_{y,1} \in \mathrm{Irr}^2(b(C'))$ or $\varphi_{y,1} \in \mathrm{Irr}^2(b(C))$, then $C_G(y) = T$, $\psi_{y,1}^{\beta} = \psi_{y^\beta,1}$ and $\varphi_{y,1}^{\beta} = \varphi_{y^\beta,1}$.

(3) Let $H = N(C(1)_1) = \mathrm{SO}(U_4)$ and b_H the principal block of H. By [8, Theorem 13], $b_H = \mathcal{E}_2(N(C), (1))$. Let s be a 2-element of G and $L = C(s)$. If $|s| \geq 4$, then we may suppose $Z(H) = \Omega_1(\langle s\rangle)$, since G has only one class of involutions. So $L = C_H(s)$. Thus $L \in \{\mathrm{GL}_2^{\epsilon}(q), T_\epsilon, T_3^{\delta}\}$ and L is a regular subgroup of H, where $\delta = \pm$. So $\pm R_L^H$ (for some sign) induces a bijection between $\mathcal{E}(L, (s))$ and $\mathcal{E}(H, (s))$. As in Section 3, we let

$$\mathcal{Y}(Q) = \mathcal{Y}(b_H, Q) = \mathrm{Irr}(b_H) \cap \left(\bigcup_s \mathcal{E}(H, (s))\right),$$

where s runs over 2-element of H such that $L \simeq Q$.

First, we consider the set $\mathcal{Y}(\mathrm{Z}_{q^2-1})$. If $\mathcal{E}(H, (s)) \subseteq \mathcal{Y}(\mathrm{Z}_{q^2-1})$, then $L = C(s) \simeq \mathrm{Z}_{q^2-1}$. Suppose s' is another 2-element of H conjugate to s in G. Then $h^{-1}sh = s'$ for some $h \in G$. Since $C(s')$ is cyclic and $C_H(s') \leq C_G(s')$, it follows that $C_H(s') \simeq \mathrm{Z}_{q^2-1}$, so that $Z(H) = \Omega_1(\langle s'\rangle)$. But $Z(H) = \Omega_1(\langle s\rangle)$, so h centralizes $Z(H)$ and $h \in H$. Thus $(s)_G \cap H$ is a single class of H, where $(s)_G$ is the conjugacy G-class containing s. Since G has $\frac{1}{2}2^a$ classes of such 2-elements s (see [24, 2.2.1 and 2.3.1]) and since $\mathcal{E}(L, (s)_L)$ has only one character, it follows that $|\mathcal{Y}(\mathrm{Z}_{q^2-1})| = 2^{a-1}$. In addition, all of them have height $a+1$ since $\zeta(1) = (H\colon L)_{p'}$ for $\zeta \in \mathcal{Y}(\mathrm{Z}_{q^2-1})$.

Suppose $\mathcal{E}(H, (s)) \subseteq \mathcal{Y}(J_\epsilon)$, so $L = J_\epsilon$ or $J_\epsilon{}^*$. Thus L is a regular subgroup of H. Since s and s^{-1} are conjugate in H, H has $\frac{1}{2}(2^a - 2)$ classes of 2-elements s such that $C_G(s) = L$. For each class $(s)_H$, $\mathcal{E}(L, (s)_L)$ has two characters with degrees 1 and q, respectively. Since H has two conjugacy classes of regular subgroups isomorphic to J_ϵ, it follows that $\mathcal{Y}(J_\epsilon)$ has $(2^a - 2)$ characters of degree $(H\colon L)_{p'}$ and $(2^a - 2)$ characters of degree $q(H\colon L)_{p'}$. In particular, $|\mathcal{Y}(J_\epsilon)| = 2(2^a - 2)$ and each character has height 1.

