Spectral Theory of Wiener-Hopf Operators and Functional Model.

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Abstract

Using the Nagy-Foias functional model for contractions we reduce the spectral problem for Wiener-Hopf Operators with rational symbols to the spectral problem for finite matrices. In particular we suggest a simple approach to calculation of Wiener-Hopf determinants for analytic symbols.

1 Introduction

In [10] the general approach for asymptotic calculation of classical Szegö-Kac determinants [1, 2, 6] was proposed. This approach permits investigation of the asymptotics of the Fredholm determinants of the Wiener-Hopf operator T, defined on the finite interval $(0, a)$ as a reduction of integral operator with difference kernel:

$$\begin{array}{rccc} T \equiv T_a(g): & L_2(0,a) & \longrightarrow & L_2(0,a) \\ & u(x) & \longmapsto & u(x) + \int_0^a g(x-s)u(s)\,ds. \end{array}$$

More specifically (see [11]), the asymptotics as $a \to \infty$ can be calculated provided the symbol $\sigma = 1 + \int e^{iks} g(s) ds$ possesses real roots. In [13], an elegant description is given for the oscillating terms in the asymptotics, in the special case that the symbol has only two real roots. Recovering similar asymptotics in the more general case of matrix integral operators (see for instance [8, 9]), requires a more direct and general approach to the problem, which can be supplied by the Lax-Phillips version of analytic semigroup theory suggested first in [4] for scattering problems. This approach is equivalent [5], to the construction of functional models of dissipative operators [7].

In this paper we consider Wiener-Hopf operators on the finite interval $(0, a)$ whose symbols σ are analytic in the upper half plane or rational functions. In the first case a straightforward procedure based on the Functional Model for calculating the determinant of the operator is suggested. In the second case we just reduce the spectral problem for Wiener- Hopf Operator to the similar problem for a finite matrix . Our approach can be considered as an alternative to the classical approach by Gohberg ,which differs from the prototype [12] in two respects : our simple geometric mechanics remains practically unchanged (but just more boresome) for matrix case and it is applicable to similar problems on multiconnected domains. These interesting subjects will be discussed soon elswhere.

2 Determinants of Wiener-Hopf operators with analytic symbols — sketch of the plan and the headlights.

Let T be the above Wiener-Hopf operator, and set $\rho = \sigma - 1 = \mathcal{F}^* g$, where $\mathcal{F}^*$ is the inverse Fourier transform. (The operator of multiplication by σ will also be denoted by σ — the meaning is always clear from the context). Our approach to the calculation of the determinant of T is based on the fact that T can be approximated by some other operators W_β whose eigenvalues are known exactly.

Recall that the inverse Fourier transform $\mathcal{F}^*$ maps $L_2[0,\infty]$ unitarily to the Hardy space H_2^+ of the upper half plane[1]. Further, $\mathcal{F}^*(L_2[0,a]) = H_2^+ \ominus e^{ika} H_2^+$, and of course the convolution operator becomes the multiplication operator. So denoting $H_2^+ \ominus e^{ika} H_2^+$ by K_a, and writing P_a for orthogonal projection $H_2^+ \to K_a$, we see that T is unitarily equivalent to $W \equiv W_a(\sigma) = P_a \sigma P_a$. In other words, we have reduced our original problem to the calculation of

$$\det(P_a \sigma P_a). \tag{*}$$

The exponential e^{ika} is a singular function, so we can find a sequence of Blaschke products tending uniformly to it on the upper half plane. A good choice turns out to be

$$\Pi_\beta(k) = \frac{e^{ika} - e^{-\beta}}{1 - e^{ika} e^{-\beta}},$$

which does indeed tend uniformly (on the upper half plane) to e^{ika} as $\beta \to \infty$. It's convenient to note here that the zeroes of $\Pi_\beta(k)$ occur at the points $k_l = 2\pi l/a + i\beta/a$, $l \in \mathbb{Z}$.

Let $K_\beta = H_2^+ \ominus \Pi_\beta H_2^+$, and $P_\beta =$ orthogonal projection $H_2^+ \to K_\beta$. We consider the operator

$$W_\beta = P_\beta \sigma P_\beta$$

as an approximation for W, because in a sense to be made precise later, $K_\beta \to K_a$ and $P_\beta \to P_a$ as $\beta \to \infty$. The idea is that instead of (*), we can use

$$\lim_{\beta \to \infty} \det(P_\beta \sigma P_\beta), \tag{†}$$

provided that the W_β approximate W well enough. But the whole point of this approach is that the operator $P_\beta \sigma P_\beta$ turns out to have a remarkably simple form, provided the function σ is *analytic in the upper half plane*: its eigenvectors form a complete set (and even a Riesz basis) in K_β, and the eigenvalues are just $\sigma(k_l)$. This fact immediately gives an explicit expression for (†).

The straightforward plan outlined here meets some minor obstacles, such as the fact that the operators W and W_β are close in operator norm but not in trace norm. Therefore we need an intermediate operator, which is similar [2] to W_β (and therefore has the same determinant), but close to W in trace norm. This intermediate operator will be constructed as the image of W_β under the multiplication operator of an entire function f_β, which is bounded and invertible as an operator in $L_2(\mathbb{R})$. In summary, the plan is realised as the following chain of statements, which sketch the way of using Semigroup Theory (or the functional calculus for shift operators) for calculating Szegö-Kac determinants.

[1] An excellent reference to material on Hardy spaces and Blaschke products is [3]

[2] In the technical sense, i.e. equal to $A\,W_\beta\,A^{-1}$ for some A.

3 Proof of the main theorem

In all the following results, $a > 0$ and $\beta > 1$. We first state a well-known result from the theory of semigroups.

Proposition 1 *Let Π_β be the family of Blaschke products*

$$\Pi_\beta(k) = \frac{e^{ika} - e^{-\beta}}{1 - e^{ika}e^{-\beta}},$$

approaching the singular function $e^{ika} \equiv \theta_a$ uniformly in upper half plane as $\beta \to \infty$. Consider the generators B_β of the contracting semigroup

$$Z_\beta(t) = P_\beta e^{ikt} P_\beta \equiv e^{iB_\beta t}, \ \ t > 0,$$

which arises as a compression of the shift group onto the coinvariant subspaces

$$K_\beta = H_+^2 \ominus \Pi_\beta H_+^2.$$

Then the B_β are simple dissipative operators, with eigenfunctions given by

$$\psi_l(k) = \frac{\Pi_\beta(k)}{k - k_l}, \ \ l \in \mathbb{Z}$$

and corresponding eigenvalues $k_l = 2\pi l/a + i\beta/a$.

Proof: This is a well-known result, but we provide a proof here for completeness.

