TECHNION -- ISRAEL INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MATHEMATICS

# Mean value interpolation for points in general position

Shayne Waldron* (*waldron@math.auckland.ac.nz*)

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**ABSTRACT**

The mean value interpolation operators are extended to less smooth functions when the points defining them are in general position, and the corresponding interpolation conditions are described. In particular, it is shown that Kergin interpolation to points in general position in $\mathbb{R}^d$ has a (continuous) extension to $C^{d-1}(\mathbb{R}^d)$. Applications to the finite element method are given.

**Key Words:** mean value interpolation, Kergin interpolation, Hakopian interpolation, multivariate interpolation, finite element method

**AMS (MOS) Subject Classifications:** primary 41A05, 41A63, secondary 41A10, 41A35

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## 1. Introduction and preliminary results

The mean value interpolation operators $\mathcal{H}_\Theta^{(m)}$, $0 \le m \le d-1$ (Kergin and Hakopian interpolation are particular cases) are extended to less smooth functions when the $n \ge m+1$ points $\Theta \subset \mathbb{R}^d$ defining them are in general position, and the corresponding interpolation conditions are described. The main result, Theorem 2.1, says that for points in general position $\mathcal{H}_\Theta^{(m)}$ can be extended from $C^{n-m-1}$ to $C^{d-1-m}$ functions. For example, Kergin interpolation ($m=0$) at $n > d$ points can be extended from $C^{n-1}(\mathbb{R}^d)$ to $C^{d-1}(\mathbb{R}^d)$.

Below we give the required background, and some lemmas used to construct the extension. In Section 2, we give the extension of $\mathcal{H}_\Theta^{(m)}$ and discuss some of its applications.

A set/sequence of points in $\mathbb{R}^d$ is said to be in general position if every subset $\Theta$ has the maximum affine dimension that its cardinality $\#\Theta$ allows. For $\Theta = (\theta_0, \ldots, \theta_k)$ and $f$ continuous on $\operatorname{conv}\Theta$ (the convex hull of $\Theta$), let

$$\int_\Theta f := \int_0^1 \int_0^{s_1} \ldots \int_0^{s_{k-1}} f(\theta_0 + s_1(\theta_1 - \theta_0) + \cdots + s_k(\theta_k - \theta_{k-1}))\, ds_k \cdots ds_2\, ds_1.$$

The Newton form of $\mathcal{H}_\Theta^{(m)} : C^{n-m-1}(\mathbb{R}^d) \to \Pi_{n-m-1}(\mathbb{R}^d)$ the mean value interpolation operator for any sequence $\Theta \subset \mathbb{R}^d$ of $n \ge m+1$ points (possibly all coincident) is

$$\mathcal{H}_\Theta^{(m)} f(x) = m! \sum_{j=m+1}^{n} \sum_{\substack{\tilde\Theta \subset \Theta_{j-1} \\ \#\tilde\Theta = m}} \int_{\Theta_j} D_{x-\Theta_{j-1}\backslash\tilde\Theta} f. \tag{1.1}$$

Here $\tilde\Theta$ is an $m$–subsequence of $\Theta_{j-1}$ which is the first $j-1$ points of $\Theta$, and $D_{x-\Theta_{j-1}\backslash\tilde\Theta} f$ is the $j-1-m$ order derivative

$$D_{x-\Theta_{j-1}\backslash\tilde\Theta} f = \Big( \prod_{\substack{\theta \in \Theta_{j-1} \\ \theta \notin \tilde\Theta}} D_{x-\theta} \Big) f,$$

with $D_y$ the directional derivative in the direction $y$. The presentation (1.1) is due to Goodman [G83], with $m=0$ giving Kergin interpolation, and $m = d-1$ giving Hakopian interpolation. As usual, let $\Pi_k := \Pi_k(\mathbb{R}^d)$ and $\Pi_k^0 := \Pi_k^0(\mathbb{R}^d)$ denote the polynomials defined on $\mathbb{R}^d$ that are of degree $\le k$ and homogeneous of degree $k$, respectively.