Suppose $\mathcal{E}(H, (s)) \subseteq \mathcal{Y}(T_\epsilon)$, so $L = T_\epsilon$. Let $Q = N_H(\langle s\rangle)$. Then we may suppose $Q = \langle T_\epsilon, \rho, \tau\sigma\rangle$. Let $L = \mathrm{Z}_{q-\epsilon} \times \mathrm{Z}_{q-\epsilon}$ and $x \in \mathrm{Z}_{q-\epsilon}$. Each element y of $\Omega_1(O_2(L))$ is stabilized by a non-trivial element of Q/L. Thus $L < C_H(y)$ and so $C_H(y) \not\simeq T_\epsilon$. Similarly, each element $y \in \{x \times x, x \times x^{-1}\}$ is stabilized by a non-trivial element of Q/L. Thus H contains $(2^{2a} - (2^a - 2) - (2^a - 2) - 4)$ 2-elements s such that $C_H(s) = L$. Since $|Q/L| = 4$, H contains exactly $\frac{1}{4}(2^{2a} - 2^{a+1})$ classes of such 2-elements s. But $\mathcal{E}(L, (s)_L)$ has only one character, so $|\mathcal{Y}(T_\epsilon)| = \frac{1}{4}(2^{2a} - 2^{a+1})$ and each character has height 2.

Suppose $\mathcal{E}(H, (s)) \subseteq \mathcal{Y}(Z(H))$, so that $s = 1$ or $\langle s\rangle = Z(H)$. There exists a bijection between $\mathcal{E}(H, (s))$ and $\mathcal{E}(H, (1))$ preserving degrees. Now H has four unipotent irreducible characters with degrees 1, q, q, q^2, and $|Z(H)| = 2$. So $\mathcal{Y}(Z(H))$ has two characters of degree 1, four of degree q and two of degree q^2, and all of them have height 0.

Finally, let $\mathcal{Y}_0 = \cup_s \mathcal{E}(H,(s))$, where s runs over non-central involutions of H. So $N = C_H(s) = \langle L_0, \rho\rangle$, where $L_0 = \mathrm{SO}^{\delta}(2,q)\times\mathrm{SO}^{\delta}(2,q)$, $\delta = \pm$ and ρ is an involution. Since $L_0 \simeq \mathrm{GL}(1,\delta q)\times\mathrm{GL}(1,\delta q)$, it follows that $\mathcal{E}(L_0,(1))$ contains only the trivial character of L_0 and $N/L_0 \simeq \mathrm{Z}_2$ acts trivially on $\mathcal{E}(L_0,(1))$. The stabilizer in N/L_0 of the trivial character is N/L_0 itself. By [27, Proposition 5.1], $\mathcal{E}(H,(s))$ contains exactly 2 characters. Since H has two classes of non-central involutions, it follows that $\mathcal{Y}_0$ has 4 characters.

In the notation of (2.5), $H = \langle M_1 \circ M_2, g\rangle$ and we identify M_1 and M_2 with $\mathrm{SL}_2(q)$. As shown in the proof of [19, Section 5.2] $\mathrm{SL}_2(q)$ has two irreducible characters of degree $\frac{1}{2}(q+1)$ and two of degree $\frac{1}{2}(q-1)$ which are permuted by g_i respectively, since $g_i \in \mathrm{GL}_2(q)\backslash\mathrm{SL}_2(q)$ (given before (2.5)). Moreover, these four characters lie in $B_0(\mathrm{SL}_2(q))$.

Let $\chi' \in \mathrm{Irr}(\mathrm{SL}_2(q))$ with $\chi'(1) = \frac{1}{2}(q-\epsilon)$, and let $\chi'' = (\chi')^{g_i}$. Then χ' and χ'' induce the same linear character on $Z(\mathrm{SL}_2(q))$. Thus $\chi'\otimes\chi''$ is an irreducible character of $M_1\circ M_2$ and $(\chi'\otimes\chi'')^g = \chi''\otimes\chi'$. Similarly, $\chi'\otimes\chi' \in \mathrm{Irr}(M_1\circ M_2)$ and

$$(\chi'\otimes\chi')^g = \chi''\otimes\chi''.$$

Thus H has two irreducible characters χ_1 and χ_2 of degree $\frac{1}{2}(q-\epsilon)^2$. Let $\varphi' \in \mathrm{Irr}(\mathrm{SL}_2(q))$ with $\varphi'(1) = \frac{1}{2}(q+\epsilon)$, and let $\varphi'' = (\varphi')^{g_i}$. A proof similar to above shows that H has two irreducible characters φ_1 and φ_2 of degree $\frac{1}{2}(q+\epsilon)^2$. It follows from the degrees of characters given above that $\chi_i, \varphi_i \in \mathcal{Y}_0$ for $i = 1, 2$. Thus

$$\mathcal{Y}_0 = \{\chi_1, \chi_2, \varphi_1, \varphi_2\}, \tag{4.2}$$

and each χ_i has height $2a-1$ and each φ_i has height 1.