Denote by $\langle \cdot, \cdot \rangle$ the standard inner product on $L_2(\mathbb{R})$ (and inherited by H_2^+). First observe that $\frac{\Pi_\beta}{k-k_l} \in K_\beta$, since for any $g \in H_2^+$ we have

$$\begin{aligned} \left\langle \tfrac{\Pi_\beta}{k-k_l}, g\Pi_\beta \right\rangle &= \int_{-\infty}^{\infty} \tfrac{\Pi_\beta}{k-k_l} \overline{\Pi_\beta g} \, dk \\ &= \left\langle \tfrac{1}{k-k_l}, g \right\rangle \qquad \text{as } \Pi_\beta \bar{\Pi}_\beta \equiv 1 \text{ on real axis} \\ &= 0 \qquad \text{as } g \in H_2^+ \text{ and } \tfrac{1}{k-k_l} \in H_2^- \end{aligned}$$

So to prove ψ_l is an eigenvector with eigenvalue k_l, it remains to show that

$$P_{K_\beta} \left[(e^{ikt} - e^{ik_l t}) \frac{\Pi_\beta}{k - k_l} \right] = 0.$$

But this is equivalent to $(e^{ikt} - e^{ik_l t}) \frac{\Pi_\beta}{k-k_l} \in \Pi_\beta H_2^+$, or $\frac{e^{ikt} - e^{ik_l t}}{k-k_l} \in H_2^+$, which is obviously true since $t > 0$. ■

A similar statement is valid for the systems of eigenvectors of the adjoint operators B_β^*. Actually in this case the eigenvectors conveniently coincide with the H_2^+ reproducing kernels: $\varphi_l(k) = \frac{1}{k-\bar{k}_l}$. In other words, we have,

Proposition 2 *The eigenvectors of B_β^* are*

$$\varphi_l(k) = \frac{1}{k - \bar{k}_l}, \ \ l \in \mathbb{Z}$$

with eigenvalues $\bar{k}_l = 2\pi l - i\beta$.

Proof: We need only check that $\{\varphi_l\}$ and $\{\psi_l\}$ are biorthogonal sets. To do this we need a simple result stating the reproducing kernel for H_2^+:

Let $c \in$ upper half plane, and $f \in H_2^+$. Then $\langle f, \frac{1}{k-\bar{c}}\rangle = 2\pi i f(c)$. (The proof of this result is simple: close the contour in the upper half plane after using the fact that $k = \bar{k}$ on the real axis.)

So if $l \neq m$, $\langle \psi_l, \varphi_m\rangle = 2\pi i \frac{\Pi_\beta(k_m)}{k_m - k_l} = 0$. And if $l = m$, $\langle \psi_l, \varphi_l\rangle = 2\pi i \prod_{l\neq m}(k_m - k_l) \neq 0$ as an infinite Blaschke product converges to a non-zero complex number except at zeroes of its factors. ■

We also have the following fact which will be crucial for our proofs later on.

Proposition 3 *The sets $\{\varphi_l\}$ and $\{\psi_l\}$ both form Riesz bases[3] for the subspace K_β.*

Proof: Since we showed in Proposition 2 that $\{\varphi_l\}$ and $\{\psi_l\}$ are biorthogonal sets, it will be enough to show this for the $\{\varphi_l\}$ only.

First we show the $\{\varphi_l\}$ are complete in K_β. So suppose $f \in K_\beta$, and $\langle f, \frac{1}{k-\bar{k}_l}\rangle = 0$, each $l \in \mathbb{Z}$. Then by the result used in the proof of Proposition 2 , $f(k_l) = 0$ for all $l \in \mathbb{Z}$. So $f \in \Pi_\beta H_2^+$. But $f \in K_\beta$, hence $f \equiv 0$.

Finally we show the complete set φ_l is in fact a Riesz basis. Fortunately, this problem was solved for us many years ago by Carleson [21, 15], in the context of interpolation by analytic functions. The well-known Carleson condition states that the family $\{\varphi_l\}$ is a Riesz basis [4] iff

$$\inf_m \prod_{l\neq m} \left|\frac{k_m - k_l}{k_m - \bar{k}_l}\right| > 0.$$

A quick calculation shows that this condition holds for our set $\{\varphi_l\}$. ■

In the next theorem we describe an important automorphism of $L_2(\mathbb{R})$ which maps K_β to K_a.

Theorem 4 *Write $\theta(k) = e^{ika}$, and let $f_\beta(k)$ be the entire function of exponential type defined by*

$$f_\beta(k) = 1 - e^{-\beta} e^{ika}.$$

Then the multiplication operator $u \mapsto f_\beta u$ is a bounded and invertible operator on $L_2(\mathbb{R})$, transforming the orthogonal sum

$$L_2(\mathbb{R}) = H_-^2 \oplus K_\beta \oplus \Pi_\beta H_+^2$$

into the direct sum

$$\overline{\Pi}_\beta \theta H_-^2 + K_a + \Pi_\beta H_+^2,$$

where $K_a = H_+^2 \ominus \theta H_+^2$ is a coinvariant subspace of the shift group corresponding to the singular function θ. The entire functions

$$\Phi_l = f_\beta \varphi_l$$

form a Riesz basis in K_a for each $\beta > 1$.

[3] By a Riesz basis, we mean a basis obtained from an orthonormal basis by an invertible, bounded, linear transformation.

[4] Actually, the Carleson condition guarantees only that we have a so-called *unconditional basis*. A Riesz basis must also satisfy $\inf \|\varphi_l\| > 0$ and $\sup \|\varphi_l\| < \infty$, but these conditions are clearly fulfilled here.

Proof: Recall that $a > 0$ and $\beta > 1$ — we will use this fact repeatedly without mentioning it. Multiplication by f_β is clearly bounded and invertible since $1 - e^{-\beta} \leq |f_\beta| \leq 1 + e^{-\beta}$ on the real axis. We check each component of the claimed transformation separately.

1st component: To see $f_\beta H_2^- \subseteq \overline{\Pi_\beta}\theta H_2^-$, suppose $g \in H_2^-$. Then

$$f_\beta g = (1 - e^{-\beta}e^{ika})g = \overline{\Pi_\beta}\theta(1 - e^{-\beta}e^{-ika})g \in \overline{\Pi_\beta}\theta H_2^-.$$

To see $f_\beta^{-1}\overline{\Pi_\beta}\theta H_2^- \subseteq H_2^-$, suppose again that $g \in H_2^-$. Then

$$f_\beta^{-1}\overline{\Pi_\beta}\theta g = \frac{1}{1 - e^{-ika}e^{-\beta}} g \in H_2^-.$$

2nd component: To see $K_\beta \xrightarrow{f_\beta} K_a$ is a bijection, it will be enough (by Proposition 1) to show that $\{f_\beta\varphi_l\}_{l\in\mathbb{Z}}$ is a complete set for K_a. First we claim that $f_\beta\varphi_l \in K_a$:

$$f_\beta\varphi_l = \frac{1 - e^{-\beta}e^{ika}}{k - \overline{k}_l}$$

which is clearly in H_2^+, so suppose $g \in H_2^+$. Then

$$\begin{aligned} \langle f_\beta\varphi_l, \theta g\rangle &= \left\langle \frac{1 - e^{-\beta}e^{ika}}{k - \overline{k}_l}, e^{ika}g \right\rangle \\ &= \left\langle \frac{e^{-ika} - e^{-\beta}}{k - \overline{k}_l}, g \right\rangle \\ &= 0 \end{aligned}$$

as

$$\frac{e^{-ika} - e^{-\beta}}{k - \overline{k}_l} \in H_2^-.$$

It's entire function , with decreasing modulus as $\Im k \to -\infty$, whereas $g \in H_2^+$.