To describe the interpolation conditions of the extended mean value interpolation maps we need "the derivatives of order $k$ which are (in directions) orthogonal to $\operatorname{aff}\Theta$ the affine hull of $\Theta \subset \mathbb{R}^d$", defined as follows:

$$D^k \perp \operatorname{aff}\Theta := \operatorname{span}\{D_{\mathbf{n}_1} D_{\mathbf{n}_2} \cdots D_{\mathbf{n}_k} : \mathbf{n}_1, \ldots, \mathbf{n}_k \perp \operatorname{aff}\Theta\}, \tag{1.2}$$

with $D^0 \perp \operatorname{aff}\Theta := \operatorname{span}\{1\}$. This space has dimension

$$\dim(D^k \perp \operatorname{aff}\Theta) = \dim(\Pi_k^0(\mathbb{R}^{d-n+1})) = \binom{k+d-n}{d-n}, \qquad n := \dim(\operatorname{aff}\Theta) + 1. \tag{1.3}$$

Hopefully the reader will find the (nonstandard) notation $D^k \perp \operatorname{aff}\Theta$ suitably suggestive. Recall (see Micchelli [M80:Th.6]) that if $y \in \operatorname{aff}\Theta$, where $\Theta = (\theta_1, \ldots, \theta_r) \subset \mathbb{R}^d$, i.e., can be expressed in the form

$$y = \sum_{i=1}^r \lambda_i \theta_i, \qquad \lambda_i \in \mathbb{R},\ \sum_{i=1}^r \lambda_i = 0, \tag{1.4}$$

then

$$\int_\Theta D_y f = -\sum_{i=1}^r \lambda_i \int_{\Theta\backslash\theta_i} f. \tag{1.5}$$

This will be used as follows. Let $g(D)$ denote the (homogeneous) differential operator induced by $g \in \Pi_k^0(\mathbb{R}^d)$.

**Lemma 1.6.** *Suppose that $\Theta \subset \mathbb{R}^d$ is a sequence of $r \geq 2$ points, and $g(D)$ is a homogeneous differential operator of order $k \geq 1$ ($g \in \Pi_k^0$). Then the linear functional*

$$\lambda : f \mapsto \int_\Theta g(D) f$$

*belongs to $A + B$ where*

$$\begin{aligned} A &:= \operatorname{span}\{f \mapsto \int_{\tilde\Theta} p(D) f : \tilde\Theta \subset \Theta,\ \#\tilde\Theta = r-1,\ p \in \Pi_{k-1}^0\}, \\ B &:= \operatorname{span}\{f \mapsto \int_\Theta q(D) f : q(D) \in D^k \perp \operatorname{aff}\Theta\}. \end{aligned}$$

**Proof:** It is sufficient to assume that $g(D) = D_{\xi_1} D_{\xi_2} \cdots D_{\xi_k}$ (since these derivatives span the space homogeneous differential operators of order $k$). Express $\xi_1$ as

$$\xi_1 = y + \mathbf{n}, \qquad y \in \operatorname{aff}\Theta,\ \mathbf{n} \perp \operatorname{aff}\Theta.$$

Since $y$ is in the affine hull of $\Theta := (\theta_1, \ldots, \theta_r)$, it can be expressed in the form (1.4). Thus, by (1.5) we obtain that

$$\int_\Theta g(D) f = -\sum_{i=1}^r \lambda_i \int_{\Theta\backslash\theta_i} D_{\xi_2} \cdots D_{\xi_k} f + \int_\Theta D_{\mathbf{n}} D_{\xi_2} \cdots D_{\xi_k} f, \tag{1.7}$$

and so $\lambda$ can be expressed as the sum of an element of $A$ and the linear functional

$$f \mapsto \int_\Theta D_{\mathbf{n}} D_{\xi_2} \cdots D_{\xi_k} f.$$

Repeating this process (on the above functional) an additional $k-1$ times (for $\xi_2, \ldots, \xi_k$) shows that $\lambda$ is the sum of an element of $A$ and a linear functional of the form

$$f \mapsto \int_\Theta D_{\mathbf{n}_1} D_{\mathbf{n}_2} \cdots D_{\mathbf{n}_k} f, \qquad \mathbf{n}_1, \ldots, \mathbf{n}_k \perp \operatorname{aff}\Theta,$$

which belongs to $B$. □

To determine the space of interpolation conditions for the extended mean value interpolation operators we will need the following combinatorial result.