If $p \neq 3$, then $\alpha(M_i) = M_i$ and α induces a field automorphism of M_i for $i = 1, 2$. Thus $(\chi')^{\alpha} \in \{\chi', \chi''\}$, so that α stabilizes each χ_i. Similarly, α stabilizes each φ_i.

Suppose $p = 3$. By (2C), β' interchanges M_1 and M_2. Since $g^{\beta'} = h_1(-1)g$, it follows that β' stabilizes each character of $\mathcal{Y}_0$.

It follows that $\mathrm{Irr}^{2a-1}(b_H) = \{\chi_1, \chi_2\}$ and β' stabilizes each character of $\mathrm{Irr}^{2a-1}(b_H)$. Since $\mathrm{Irr}^{2a-1}(B_1) = \{X_{17}, X_{18}\}$ or $\{X_{15}, X_{16}\}$ according as $\epsilon = +$ or $-$, it follows by (3A) that β stabilizes each character of $\mathrm{Irr}^{2a-1}(B_1)$. Thus there exists a bijection Ψ from $\mathrm{Irr}^{2a-1}(b_H)$ to $\mathrm{Irr}^{2a-1}(B_1)$ such that w stabilizes $\Psi(\varphi)$ if and only if w stabilizes $\varphi \in \mathrm{Irr}^{2a-1}(b_H)$.

By the proof above, $\mathrm{Irr}^0(b_H) = \{\varphi_{1i}, \varphi_{2i} : 1 \le i \le 4\} = \mathcal{Y}(Z(H))$, where $\varphi_{1i}(1) = \varphi_{2i}(1)$, $\varphi_{11}(1) = 1$, $\varphi_{12}(1) = q^2$, and $\varphi_{13}(1) = \varphi_{14}(1) = q$. Each character of degree q is an extension of a central product of the trivial character 1 and the Steinberg character St of $\mathrm{SL}_2(q)$. Since $M_i^{\alpha} = M_i$, it follows that α stabilizes each character of $\mathrm{Irr}^0(b_H)$. If $p = 3$, then $M_1^{\beta'} = M_2$, so that $(1\otimes\mathrm{St})^{\beta'} = \mathrm{St}\otimes 1$ and β' stabilizes no characters of degree q, so $\varphi_{13}^{\beta'} = \varphi_{14}$ and $\varphi_{23}^{\beta'} = \varphi_{24}$. But $\mathrm{Irr}^0(B_1) = \{X_{1i}, X_{2i} : 1 \le i \le 4\}$, so α stabilizes each character of $\mathrm{Irr}^0(B_1)$. If $p = 3$, then $X_{13}^{\beta} = X_{14}$ and $X_{23}^{\beta} = X_{24}$, it follows that the map $\Psi(\varphi_{ij}) = X_{ij}$ is a bijection between $\mathrm{Irr}^0(b_H)$ and $\mathrm{Irr}(B_1)$ and moreover, w stabilizes $\Psi(\varphi)$ if and only if w stabilizes $\varphi \in \mathrm{Irr}^0(b_H)$.

Similarly, $\mathrm{Irr}^1(b_H) = \{\varphi_1, \varphi_2\} \cup \mathcal{Y}(b_H, J_\epsilon)$ and $\mathrm{Irr}^1(B_1) = \{X_{15}, X_{16}\} \cup \mathcal{Y}(B_1, J_\epsilon)$ or $\{X_{17}, X_{18}\} \cup \mathcal{Y}(B_1, J_\epsilon)$ according as $\epsilon = +$ or $-$. Define $\Psi(\varphi_1) = X_{15}$ or X_{17}, $\Psi(\varphi_2) = X_{16}$ or X_{18} according as $\epsilon = +$ or $-$, and $\Psi(\varphi_{y,\mu}) = \chi_{y,\mu}$, where $\varphi_{y,\mu} \in \mathcal{Y}(b_H, J_\epsilon)$ such that $\varphi_{y,1}(1) = (H\colon J_\epsilon)_{p'}$ and $\varphi_{y,\mathrm{St}}(1) = q(H\colon J_\epsilon)_{p'}$. Thus Ψ is a required bijection from $\mathrm{Irr}^1(b_H)$ to $\mathrm{Irr}^1(B_1)$.