So we have shown $f_\beta\varphi_l \in K_a$; now we need to show $\{f_\beta\varphi_l\}$ is complete in K_a. Actually, it will be more convenient to show $\{\theta^{-1}f_\beta\varphi_l\}$ is complete in $\theta^{-1}K_a$, which amounts to the same thing since θ is unitary. Note[5] that $\theta^{-1}K_a = H_2^- \ominus e^{-ika}H_2^-$. So suppose $g \in H_2^-$, and $\langle g, \theta^{-1}f_\beta\varphi_l\rangle = 0$, each $l \in \mathbb{Z}$. We need to show $g \in e^{-ika}H_2^-$. But

$$\begin{aligned} g \in e^{-ika}H_2^- \quad &\Leftrightarrow \quad e^{ika}g \in H_2^- \\ &\Leftrightarrow \quad \left\langle e^{ika}g, H_2^+ \right\rangle = 0 \\ &\Leftrightarrow \quad \left\langle e^{ika}g, K_\beta \right\rangle = 0 \text{ and } \left\langle e^{ika}g, \Pi_\beta H_2^+ \right\rangle = 0. \end{aligned}$$

We can check each of these last conditions separately. For the first, recall we assumed $\langle g, \theta^{-1}f_\beta\varphi_l\rangle = 0$, i.e. $\left\langle g, \frac{e^{-ika} - e^{-\beta}}{k - \overline{k}_l} \right\rangle = 0$. But $\left\langle g, \frac{e^{-\beta}}{k - \overline{k}_l} \right\rangle = 0$ since $g \in H_2^-$ and $\frac{1}{k - \overline{k}_l} \in H_2^+$, so we have $\left\langle g, \frac{e^{-ika}}{k - \overline{k}_l} \right\rangle = 0$ and hence $\langle g, e^{-ika}K_\beta\rangle = 0$ as $\left\{\frac{1}{k - \overline{k}_l}\right\}_{l\in\mathbb{Z}}$ are complete in K_β.

[5] To see this, observe that $K_a = L_2 \ominus e^{ika}H_2^+ \ominus H_2^-$, so $e^{-ika}K_a = L_2 - \ominus H_2^+ \ominus e^{-ika}H_2^- = H_2^- \ominus e^{-ika}H_2^-$.

We prove the second condition by contradiction. Suppose $\langle e^{ika}g, \Pi_\beta f\rangle \neq 0$ for some $f \in H_2^+$. Then $e^{ika}g = q + \Pi_\beta p$ for some $q \in H_2^-$ and some *non-zero* $p \in H_2^+$. It is easy to see that $q = e^{ika}r$ for some $r \in H_2^-$, so rearranging gives $\Pi_\beta p = e^{ika}s$, for some $s \in H_2^-$. Thus $s = e^{-ika}\Pi_\beta p = \frac{1-e^{-ika}e^{-\beta}}{1-e^{ika}e^{-\beta}}p$, or

$$\frac{s}{1-e^{-ika}e^{-\beta}} = \frac{p}{1-e^{ika}e^{-\beta}}.$$

But the left side here is in H_2^-, whereas the right side is in H_2^+. Thus $p \equiv s \equiv 0$, contradicting the fact that p was not identically zero.

3rd component: To see $f_\beta(\Pi_\beta H_2^+) \subseteq \Pi_\beta H_2^+$, suppose $g \in H_2^+$. Then $f_\beta \Pi_\beta g = (e^{ika} - e^{-\beta})g \in H_2^+$. And for the reverse inclusion, $f_\beta^{-1}\Pi_\beta g = \Pi_\beta \frac{g}{1-e^{-\beta}e^{ika}} \in \Pi_\beta H_2^+$.

Finally we remark that the $\{\Phi_l\}$ are certainly a Riesz basis of K_a as they were obtained from the Riesz basis $\{\varphi_l\}$ of K_β by a bounded invertible linear transformation. ■

It is interesting to note here that in fact the functions Φ_l are Fourier images of the projections of exponentials $e^{-i\bar{k}_l x}$ in $L_2(\mathbb{R})$ onto $L_2(0,a)$.

As explained in the introduction, it will turn out that the operators $W_\beta \equiv P_\beta \sigma P_\beta$ do not approximate the operator $W_a \equiv P_a \sigma P_a$ well enough for our purposes. Therefore in the next theorem we introduce the operators $\mathcal{W}_\beta$ which have the same determinants as the W_β. Then in Theorem 7 we show that $\mathcal{W}_\beta$ is a good enough approximation to W_a.

Theorem 5 *The operator*

$$\mathcal{P}_a^\beta = f_\beta P_\beta f_\beta^{-1}$$

is a skew (i.e. nonorthogonal) projection onto K_a parallel to the sum of subspaces

$$\overline{\Pi}_\beta \theta H_-^2 + \Pi_\beta H_+^2.$$

For each essentially bounded function σ defined on the real axis, the operator $W_\beta \equiv P_\beta \sigma P_\beta$ is bounded and is similar to the operator $\mathcal{W}_\beta \equiv \mathcal{P}_a^\beta \sigma \mathcal{P}_a^\beta$ acting in the subspace K_a; in particular, $\mathcal{W}_\beta$ and W_β have the same determinants.

Proof: The effect of f_β described in Theorem 4 means precisely that $\mathcal{P}_a^\beta$ is zero on $\Pi_\beta \theta H_2^- + \Pi_\beta H_2^+$, and the identity on K_a; this is the definition of a skew projection so the first statement is proved.

Now it is not generally true for infinite-dimensional determinants that

$$\det(PQP^{-1}) = \det Q,$$

but this formula does hold if P maps some Riesz basis of dom Q to a Riesz basis of dom PQP^{-1}. But we proved in Theorem 4 that f_β maps the Riesz basis $\{\varphi_l\}$ of K_β to the Riesz basis $\{\Phi_l\}$ of K_a. By the definition of $\mathcal{W}_\beta$ we have

$$\mathcal{W}_\beta|_{K_a} = f_\beta P_\beta \sigma P_\beta f_\beta^{-1}|_{K_a} = f_\beta W_a(\sigma) f_\beta^{-1}|K_a,$$

so the above remarks tell us that $\det \mathcal{W}_\beta = \det W_\beta$. ■

Our next task is to estimate the norms of the various operators defined so far. Define

$$\epsilon_\beta = \sup_{k\in\mathbb{R}} (\Pi_\beta(k) - \theta(k)) = \sup_{k\in\mathbb{R}} \left(1 - \overline{\Pi}_\beta(k)\theta(k)\right).$$

(It is easy to see these suprema are equal, and in fact that $\epsilon_\beta \le \frac{2e^{-\beta}}{1-e^{-\beta}}$. In particular, we see that $\epsilon_\beta \to 0$ as $\beta \to \infty$.)