**Lemma 1.8 (Vandermonde convolution).** *For $0 \le m \le d-1$, $n \ge m+1$,*

$$\dim(\Pi_{n-m-1}(\mathbb{R}^d)) = \binom{n-m-1+d}{d} = \sum_{j=m+1}^{d} \binom{d-m-1}{d-j}\binom{n}{j}.$$

**Proof:** By the Vandermonde convolution identity for binomial coefficients

$$\sum_{j=m+1}^{d} \binom{d-m-1}{d-j}\binom{n}{j} = \sum_{j=0}^{d} \binom{d-m-1}{d-j}\binom{n}{j} = \binom{d-m-1+n}{d}.$$

□

## 2. The extension of $\mathcal{H}_\Theta^{(m)}$

If all the points in $\Theta$ coincide, then the mean value interpolation operator

$$\mathcal{H}_\Theta^{(m)} : C^{n-m-1}(\mathbb{R}^d) \to \Pi_{n-m-1}(\mathbb{R}^d)$$

is Taylor interpolation by polynomials of degree $n-m-1$ at that common point, and so it cannot be extended to less smooth functions. On the other hand, we now show that if the points are in general position, then the mean value interpolation operators can be extended. The argument below can be viewed as the multivariate analogue of taking the Newton form for Hermite interpolation at points in general position in $\mathbb{R}$ (no points repeated) and using the divided difference identity (cf Lemma 1.6) to obtain the Lagrange form.

**Theorem 2.1.** *Suppose that the $n$ points $\Theta \subset \mathbb{R}^d$ are in general position. Then the mean value interpolation map*

$$\mathcal{H}_\Theta^{(m)} : C^{n-m-1}(\mathbb{R}^d) \to \Pi_{n-m-1}(\mathbb{R}^d), \qquad 0 \le m \le d-1,\ n \ge m+1$$

*has a (unique) continuous extension $\mathcal{H} : C^{d-1-m}(\operatorname{conv}\Theta) \to \Pi_{n-m-1}(\mathbb{R}^d)$, which is determined by the interpolation conditions*

$$\int_{\tilde\Theta} g(D)\mathcal{H}f = \int_{\tilde\Theta} g(D)f, \qquad \tilde\Theta \subset \Theta,\ m+1 \le \#\tilde\Theta \le d,\ g(D) \in D^{\#\tilde\Theta-m-1} \perp \operatorname{aff}\tilde\Theta. \tag{2.2}$$

*In particular, Kergin interpolation to points in general position in $\mathbb{R}^d$ has an extension to $C^{d-1}(\mathbb{R}^d)$. Further, by choosing the $g(D)$ in (2.2) to be the elements of some basis for each $D^{\#\tilde\Theta-m-1} \perp \operatorname{aff}\tilde\Theta$, one obtains a basis for the interpolation conditions of $\mathcal{H}$.*

**Proof:** First, observe that since the points in $\Theta$ are in general position, (1.3) gives

$$\dim(D^{\#\tilde\Theta-m-1} \perp \operatorname{aff}\tilde\Theta) = \binom{d-m-1}{d-\#\tilde\Theta},$$

and so by Lemma 1.8, the space of linear functionals that $\mathcal{H}$ matches in (2.2),

$$\Lambda := \operatorname{span}\{f \mapsto \int_{\tilde\Theta} g(D)f : \tilde\Theta \subset \Theta,\ m+1 \le \#\tilde\Theta \le d,\ g(D) \in D^{\#\tilde\Theta - m-1} \perp \operatorname{aff}\tilde\Theta\}$$

has dimension

$$\dim\Lambda \le \sum_{j=m+1}^{d} \binom{d-m-1}{d-j}\binom{n}{j} = \dim(\Pi_{n-m-1}(\mathbb{R}^d)), \tag{2.3}$$

and these functionals are continuous on $C^{d-1-m}(\operatorname{conv}\Theta)$.