Now $\mathrm{Irr}^{a+1}(b_H) = \mathcal{Y}(b_H, \mathrm{Z}_{q^2-1})$, $\mathrm{Irr}^{a+1}(B_1) = \mathcal{Y}(B_1, \mathrm{Z}_{q^2-1})$ and $|\mathrm{Irr}^{a+1}(b_H)| = |\mathrm{Irr}^{a+1}(B_1)| = 2^{a-1}$. If $\varphi_{s,1} \in \mathcal{Y}(b_H, \mathrm{Z}_{q^2-1})$, then define $\Psi(\varphi_{s,1}) = \chi_{s,1} \in \mathcal{Y}(B_1, \mathrm{Z}_{q^2-1})$, so that Ψ is a required bijection.

Finally, $\mathrm{Irr}^2(b_H) = \mathcal{Y}(b_H, T_\epsilon)$, $|\mathrm{Irr}^2(b_H)| = \frac{1}{4}(2^{2a} - 2^{a+1})$ and if $\zeta_{y,1} \in \mathrm{Irr}^2(b_H)$, we may suppose $y = \mathrm{diag}\,\{y_1, y_2, y_3\} \in T_\epsilon$ such that $y_2 \ne \{y_1, y_1^{-1}\}$. By [24], $\mathrm{Irr}^2(B_1) = \mathcal{Y}(B_1, T_\epsilon)$, $|\mathrm{Irr}^2(B_1)| = \frac{1}{12}(2^a - 4)(2^a - 2)$ and if $\chi_{y,1} \in \mathrm{Irr}^2(B_1)$, then $C_G(y) = T_\epsilon$. Define

$$\Psi : \mathrm{Irr}^2(b(C)) \longrightarrow \mathrm{Irr}^2(B_1)$$

such that $\Psi(\varphi_{y,1}) = \chi_{y,1}$ for $\varphi_{y,1} \in \mathrm{Irr}^2(b(C))$ and also define $\Psi : \mathrm{Irr}^2(b(C')) \longrightarrow \mathrm{Irr}^2(b_H)$ such that $\Psi(\psi_{y,1}) = \zeta_{y,1}$, where $\mathrm{Irr}^2(b(C))$ and $\mathrm{Irr}^2(b(C'))$ are given by (2). Then Ψ is a required bijection. It follows by (2) and (3) that

$$\Psi : \mathrm{Irr}(b(C')) \cup \mathrm{Irr}(B_1) \longrightarrow \mathrm{Irr}(b_H) \cup \mathrm{Irr}(b(C))$$

is a bijection such that $w \in \langle\alpha\rangle$ stabilizes $\Psi(\varphi)$ if and only if w stabilizes φ. Thus

$$\mathrm{k}(G, B_1, d, u) + \mathrm{k}(N(C'), B_1, d, u) = \mathrm{k}(H, B_1, d, u) + \mathrm{k}(N(C), B_1, d, u).$$

(4) Suppose $C \in \{C(3)_1, C(3), C(4)_2, C(4)\}$, so that $N(C) = N_{G_2(p)}(C)$. If $p \ne 3$, then $N_A(C) = N(C) \times \langle\alpha\rangle$ and we can apply the proofs (3) and (4) of [4, (4C)], so that (4D) holds. Suppose $p = 3$, so that $N_A(C) = N_{G_2(3):2}(C) \times \langle\alpha\rangle$, where $G_2(3)\colon 2$ is the automorphism of $G_2(3)$. Thus we may suppose $q = p = 3$, so that $A = G_2(3)\colon 2$.

Let $C = C(3)_1 : 1 < E_8$ and $C' = C(3) : 1 < E_8 < E$, where $E = 2_+^{1+4}$. Then $N(E_8) = E_8.L_3(2)$ and $N_A(E_8) = E_8.L_3(2)\colon 2$. Since $C_{N(C)}(E_8) = E_8 \trianglelefteq N(C)$, it follows that $\mathrm{Irr}(b(C)) = \mathrm{Irr}(N(C))$. Similarly, $\mathrm{Irr}(b(C')) = \mathrm{Irr}(N(C'))$.