We will need the following technical lemma.

Lemma 6 *(i) The operator norm of $\mathcal{P}_a^\beta$ is estimated by*

$$\|\mathcal{P}_a^\beta\|_{op} \le \frac{1}{1-\sqrt{2}\epsilon_\beta}.$$

(ii) The following uniform estimate holds.

$$\|\mathcal{P}_a^\beta - P_a\|_{op} \le \frac{\sqrt{2}\epsilon_\beta}{1-\sqrt{2}\epsilon_\beta}.$$

(iii) If $\rho \in L_\infty \cap L_2$, then $P_a \rho P_a$ is a Hilbert-Schmidt operator with Hilbert-Schmidt norm not greater than $\|\rho\|_2$.

(iv) If $\rho \in L_\infty \cap L_2$, then $\mathcal{P}_a^\beta \rho \mathcal{P}_a^\beta$ is a Hilbert-Schmidt operator with Hilbert-Schmidt norm not greater than

$$\left(\frac{1}{1-\sqrt{2}\epsilon_\beta}\right)^2 \|\rho\|_2.$$

Proof:

(i) Given $f \in L_2$ with $\|f\| = 1$, write $f = \mathcal{P}_a^\beta f + \overline{\Pi}_\beta \theta f_- + \Pi_\beta f_+$, where $f_\pm \in H_2^\pm$. Let $g = \mathcal{P}_a^\beta f + f_- + \theta f_+$. ($g$ is the result of "straightening" our decomposition of f to make it an orthogonal decomposition.) Then

$$\begin{aligned} \|f-g\| &= \|\left(\overline{\Pi}_\beta - 1\right) f_- + (\Pi_\beta - \theta) f_+\| \\ &\le \epsilon_\beta \left(\|f_-\| + \|f_+\|\right) \\ &\le \sqrt{2}\epsilon_\beta \left(\|f_-\|^2 + \|f_+\|^2\right)^{1/2}. \end{aligned}$$

So

$$\|g\| \le 1 + \sqrt{2}\epsilon_\beta \left(\|f_-\|^2 + \|f_+\|^2\right)^{1/2} \quad \text{(recall } \|f\| = 1\text{)}.$$

Also, $\|g\|^2 = \|\mathcal{P}_a^\beta f\|^2 + \|f_-\|^2 + \|f_+\|^2$, so $\left(\|f_-\|^2 + \|f_+\|^2\right)^{1/2} \le \|g\|$. Thus $\|g\| \le 1 + \sqrt{2}\,\epsilon_\beta \|g\|$, i.e. $\|g\| \le \frac{1}{1-\sqrt{2}\epsilon_\beta}$. Hence

$$\|\mathcal{P}_a^\beta f\| \le \|g\| \le \frac{1}{1-\sqrt{2}\epsilon_\beta},$$

as claimed.

(ii) Given $f \in L_2$ with $\|f\| = 1$, write $f = P_a f + f_- + \theta f_+$, where $f_\pm \in H_2^\pm$. Set $g = P_a f + \overline{\Pi}_\beta \theta f_- + \Pi_\beta f_+$. (We have "bent" the orthogonal components of f so that $\mathcal{P}_a^\beta g = P_a f$.) Then

$$\begin{aligned} \|f-g\| &= \left\|\left(\overline{\Pi}_\beta \theta - 1\right) f_- + (e^{ika} - \Pi_\beta) f_+\right\| \\ &\le \epsilon_\beta (\|f_-\| + \|f_+\|) \\ &\le \sqrt{2}\epsilon_\beta \left(\|f_-\|^2 + \|f_+\|^2\right)^{1/2} \end{aligned}$$

Now we can obtain the desired estimate:

$$\begin{aligned}
\|(\mathcal{P}_a^\beta - P_a)f\| &= \left\|\mathcal{P}_a^\beta(f-g) + \mathcal{P}_a^\beta g - P_a f\right\| \\
&= \|\mathcal{P}_a^\beta(f-g)\| \qquad (\text{since } \mathcal{P}_a^\beta g = P_a f) \\
&\le \|\mathcal{P}_a^\beta\|_{\mathrm{op}}\|f-g\| \\
&\le \frac{1}{1-\sqrt{2}\epsilon_\beta}\sqrt{2}\,\epsilon_\beta\left(\|f_-\|^2 + \|f_+\|^2\right)^{1/2} \\
&\le \frac{\sqrt{2}\epsilon_\beta}{1-\sqrt{2}\epsilon_\beta}\|f\|.
\end{aligned}$$

(iii) Taking the Fourier transform of $P_a \rho P_a$, we have an integral operator whose kernel K is non-zero only on $[0,a]\times[0,a]$. Thus

$$\|k\|_{L_2\times L_2} = \int_0^a\int_0^a |k(x-y)|^2 dx\, dy \le a\int_{-\infty}^{\infty}|k|^2 dx < \infty,$$

as $\rho \in L_2$ so $k \in L_2$ too. A standard theorem on integral equations now tells us the operator is Hilbert-Schmidt[6].

(iv) Use a similar argument to (iii), together with (i). ■

The next theorem states that the intermediate operator $\mathcal{W}_\beta$ is actually close to W in trace norm.

Theorem 7 *Let σ be a bounded analytic function in the upper half plane, set $\rho = \sigma - 1$ and suppose ρ can be expressed as the product of 3 functions, each in the intersection of the Hardy classes $H_\infty^+ \cap H_2^+$:*

$$\rho = \rho_1\rho_2\rho_3, \quad \rho_j \in H_\infty^+ \cap H_2^+ \tag{1}$$

Then

$$\left\|P_a\rho P_a - \mathcal{P}_a^\beta \rho \mathcal{P}_a^\beta\right\|_{\mathit{Trace}} \le \frac{\sqrt{2}\epsilon_\beta}{\left(1-\sqrt{2}\epsilon_\beta\right)^3}\ \mathit{const},$$

where the constant depends only on the L_2 and L_∞ norms of the factors ρ_j. In particular, $\mathcal{P}_a^\beta\rho\mathcal{P}_a^\beta \to P_a\rho P_a$ in trace norm as $\beta\to\infty$.

Proof: We will need three basic facts about operators.

Fact A. An operator is of trace class if and only if it can be expressed as the product of two Hilbert-Schmidt operators.