By using (for simplicity) the basis $\{D^\alpha : |\alpha| \le k\}$ for the space of differential operators of order $k$, the Newton form (1.1) can be expanded as

$$\mathcal{H}_\Theta^{(m)} f(x) = \sum_{j=m+1}^{n} \sum_{|\alpha|=j-1-m} p_\alpha(x) \int_{\Theta_j} D^\alpha f, \tag{2.4}$$

where $p_\alpha \in \Pi_{n-m-1}$ (the $\alpha$ are multiindices). Since the number of terms in (2.4) is

$$\sum_{k=0}^{n-m-1} \#\{D^\alpha : |\alpha| = k\} = \dim(\Pi_{n-m-1}(\mathbb{R}^d)),$$

and $\mathcal{H}_\Theta^{(m)}$ is a linear projector, the polynomials $(p_\alpha)$ form a basis for $\Pi_{n-m-1}(\mathbb{R}^d)$ with the $\lambda_\alpha : f \mapsto \int_{\Theta_j} D^\alpha f$ giving the dual basis. Thus, it is sufficient to prove that each $\lambda_\alpha \in \Lambda$, with

$$\mathcal{H} f := \sum_{j=m+1}^{n} \sum_{|\alpha|=j-1-m} p_\alpha \lambda_\alpha(f), \qquad \forall f \in C^{d-1-m}(\operatorname{conv}\Theta),$$

then giving the desired extension of $\mathcal{H}_\Theta^{(m)}$. Since (the restriction of) $C^{n-m-1}(\mathbb{R}^d)$ is dense in $C^{d-1-m}(\operatorname{conv}\Theta)$, this extension is unique. Moreover, in view of (2.3), this implies that $\dim\Lambda = \dim(\Pi_{n-m-1}(\mathbb{R}^d))$, and so choosing the $g(D)$ in (2.2) as suggested gives a basis for (the interpolation conditions) $\Lambda$.

We now show that the $\lambda_\alpha \in \Lambda$. Since the $j$ points of $\Theta_j$ are in general position, $D^k \perp \operatorname{aff}\tilde\Theta = \{0\}$ whenever $\#\tilde\Theta \ge d+1$, and so applying Lemma 1.6 $j-1-m$ times to $\lambda_\alpha : f \mapsto \int_{\Theta_j} D^\alpha f$ gives

$$\lambda_\alpha \in \operatorname{span}\{f \mapsto \int_{\tilde\Theta} g(D)f : \tilde\Theta \subset \Theta_j,\ m+1 \le \#\tilde\Theta \le d,\ g(D) \in D^{\#\tilde\Theta - m-1} \perp \operatorname{aff}\tilde\Theta\} \subset \Lambda,$$

which completes the proof. $\square$

For each $\tilde\Theta$, let $B_{\tilde\Theta}$ be some (natural) basis for $D^{\#\tilde\Theta-m-1} \perp \operatorname{aff}\tilde\Theta$. Then, by the basis property of (2.2), the extension $\mathcal{H}_\Theta^{(m)} := \mathcal{H}$ can be represented in the *Lagrange form*

$$\mathcal{H}_\Theta^{(m)} f(x) = \sum_{j=m+1}^{d} \sum_{\substack{\tilde\Theta\subset\Theta \\ \#\tilde\Theta=j}} \sum_{g(D)\in B_{\tilde\Theta}} p_{\tilde\Theta,g}(x) \int_{\tilde\Theta} g(D)f, \qquad \forall f \in C^{d-1-m}(\operatorname{conv}\Theta), \tag{2.5}$$

where $(p_{\tilde\Theta,g})$ is the basis of $\Pi_{n-m-1}(\mathbb{R}^d)$ which is dual to the interpolation conditions $f \mapsto \int_{\tilde\Theta} g(D)f$. We refer to (2.5) as the Lagrange form, as opposed to the Newton form (1.1), since it involves derivatives of $f$ of the lowest possible orders. It it also then natural to call the $p_{\tilde\Theta,g}$ *Lagrange polynomials*. Before describing how Theorem 2.1 fits into the existing literature, it is instructive to consider some examples.