The character table of $N(C) = E_8.L_3(2)$ is given by [25, Table III], but the actions of an outor automorphism $\theta \in E_8.L_3(2)\colon 2 \backslash E_8.L_3(2)$ on the characters of $\mathrm{Irr}(N(E_8))$ is incorrect.

If $t \in N(E_8)$ is an element of order 4, then $t \in (4A)_G$ or $(4B)_G$, where $(4A)_G$ and $(4B)_G$ are the G-classes of elements of order 4 given by [11, p.60]. By [11, p.60], β fuses the two irreducible characters of degree 273 and so β fuses the classes $(4A)_G$ and $(4B)_G$, since $\chi^\beta(M) = \chi(M^\beta)$ for each G-class M. It follows that t^θ and t are not G-conjugate, so that t^θ and t are not $N(E_8)$-conjugate. Thus θ fuses the only two $N(E_8)$-classes of elements of order 4. Similarly, since $(8A)_G^\beta = (8B)_G$, it follows that θ fuses the two $N(E_8)$-classes of elements of order 8. By the values of the irreducible characters χ_i on these classes we have that θ interchanges the characters of each pair (χ_7, χ_8) and (χ_{10}, χ_{11}), where $\chi_i \in \mathrm{Irr}(N(E_8))$ are given by [25, Table III].

Suppose θ fuses the two irreducible characters χ_2 and χ_3 of degree 3. Then θ induces an outor automorphism of $L_3(2)$, since E_8 is in the kernel of χ_2 and χ_3 and they can be viewed as characters of $L_3(2)$. Suppose $E = 2_+^{1+4}$ and $Q = \langle \Omega_2(O_2(T_\epsilon)), \rho \rangle$. As shown in the proof of (2.12) we may suppose $E, Q \in \Phi(E_8.L_3(2), 2)$, so that E/E_8 and Q/E_8 are two non-conjugate $L_3(2)$-classes of unipotent radicals of order 4. Since θ induces an outor automorphism, it follows by [11, p.3] that θ fuses E/E_8 and Q/E_8, so that E and Q are A-conjugate. This is impossible, since $Z(E) = \mathrm{Z}_2$ and $Z(Q) = \mathrm{Z}_2 \times \mathrm{Z}_2$. Then θ stabilizes both χ_2 and χ_3. By the degrees of characters $\chi \in \mathrm{Irr}(E_8.L_3(2))$, θ interchanges exactly the characters of each pair (χ_7, χ_8) and (χ_{10}, χ_{11}), and stabilizes the other characters. It follows that

$$\mathrm{k}(N(C(3)_1), B_0, d, u) = \begin{cases} 4 & \text{if } d = 6 \text{ and } u = 2, \\ 4 & \text{if } d = 6 \text{ and } u = 1, \\ 2 & \text{if } d = 5 \text{ and } u = 2, \\ 1 & \text{if } d = 3 \text{ and } u = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

The character table of $N(C') = 2_+^{1+4}.S_3$ is given below, which can be calculated directly using (2.12) or obtained using computer.

The ordinary character table of the group $2_+^{1+4}.S_3$

class	1	2	3	4	5	6	7	8	9	10	11	12	13
conjg	1	1	6	12	24	32	6	6	12	12	32	24	24
order	1	2	2	2	2	3	4	4	4	4	6	8	8
ξ_1	1	1	1	1	1	1	1	1	1	1	1	1	1
ξ_2	1	1	1	1	-1	1	1	1	-1	-1	1	-1	-1
ξ_3	2	2	2	2	0	-1	2	2	0	0	-1	0	0
ξ_4	3	3	-1	-1	-1	0	3	-1	1	1	0	1	-1
ξ_5	3	3	-1	-1	-1	0	-1	3	1	1	0	-1	1
ξ_6	3	3	-1	-1	1	0	-1	3	-1	-1	0	1	-1
ξ_7	3	3	-1	-1	1	0	3	-1	-1	-1	0	-1	1
ξ_8	3	3	3	-1	-1	0	-1	-1	-1	-1	0	1	1
ξ_9	3	3	3	-1	1	0	-1	-1	1	1	0	-1	-1
ξ_{10}	4	-4	0	0	0	1	0	0	-2	2	-1	0	0
ξ_{11}	4	-4	0	0	0	1	0	0	2	-2	-1	0	0
ξ_{12}	6	6	-2	2	0	0	-2	-2	0	0	0	0	0
ξ_{13}	8	-8	0	0	0	-1	0	0	0	0	1	0	0