Fact B. The set of Hilbert-Schmidt operators is an ideal in the algebra of bounded operators. So if A is Hilbert-Schmidt, and B is bounded, then AB and BA are both Hilbert-Schmidt. Moreover, $\|BA\|_{\mathrm{HS}} \le \|B\|_{\mathrm{op}}\|A\|_{\mathrm{HS}}$ and similarly for $\|AB\|_{\mathrm{HS}}$.

Fact C. Fact B holds if we replace Hilbert-Schmidt by trace class throughout.

To get the proof started, we also need the following result, which allows us to insert P_a and $\mathcal{P}_a^\beta$ in certain places without affecting anything. Let ρ_1, ρ_2 be functions in H_2^+. Then

$$P_a\rho_1\rho_2P_a = P_a\rho_1P_a\rho_2P_a, \qquad \text{and} \qquad \mathcal{P}_a^\beta\rho_1\rho_2\mathcal{P}_a^\beta = \mathcal{P}_a^\beta\rho_1\mathcal{P}_a^\beta\rho_2\mathcal{P}_a^\beta.$$

[6]See, for example, Proposition 4.8 of [20] — actually the statement there is that the operator is compact, but exactly the same proof shows the operator is Hilbert-Schmidt.

To see this, let $f \in L_2(\mathbb{R})$. Write $\rho_2 P_a f = P_a(\rho_2 P_a f) + \theta f_+$, where $f_+ \in H_2^+$. Then $P_a\rho_1\rho_2 P_a f = P_a\rho_1 P_a(\rho_2 f) = P_a\rho_1 P_a\rho_2 f$. The second statement follows from a similar calculation.

Now we can proceed with the proof of the theorem. Applying the above result immediately, and then adding and subtracting some terms, we get

$$\begin{aligned}
&P_a\rho P_a - \mathcal{P}_a^\beta\rho\mathcal{P}_a^\beta \\
&\quad = P_a\rho_1 P_a\rho_2 P_a\rho_3 P_a - \mathcal{P}_a^\beta\rho_1\mathcal{P}_a^\beta\rho_2\mathcal{P}_a^\beta\rho_3\mathcal{P}_a^\beta \\
&\quad = \left(P_a - \mathcal{P}_a^\beta\right)\rho_1 P_a\rho_2 P_a\rho_3 P_a \\
&\qquad + \mathcal{P}_a^\beta\rho_1\left(P_a - \mathcal{P}_a^\beta\right)\rho_2 P_a\rho_3 P_a \\
&\qquad + \mathcal{P}_a^\beta\rho_1\mathcal{P}_a^\beta\rho_2\left(P_a - \mathcal{P}_a^\beta\right)\rho_3 P_a \\
&\qquad + \mathcal{P}_a^\beta\rho_1\mathcal{P}_a^\beta\rho_2\mathcal{P}_a^\beta\rho_3\left(P_a - \mathcal{P}_a^\beta\right)
\end{aligned}$$

We can estimate the trace norm of each term separately; we show only the second term here as an example.

$$\begin{aligned}
&\left\|\mathcal{P}_a^\beta\rho_1\left(P_a - \mathcal{P}_a^\beta\right)\rho_2 P_a\rho_3 P_a\right\|_{\mathrm{Trace}} \\
&\quad = \| \underbrace{\left(\mathcal{P}_a^\beta\rho_1\mathcal{P}_a^\beta\right)}_{\text{Hilbert-Schmidt}} \underbrace{\left(P_a - \mathcal{P}_a^\beta\right)\rho_2}_{\text{bounded}} \underbrace{(P_a\rho_3 P_a)}_{\text{Hilbert-Schmidt}} \|_{\mathrm{Trace}} \\
&\quad \le \left\|\mathcal{P}_a^\beta\rho_1\mathcal{P}_a^\beta\right\|_{\mathrm{HS}} \left\|\left(P_a - \mathcal{P}_a^\beta\right)\rho_2\right\|_{\mathrm{op}} \|P_a\rho_3 P_a\|_{\mathrm{HS}} \\
&\quad \le \left(\frac{1}{1-\sqrt{2}\epsilon_\beta}\right)^2 \|\rho_1\|_2 \left(\frac{\sqrt{2}\epsilon_\beta}{1-\sqrt{2}\epsilon_\beta}\right) \|\rho_2\|_\infty \|\rho_3\|_2
\end{aligned}$$

The first line here used Lemma 6 (iii) and (iv), together with a slight variant of the result above for inserting $\mathcal{P}_a^\beta$ and P_a. The second line uses Facts A, B and C, and the third line follows from Lemma 6 (ii)-(iv).

After estimating the other 3 terms in the same way and summing the results, we get precisely the formula in the theorem. ∎

At last we are in a position to prove our main theorem, obtaining a formula for $\det W$.

Theorem 8 *Suppose σ is an analytic function in the upper half plane, and $\sigma - 1$ satisfies the 3-factor condition (1) of the previous theorem. Then*

$$\det W_a(\sigma) = \lim_{\beta\to\infty} \prod_{l\in\mathbb{Z}} \sigma(k_l), \tag{2}$$

where $k_l = 2\pi l/a + i\beta/a$.

Proof: We apply our previous results to carry out the plan outlined in the introduction. Note that in the first line we will use the fact that det is continuous with respect to the trace norm. (This is proved in [11], for example). We have

$$\begin{aligned}
\det W_a(\sigma) &= \lim\nolimits_{\beta\to\infty} \det \mathcal{W}_\beta, && \text{by Theorem 7} \\
&= \lim\nolimits_{\beta\to\infty} \det W_\beta, && \text{by Theorem 5} \\
&= \lim\nolimits_{\beta\to\infty} \textstyle\prod(\text{evals of } W_\beta), && \text{by definition of det} \\
&= \lim\nolimits_{\beta\to\infty} \textstyle\prod_{l\in\mathbb{Z}} \sigma(k_l), && \text{by Proposition 1}
\end{aligned}$$

■

4 A generalisation

Using perturbation theory, we can extend the approach described above to the case where the symbol is of the form

$$\sigma(k)\pi^{-1}(k),$$

where σ is a bounded analytic function in the upper half plane and π is a finite Blaschke product whose zeroes are in the upper half plane. (Note this means π^{-1} has *poles* in the upper half plane.) The corresponding Wiener-Hopf operator on a finite interval $(0,a)$ is Fourier equivalent to the operator $P_a\sigma\pi^{-1}|_{K_a}$, and approximated in trace class by the operator $f_\beta P_\beta \sigma\pi^{-1}P_\beta f_\beta^{-1}$, which can't be diagonalized using the semigroup approach as before. Nevertheless its determinant can be calculated by the following method.