As shown above $\theta \in N_A(E_8)\backslash N(E_8)$ fuses the two $N(E_8)$-classes $(8A)_{N(E_8)}$ and $(8B)_{N(E_8)}$, as well as $(4A)_{N(E_8)}$ and $(4B)_{N(E_8)}$. We may suppose $Z(\mathrm{SO}(U_4)) = \langle z\rangle \le E_8$ and $2_+^{1+4} = O_2(\mathrm{SO}(U_4))$, where $\mathrm{SO}(U_4) = \mathrm{SO}_4^+(3)$. Then $z^\theta \in E_8$ and so $z^{\theta y} = z$ for some $y \in N(E_8)$, since $N(E_8)$ acts transitively on E_8. If $\theta' = \theta y$, then $\theta' \in N_A(E_8)\backslash N(E_8)$ and $z^{\theta'} = z$. Thus $\theta' \in \mathrm{SO}(U_4).2\backslash \mathrm{SO}(U_4)$. Since $2_+^{1+4} = O_2(\mathrm{SO}(U_4))$, θ' normalizes $N_{N(E_8)}(2_+^{1+4}) = N(C(3))$, so that $\theta' \in N_A(C(3))\backslash N(C(3))$. Moreover, since $\theta' = \theta y$ for some $y \in N(E_8)$, it follows that $(8A)_{N(E_8)}^{\theta'} = (8B)_{N(E_8)}$ and $(4A)_{N(E_8)}^{\theta'} = (4B)_{N(E_8)}$.

Suppose A_j is the j-th class of $N(C(3))$ given above. It $t \in A_{12}$, then $|t| = 8$ and t is an element of $(8A)_{N(E_8)}$ or $(8B)_{N(E_8)}$. Since θ' fuses $(8A)_{N(E_8)}$ and $(8B)_{N(E_8)}$, it follows that $t^{\theta'}$ and t are not $N(E_8)$-conjugate. In particular, $t^{\theta'}$ and t are not $N(C(3))$-conjugate. Since $N(C(3))$ has exactly two classes of elements of order 8, it follows that $t^{\theta'} \in A_{13}$, so that θ' fuses the $N(C(3))$-classes A_{12} and A_{13}. Similarly, if $t \in A_9$, then $t \in (4A)_{N(E_8)}$ or $t \in (4B)_{N(E_8)}$, so that $t^{\theta'}$ and t are not $N(E_8)$-conjugate. Thus $t^{\theta'} \notin A_9$, and so $t^{\theta'} \in A_{10}$, since $C_{N(C(3))}(t)^{\theta'} = C_{N(C(3))}(t^{\theta'})$. If $t \in A_7$, then a similar proof shows that $t^{\theta'} \in A_8$.

If $\xi \in \mathrm{Irr}(N(C(3))$ and $\xi^{\theta'} = \xi'$, then $\xi(A_j) = \xi'(A_j)$ for all j. Since $A_{12}^{\theta'} = A_{13}$, it follows by the values of ξ_4 and ξ_5 on A_{12} and A_{13} that $\xi_4^{\theta'} = \xi_5$. Similarly, $\xi_6^{\theta'} = \xi_7$ and $\xi_{10}^{\theta'} = \xi_{11}$. Moreover, $\xi_8^\theta = \xi_8$ and $\xi_9^\theta = \xi_9$, and so θ' stabilizes each character of

$$\{\xi_j : j \in \{1, 2, 3, 8, 9, 12, 13\}\}$$

and fuses the characters of each pair of $(\xi_4, \xi_5), (\xi_6, \xi_7)$ and (ξ_{10}, ξ_{11}). Thus

$$\mathrm{k}(N(C(3)), B_1, d, u) = \begin{cases} 4 & \text{if } d = 6 \text{ and } u = 2, \\ 4 & \text{if } d = 6 \text{ and } u = 1, \\ 2 & \text{if } d = 5 \text{ and } u = 2, \\ 2 & \text{if } d = 4 \text{ and } u = 1, \\ 1 & \text{if } d = 3 \text{ and } u = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