Denoting $H_+^2 \ominus \pi H_+^2$ by K_π and the corresponding orthogonal projection by P_π, we claim that we can rewrite the operator $W_\beta = P_\beta\sigma\pi^{-1}|_{K_\beta}$ as

$$W_\beta = (P_\beta\sigma P_\beta)\left(P_\beta\pi^{-1}P_\beta\right)|_{K_\beta} + P_\beta\sigma\pi^{-1}P_\pi|_{K_\beta}.$$

To see this, first make sure that the following decomposition of the identity operator on $L_2(\mathbb{R})$ is true:

$$I = P_{\pi^{-1}H_2^-} + \pi^{-1}P_\pi\pi + P_\beta + P_{\Pi_\beta H_2^+}$$

Then we have

$$\begin{aligned}
W_\beta &= P_\beta\sigma I\pi^{-1}|_{K_\beta} \\
&= P_\beta\sigma\{\underbrace{P_{\pi^{-1}H_2^-}}_{\substack{\text{this is zero}\\ \text{on } \pi^{-1}K_\beta}} + \pi^{-1}P_\pi\pi + P_\beta + \underbrace{P_{\Pi_\beta H_2^+}}_{\substack{\text{anything in}\\ \Pi_\beta H_2^+ \text{ will}\\ \text{be killed by}\\ P_\beta}}\}\pi^{-1}|_{K_\beta} \\
&= P_\beta\sigma\{\pi^{-1}P_\pi\pi + P_\beta\}\pi^{-1}|_{K_\beta} \\
&= (P_\beta\sigma P_\beta)\left(P_\beta\pi^{-1}P_\beta\right)|_{K_\beta} + P_\beta\sigma\pi^{-1}P_\pi|_{K_\beta},
\end{aligned}$$

as claimed.

Now $\det(P_\beta\sigma P_\beta)$ can be calculated exactly as before, and $\det(P_\beta\ \pi^{-1}P_\beta)$ can be calculated using a parallel theory which exploits the fact that π^{-1} is analytic in the *lower* half plane. So the determinant of the first addend in this expression for W_β can be calculated; we claim the second addend contributes only a finite-dimensional perturbation. To see this, re-prove Proposition 1 with π in place of Π_β. The numbers k_l end up being the zeroes of π, and there are only finitely many of them! Hence K_π is finite-dimensional. Therefore, the standard theory of finite-dimensional perturbations can be used to obtain expressions for $\det W$.

5 Wiener-Hopf Operators with rational symbols and functional model

.

The complete spectral theory of Wiener-Hopf operators with rational symbols was developed in [12]. We suggest here an alternative approach, which can be easily generalized for matrix symbols and for operators acting in Hardy classes on multiconnected domains. We do not need any conditions concerning the selfadjointness of the operators considered. Nevertheless, just to fascilate the formulation of our resuts and the comparisson of them with classical ones we consider only Wiener-Hopf operators whose symbol σ are *real* rational functions

$$W \equiv W_a(\sigma) = P_a \sigma P_a,$$

where $P_a \equiv P_{K_a}, K_a = H_2^+ \ominus e^{ika} H_2^+$. The asymptotics of the corresponding determinants was calculated first in [1, 6, 2], and later for extended class of symbols by [10, 11, 8]. Some new fascinating features of asymptotics of determinants were found in [13].

The first part of our paper was aimed on the revealing connections between the Nagy-Foias Functional Model and the spectral properties of Wiener-Hopf Operators with analytic symbols. In this part we develop similar programm for rational symbols. In particular we show, that the spectral analysis of Wiener-Hopf Operators with the real rational symbol with $2N$ poles can be reduced to the spectral problem for $2N \times 2N$ matrix.

Considering the real rational symbol σ we assume, that it is represented in form of a finite sum of reproducing kernels (Cauchy kernels) with poles at prescribed complex points and the derivatives of reproducing kernels :

$$\sigma(k) = 1 + \sum_{l=1,m=0}^{L,M_l} \frac{\alpha_l^m}{(k-k_l)^m} + \frac{\bar{\alpha}_l^m}{(k-\bar{k}_l)^m} \equiv 1 + \varphi(k), \;\; \Im k_l > 0.$$

It is easy to see, that the corresponding Wiener-Hopf operator $P_a \varphi|_{K_a}$ is compact. Let us denote by $k_n^\lambda, n = -2\sum M_l, .. -2, -1, 1, 2, ..., 2\sum M_l$ the roots of the auxilliary equation $\varphi(k) = \lambda$ (counting multiplicity).

Theorem 9 *The eigenvalues of the Wiener-Hopf operator in the coinvariant subspace $K_a = H^2 \ominus \theta_a H^2$, $\theta_a(k) = e^{ika}$ coincide with the zeroes of the determinant of a finite square matrix :*

$$\det \left\{ \begin{array}{cccc} \cdots & \cdots & \cdots & \cdots \\ \cdots & \frac{\alpha_l^m}{k_n^{(\lambda} - k_l)^m} & \cdots \frac{\bar{\alpha}_l^m}{(k_n^\lambda - \bar{k}_l)^m} e^{ik_n^\lambda a} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array} \right\}$$

Proof is based on the following statement , which is well known for $m = 0$ [7], and obviously true for $m \geq 1$

Proposition The following representations are true for the resolvents of the generators of compressions shifts and adjoint shifts semigroups onto the coinvariant subspace $K_a = H^2 \ominus e^{ika} H^2$:

$$P_a \frac{1}{k-p} u = \frac{u - u(p)}{k-p}, \;\; \Im p > 0, \; u \in K_a$$

$$P_a \frac{1}{k-\bar{p}} u = \frac{u - \theta[\bar{\theta} u(\bar{p})]}{k-\bar{p}}, \;\; \Im \bar{p} < 0, \; u \in K_a.$$

The similar statement is true for the derivatives order $m,\ m>0$ of the resolvent in respect to the spectral parameter :

$$P_a\frac{1}{(k-p)^m}u=\frac{1}{(m-1)!}\frac{d^m}{dp^m}\frac{u-u(p)}{k-p},\ \Im p>0,\ u\in K_a$$
$$P_a\frac{1}{(k-\bar p)^m}u=\frac{1}{(m-1)!}\frac{d^m}{d\bar p^m}\frac{u-\theta[\bar\theta u(\bar p)]}{k-\bar p},\ \Im\bar p<0,\ u\in K_a.$$