Since $q = p = 3$, $\Omega_2(O_2(T_\epsilon)) = T_\epsilon$ and $N(C(4)_2) = N(T_\epsilon) = \langle T_\epsilon, \rho, \sigma, \tau\rangle$. Thus the characters of $\mathrm{Irr}(N(C(4)_2)$ are given by the proof (2) with $a = 2$. In particular, $\Omega_1(\mathrm{Irr}(T_\epsilon))$ has 10 extensions in $\mathrm{Irr}(N(T_\epsilon))$, 4 linear characters, 4 characters of degree 3 and 2 of degree 2. In addition, $\beta' \in N_A(T_\epsilon)\backslash N(T_\epsilon)$ stabilizes exactly 2 linear characters, 2 of degree 3 and 2 of degree 2. Moreover, $\mathrm{Irr}(N(T_\epsilon))$ contains two characters of degree 6 covering $\mathcal{Z}(\tau\sigma)$ and $\mathcal{Z}(\rho\tau\sigma)$, respectively. Since $\mathcal{Z}(\tau\sigma)^{\beta'} = \mathcal{Z}(\rho\tau\sigma)$, it follows that β' stabilizes no characters of degree 6 in $\mathrm{Irr}(N(T_\epsilon))$, and so

$$\mathrm{k}(N(C(4)_2), B_1, d, u) = \begin{cases} 4 & \text{if } d = 6 \text{ and } u = 2, \\ 4 & \text{if } d = 6 \text{ and } u = 1, \\ 2 & \text{if } d = 5 \text{ and } u = 2, \\ 4 & \text{if } d = 5 \text{ and } u = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

Similarly, since $N(C(4)) \in \mathrm{Syl}_2(E_8.L_3(2))$, we may suppose $N(C(4)) = \langle T_\epsilon, \rho, \tau\sigma\rangle$. As shown in the proof (2) above, $\mathrm{Irr}(N(C(4))$ has 10 characters covering characters of $\Omega_1(\mathrm{Irr}(T_\epsilon))$, 8 linear and 2 of degree 2. In addition, $\beta' \in N_A(C(4))\backslash N(C(4))$ stabilizes exactly 4 linear characters and 2 of degree 2. Moreover, $\mathrm{Irr}(N(C(4)))$ contains two characters of degree 2 covering $\mathcal{Z}(\tau\sigma)$ and $\mathcal{Z}(\rho\tau\sigma)$, respectively. Since $\mathcal{Z}(\tau\sigma)^{\beta'} = \mathcal{Z}(\rho\tau\sigma)$, it follows that β' stabilizes no characters of degree 2 in $\mathrm{Irr}(N(C(4)))$. Also $N(C(4))$ contains 2 irreducible characters $\xi_{y,1}$ of degree 4 covering characters $\phi(y) \in \mathrm{Irr}(T_\epsilon)\backslash(\Omega_1(\mathrm{Irr}(T_\epsilon)) \cup \mathcal{Z}(\tau\sigma) \cup \mathcal{Z}(\rho\tau\sigma))$. In particular, $C(y) = T_\epsilon$. So $y^{\beta'}$ and y are not $N(E_8)$-conjugate and $\xi_{y,1} \neq \xi_{y^{\beta'},1}$. Since $\xi_{y,1}^{\beta'} = \xi_{y^{\beta'},1}$ and since $\mathrm{Irr}(N(C(4)))$ has exactly two characters of degree 4, β' fuses the two character of degree 4. It follows that

$$\mathrm{k}(N(C(4)), B_1, d, u) = \begin{cases} 4 & \text{if } d = 6 \text{ and } u = 2, \\ 4 & \text{if } d = 6 \text{ and } u = 1, \\ 2 & \text{if } d = 5 \text{ and } u = 2, \\ 4 & \text{if } d = 5 \text{ and } u = 1, \\ 2 & \text{if } d = 4 \text{ and } u = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

By (4.3)-(4.6),

$$\sum_{C \in \{C(3)_1, C(3), C(4)_2, C(4)\}} (-1)^{|C|}\, \mathrm{k}(N(C), B_1, d, u) = 0,$$

which implies (4D).

Finally, suppose $q = p = 3$ and $C \in \{C(3)_0, C(3)_1, C(3)^1, C(3)\}$. If $C \neq C(3)_0$ or $C(3)^1$, then $\mathrm{k}(N(C), B_1, d, u)$ are given by (4.3) and (4.4). If $C = C(3)_0$, then $N(C) = G$ and by [11, p.60]

$$\mathrm{k}(G, B_1, d, u) = \begin{cases} 4 & \text{if } d = 6 \text{ and } u = 2, \\ 4 & \text{if } d = 6 \text{ and } u = 1, \\ 2 & \text{if } d = 5 \text{ and } u = 2, \\ 4 & \text{if } d = 5 \text{ and } u = 1, \\ 2 & \text{if } d = 3 \text{ and } u = 2, \\ 2 & \text{if } d = 3 \text{ and } u = 1, \\ 0 & \text{otherwise.} \end{cases}$$

If $C = C(3)^1$, then $N(C) = \mathrm{SO}_4^+(3) = \mathrm{SO}(U_4)$ and the characters of $\mathrm{Irr}(N(C))$ can be obtained by the proof (3). So $\mathrm{Irr}(\mathrm{SO}_4^+(3))$ has 2 characters of degree 8 in $\mathcal{Y}(\mathrm{Z}_8)$ and each is of the form $\pm R_{T_\delta{}^3}^{\mathrm{SO}(U_4)}(\phi(y))$ for some sign $\pm$ and some element $y \in \mathrm{SO}(U_4)$ of order 8. In particular, $C(y) = C_{\mathrm{SO}(U_4)}(y) \simeq \mathrm{Z}_8$. Since β' fuses the two G-classes of maximal tori $T_\delta{}^3 \simeq \mathrm{Z}_8$, it follows that

$$R_{T_\delta{}^3}^{\mathrm{SO}(U_4)}(\phi(y))^{\beta'} = R_{T_\delta{}^3}^{\mathrm{SO}(U_4)}(\phi(y^{\beta'})) \neq R_{T_\delta{}^3}^{\mathrm{SO}(U_4)}(\phi(y))$$

and so β' fuses the two characters in $\mathcal{Y}(\mathrm{Z}_8)$.

In addition, $\mathrm{SO}(U_4)$ has 4 characters in $\mathcal{Y}(J_\epsilon)$, two characters of degree 2 and two of degree 6. The two characters of degree 2 are labelled by $(y, 1)$ and $(y', 1)$, respectively, where $C(y) = J_\epsilon$ and $C(y') = {J_\epsilon}^*$. If χ is labelled by $(y, 1)$, then $\chi^{\beta'}$ is labelled by $(y^{\beta'}, 1)$. Since β' fuses J_ϵ and ${J_\epsilon}^*$, it follows that β' fuses the two characters of degree 2. Similarly, β' fuses the two characters of degree 6 in $\mathcal{Y}(J_\epsilon)$.

Each character χ of $\mathcal{Y}(T_\epsilon)$ is of degree 4 and labelled by $(y, 1)$, where $|y| = 4$ and $C(y) = T_\epsilon$. Since $G.2$ has exactly one class of elements of order 4, a similar proof to above shows that β' fuses the two characters of degree 4 in $\mathcal{Y}(T_\epsilon)$. Moreover, $\mathcal{Y}(Z(H))$ has 4 characters of degree $3 = q$, and β' stabilizes no characters of degree 3 in $\mathcal{Y}(Z(H))$ as shown above. Since β' stabilizes the Steinberg character of $\mathrm{SO}(U_4)$, it follows that it stabilizes each character of degree 9. As shown in the proof of (4.2) β' stabilizes each character on $\mathcal{Y}_0$. So

$$\mathrm{k}(N(C(3)^1), B_1, d, u) = \begin{cases} 4 & \text{if } d = 6 \text{ and } u = 2, \\ 4 & \text{if } d = 6 \text{ and } u = 1, \\ 2 & \text{if } d = 5 \text{ and } u = 2, \\ 4 & \text{if } d = 5 \text{ and } u = 1, \\ 2 & \text{if } d = 4 \text{ and } u = 1, \\ 2 & \text{if } d = 3 \text{ and } u = 2, \\ 2 & \text{if } d = 3 \text{ and } u = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and this implies (4D).

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Department of Mathematics
University of Auckland
Private Bag 92019
Auckland, New Zealand

E-mail: an@mat.auckland.ac.nz