The first two formulas give the analytic continuation of the resolvent of the shifts generators on the " nonphysical sheet", which is just one of halfplanes $\Lambda_\pm$. In a way these formulas realise the connection between the elementary Wiener-Hopf Operators and the functional model. One can see, that each eigenfunction u of the Wiener-Hopf operator $P_a\varphi|_{K_a}$ fulfills the corresponding homogeneous equation:

$$[\varphi(k)-\lambda]u(k)=$$
$$\sum_{l=1,m=0}^{N,M_l}\frac{1}{(m-1)!}\alpha_l^m\frac{d^m}{dp^m}\frac{u(p)}{k-p}\bigg|_{p=k_l^m}+\frac{1}{(m-1)!}\theta(k)\bar\alpha_l^m\frac{d^m}{d\bar p^m}\frac{[\bar\theta u(\bar p)]}{k-\bar p}\bigg|_{p=k_l^m}$$

which implies

$$\sum_{l=1,m=0}^{N,M_l}\frac{1}{(m-1)!}\alpha_l^m\frac{d^m}{dp^m}\frac{u(p)}{k-p}\bigg|_{p=k_l^m}+$$
$$\frac{1}{(m-1)!}\theta(k)\bar\alpha_l^m\frac{d^m}{d\bar p^m}\frac{[\bar\theta u(\bar p)]}{k-\bar p}\bigg|_{p=k_l^m}=0$$

for any $k=k_n^\lambda$. Hence the zeroes of the right and left sides coincide counting multiplicity. Considering $\frac{d^r}{dp^r}u(p)\Big|_{p=k_l}\equiv u_l^r$ and $\frac{d^r}{d\bar p^r}\bar\theta u(p)\Big|_{p=k_l}\equiv u_r^l$ as a new unknowns, we get the condition of existence of the solution in the determinant form:

$$\det\begin{Bmatrix}\cdots & \cdots & \cdots & \cdots\\ \cdots & \frac{m}{r!(m-r)!}\alpha_l^m\frac{d^r}{dp^r}\frac{1}{k^{\lambda_n}-p}\Big|_{p=k^l} & \cdot\cdot\frac{m}{r!(m-r)!}\theta(k^{\lambda_n})\bar\alpha_l^m\frac{d^r}{d\bar p^r}\frac{[\bar\theta u(\bar p)]}{k^{\lambda_n}-\bar p}\Big|_{p=k_l^m} & \\ \cdots & \cdots & \cdots & \cdots\end{Bmatrix}=0.$$

On the other hand one can check that the functions constructed on the base of new unknown found as solutions of the corresponding homogeneous linear system coincide with the eigenfunctions of the original Wiener-hopf operator with λ equal to the corresponding eigenvalue. ■

The asymptotic behaviour of the determinant for small λ^δ is described by quasypolinomial in $\lambda^\delta, \exp^{\frac{d}{\lambda^\delta}}$. The asimptotics of zeroes of these quasypolinomials for small λ can be derived with using some construction involving Newton poligones ([14, 19]).

6 An example

Consider the symbol

$$\sigma(k)\equiv\frac{4k(k^2-1)}{(k^2+1)^2}\equiv\sin 4\,\mathrm{arctan}k.$$

Denoting the main branch of the function arcsin $\frac{\lambda}{4}$ by $\varphi(\lambda)$, we can parametrize the roots $k_0, k_\infty, k_1, k_{-1}$ of the equation

$$\sigma(k) = \frac{4k(k^2-1)}{(k^2+1)^2} = \lambda.$$

approaching $0, \infty, 1, -1$ respectively when $\lambda \to 0$ the following way

$$\begin{aligned} k_0(\lambda) &= \tan\frac{\varphi(\lambda)}{4}, \\ k_\infty(\lambda) &= -\frac{1}{\tan\frac{\varphi(\lambda)}{4}}, \\ k_1(\lambda) &= \frac{1-\tan\frac{\varphi(\lambda)}{4}}{1+\tan\frac{\varphi(\lambda)}{4}}, \\ k_{-1}(\lambda) &= \frac{-1}{k_1(\lambda)} \end{aligned}$$

Then the eigenvalues of the Wiener-Hopf operator coincide with the zeroes of the determinant of the following matrix

$$\begin{pmatrix} \frac{1}{(k_0-i)^2} & \frac{1}{(k_0-i)} & \frac{1}{(k_0+i)^2}e^{iak_0} & \frac{1}{(k_0+i)}e^{iak_0} \\ \frac{k_0^2}{(1+ik_0)^2} & \frac{-k_0}{(1+ik_0)} & \frac{k_0^2}{(1-ik_0)^2}e^{-i\frac{a}{k_0}} & \frac{k_0}{(-1+ik_0)}e^{-i\frac{a}{k_0}} \\ \frac{(1+k_0)^2}{[(1-i)-k_0(1+i)]^2} & \frac{(1+k_0)}{[(1-i)-k_0(1+i)]} & \frac{(1+k_0)^2}{[(1+i)-k_0(1-i)]^2}e^{ia\frac{1-k_0}{1+k_0}} & \frac{(1+k_0)}{[(1+i)-k_0(1-i)]}e^{ia\frac{1-k_0}{1+k_0}} \\ \frac{(1-k_0)^2}{[(1+i)+k_0(1-i)]^2} & \frac{(-1+k_0)}{[(1+i)+k_0(1-i)]} & \frac{(1-k_0)^2}{[(1-i)+k_0(1+i)]^2}e^{-ia\frac{1+k_0}{1-k_0}} & \frac{(-1+k_0)}{[(1-i)+k_0(1+i)]}e^{-ia\frac{1+k_0}{1-k_0}} \end{pmatrix}$$

By direct calculation with “Mathematica” we find the determinant is equal to:

$$
\begin{gathered}
\frac{(4+4\,i)\,e^{\frac{4\,i\,a\,k}{-1+k^2}}\,(-1+k)^2\,k\,(1+k)}{(-i+k)^6}+\frac{(4-4\,i)\,e^{\frac{4\,i\,a\,k}{-1+k^2}}\,(-1+k)^2\,k^2\,(1+k)}{(-i+k)^6}+\\
\frac{8\,e^{\frac{i\,a\,\left(1+k^2\right)}{-1+k}}\,(-1+k)\,k^2\,(1+k)}{(1-i\,k)^3\,(-i+k)^3}+\frac{8\,e^{\frac{i\,a\,\left(1+k^2\right)}{1+k}}\,(-1+k)\,k^2\,(1+k)}{(1-i\,k)^3\,(-i+k)^3}+\\
\frac{(4-4\,i)\,e^{\frac{i\,a\,\left(-1+k^2\right)}{k}}\,(-1+k)^2\,k\,(1+k)}{(i+k)^6}+\frac{(4+4\,i)\,e^{\frac{i\,a\,\left(-1+k^2\right)}{k}}\,(-1+k)^2\,k^2\,(1+k)}{(i+k)^6}+\\
\frac{(4+4\,i)\,e^{\frac{i\,a\,\left(1+k^2\right)}{1+k}}\,(-1+k)^2\,k\,(1+k)}{(-i+k)^2\,(i+k)^4}+\frac{(4-4\,i)\,e^{\frac{-i\,a\,\left(1+k^2\right)}{k+k^2}}\,(-1+k)^2\,k^2\,(1+k)}{(-i+k)^2\,(i+k)^4}+\\
\frac{8\,e^{\frac{i\,a\,\left(1+k^2\right)}{-k+k^2}}\,(-1+k)\,k^2\,(1+k)}{(1+i\,k)^3\,(i+k)^3}+\frac{8\,e^{\frac{-i\,a\,\left(1+k^2\right)}{k+k^2}}\,(-1+k)\,k^2\,(1+k)}{(1+i\,k)^3\,(i+k)^3}+\\
\frac{(4-4\,i)\,e^{\frac{i\,a\,\left(1+k^2\right)}{-k+k^2}}\,(-1+k)^2\,k\,(1+k)}{(-i+k)^4\,(i+k)^2}+\frac{(4+4\,i)\,e^{\frac{i\,a\,\left(1+k^2\right)}{-1+k}}\,(-1+k)^2\,k^2\,(1+k)}{(-i+k)^4\,(i+k)^2}+\\
\frac{(-4+4\,i)\,e^{\frac{4\,i\,a\,k}{-1+k^2}}\,(-1+k)\,k\,(1+k)^2}{(-i+k)^6}+\frac{(4+4\,i)\,e^{\frac{4\,i\,a\,k}{-1+k^2}}\,(-1+k)\,k^2\,(1+k)^2}{(-i+k)^6}+\\
\frac{(4+4\,i)\,e^{\frac{i\,a\,\left(-1+k^2\right)}{k}}\,(1-k)\,k\,(1+k)^2}{(i+k)^6}+\frac{(4-4\,i)\,e^{\frac{i\,a\,\left(-1+k^2\right)}{k}}\,(-1+k)\,k^2\,(1+k)^2}{(i+k)^6}+\\
\frac{(-4+4\,i)\,e^{\frac{i\,a\,\left(1+k^2\right)}{-1+k}}\,(-1+k)\,k\,(1+k)^2}{(-i+k)^2\,(i+k)^4}+\frac{(4+4\,i)\,e^{\frac{i\,a\,\left(1+k^2\right)}{-k+k^2}}\,(-1+k)\,k^2\,(1+k)^2}{(-i+k)^2\,(i+k)^4}+\\
\frac{(4+4\,i)\,e^{\frac{-i\,a\,\left(1+k^2\right)}{k+k^2}}\,(1-k)\,k\,(1+k)^2}{(-i+k)^4\,(i+k)^2}+\frac{(4-4\,i)\,e^{\frac{i\,a\,\left(1+k^2\right)}{1+k}}\,(-1+k)\,k^2\,(1+k)^2}{(-i+k)^4\,(i+k)^2}-\\
\frac{4\,i\,e^{\frac{i\,a\,\left(1+k^2\right)}{-1+k}}\,(-1+k)^2\,k\,(1+k)^2}{(1+k^2)^3}+\frac{4\,i\,e^{\frac{i\,a\,\left(1+k^2\right)}{1+k}}\,(-1+k)^2\,k\,(1+k)^2}{(1+k^2)^3}-\\
\frac{4\,i\,e^{\frac{i\,a\,\left(1+k^2\right)}{-k+k^2}}\,(-1+k)^2\,k\,(1+k)^2}{(1+k^2)^3}+\frac{4\,i\,e^{\frac{-i\,a\,\left(1+k^2\right)}{k+k^2}}\,(-1+k)^2\,k\,(1+k)^2}{(1+k^2)^3}
\end{gathered}
$$

Letting $e(b)$ stand for e^{iab} and rearranging the terms this becomes

$$
\begin{gathered}
4\,\Big(-2\,e(-1/k)\,e(k)-e(-1/k)\,e(\tfrac{1-k}{1+k})-e(k)\,e(\tfrac{1-k}{1+k})-e(-1/k)\,e(\tfrac{1+k}{-1+k})\\
-e(k)\,e(\tfrac{1+k}{-1+k})-2\,e(\tfrac{1-k}{1+k})\,e(\tfrac{1+k}{-1+k})\Big)\;k\\
+4\,\Big(-16\,i\,e(-1/k)\,e(k)-4\,e(-1/k)\,e(\tfrac{1-k}{1+k})+4\,e(k)\,e(\tfrac{1-k}{1+k})+4\,e(-1/k)\,e(\tfrac{1+k}{-1+k})\\
-4\,e(k)\,e(\tfrac{1+k}{-1+k})+16\,i\,e(\tfrac{1-k}{1+k})\,e(\tfrac{1+k}{-1+k})\Big)\;k^2\\
+4\,\Big(58\,e(-1/k)\,e(k)-3\,e(-1/k)\,e(\tfrac{1-k}{1+k})-3\,e(k)\,e(\tfrac{1-k}{1+k})-3\,e(-\tfrac{1}{k})\,e(\tfrac{1+k}{-1+k})\\
-3\,e(k)\,e(\tfrac{1+k}{-1+k})+58\,e(\tfrac{1-k}{1+k})\,e(\tfrac{1+k}{-1+k})\Big)\;k^3\\
+4\,\Big(128\,i\,e(-\tfrac{1}{k})\,e(k)-128\,i\,e(\tfrac{1-k}{1+k})\,e(\tfrac{1+k}{-1+k})\Big)\;k^4\\
+4\,\Big(-196\,e(-\tfrac{1}{k})\,e(k)-2\,e(-1/k)\,e(\tfrac{1-k}{1+k})-2\,e(k)\,e(\tfrac{1-k}{1+k})\\
-2\,e(-1/k)\,e(\tfrac{1+k}{-1+k})-2\,e(k)\,e(\tfrac{1+k}{-1+k})-196\,e(\tfrac{1-k}{1+k})\,e(\tfrac{1+k}{-1+k})\Big)\;k^5\\
+\mathrm{O}(k^6)
\end{gathered}
$$

This function is approximated by quasypolinomial in $e^(\frac{1}{k}), k; k \to 0$. The zeroes of it have a regular asymptotics defined by the corresponding Newton poligon(see [14, 15]).

7 Conclusion. Wiener-Hopf Operator on a multiconnected domain

The approach described here for Wiener-Hopf Operator with rational sumbols on the line works for Wiener-Hopf Operators on any coinvariant subspace of the shift group in the unit disc , and even for coinvariant subspaces of the shift group on hyperelliptic Riemann Surface of finite genus.

Consider the invariant subspace of the multiplication by the analytic function on a multiconnected domain Ω_+ (the first sheet of the corresponding hyperelliptic Riemann Surface). According to results of [16] the invariant subspaces are parametrized by character-automorphic inner functions S_μ- the generating function for the invariant subspace:

$$D_+ = S_\mu H^2_{-\mu}.$$

The version of Harmonic Analysis on the multiconnected domain was developed in [17]. In particular the explicite formulas for the resolvents of shift's generators are found in [17]. Hence the representations for the Wiener - Hopf operators with simbols rational in respect to the shift genetators can be deduced in this case as well with using of results [18] describing the analytic continuation of the resolvent of the multiplication operator $z*$ onto “nonphysical sheet” .This fact permitts to write down the compression of the multiplication operator of any rational function as a linear combination of resolvents of shifts and adjoints operators restricted onto the corresponding coinvariant subspace. This permitts to use the approach described above in this case as well.

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