For $m = 0$, $\mathcal{K}_\Theta := \mathcal{H}_\Theta^{(0)}$ is the Kergin interpolation map. In $\mathbb{R}$, points in general position are simply distinct points, and $\mathcal{K}_\Theta$ is Lagrange interpolation at $\Theta$, with (1.1) giving the (classical) Newton form. In (2.2), all the $\tilde\Theta$ consist of a single point, with

$$D^{\#\tilde\Theta-m-1} \perp \operatorname{aff}\tilde\Theta = D^0 \perp \operatorname{aff}\tilde\Theta = \operatorname{span}\{1\}, \tag{2.6}$$

for which it is natural to choose the basis $B_{\tilde\Theta} = \{1\}$. Then (2.5) is the (classical) Lagrange form. In $\mathbb{R}^2$, points are in general position if no three of them are in a line. Here the extension $\mathcal{K}_\Theta$ is defined on $C^1$ functions, and the subsets $\tilde\Theta$ in (2.2) consist of either one or two points. For one point subsets we have (2.6), while for subsets $\tilde\Theta = \{\theta_s, \theta_t\}$ consisting of two points, $D^{\#\tilde\Theta-m-1} \perp \operatorname{aff}\tilde\Theta$ is the 1–dimensional space of derivatives in the direction $(\theta_t - \theta_s)^\perp$ orthogonal to the line through $\theta_s$ and $\theta_t$. By choosing the (natural) bases $B_{\tilde\Theta} = \{1\}$ and $B_{\tilde\Theta} = \{D_{(\theta_t-\theta_s)^\perp}\}$ in each case respectively, the Lagrange form (2.5) for $\Theta = (\theta_1, \ldots, \theta_n)$ becomes

$$\mathcal{K}_\Theta f = \sum_{r=1}^{n} p_r f(\theta_r) + \sum_{1\le s\le t\le n} p_{st} \int_{[\theta_s,\theta_t]} D_{(\theta_t-\theta_s)^\perp} f, \qquad \forall f \in C^1(\operatorname{conv}\Theta). \tag{2.7}$$

This extension, together with some beautiful formulæ (involving the identification of $\mathbb{R}^2$ with $\mathbb{C}$) for the Lagrange polynomials $p_r, p_{st} \in \Pi_{n-1}(\mathbb{R}^2)$, was recently obtained by Bos and Calvi [BC97:Th.2.3]. In $\mathbb{R}^3$, the extension $\mathcal{K}_\Theta$ is defined for $C^2$ functions, and in addition to interpolation conditions like those in (2.7) the information $\int_{[\theta_r,\theta_s,\theta_t]} D_{\mathbf{n}}^2 f$, where $\mathbf{n}$ is a normal to the plane passing through the points $\theta_r, \theta_s, \theta_t \in \Theta$, is matched. For $n = 4$ points the existence of this extension is proved in Lorentz [L92:Th.13.2.4] (also see the applications below).

For $m = d-1$, $\mathcal{H}_\Theta := \mathcal{H}_\Theta^{(d-1)}$ is the Hakopian interpolation map. Here only subsets $\tilde\Theta$ with $d$ elements are taken in (2.2), with (2.6) holding, and so the original presentation of Hakopian's map (which requires the points be in general position) is obtained.

In Cavaretta, Goodman, Micchelli and Sharma [CGMS83] the extension of Theorem 2.1 is given in the cases $m = d-2$ (Theorem 2) and $m = d-3$ (Theorem 3), together with some explicit formulæ for the corresponding Lagrange polynomials (Theorems 4 and

5) obtained from the lifting equation for the mean value interpolation operators. Though not observed by subsequent authors, these results do apply to Kergin interpolation in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. The general result for Kergin interpolation in $\mathbb{R}^d$ ($d > 3$) follows from Theorem 2.1. Their paper also gives an extension for the mean value interpolation operator $\mathcal{H}_\Theta^{(m)}$ when $m > d+1$ ($n \geq m+1$), the only case not covered by Theorem 2.1, namely

$$\mathcal{H} : L_1(\operatorname{conv}\Theta) \to \Pi_{n-m-1}(\mathbb{R}^d),$$

with interpolation conditions

$$\int_{\tilde{\Theta}} \mathcal{H}f = \int_{\tilde{\Theta}} f, \qquad \tilde{\Theta} \subset \Theta, \quad \#\tilde{\Theta} = m+1. \tag{2.8}$$

If the first $m-d+1$ elements of $\tilde{\Theta}$ in (2.8) are fixed, then a basis for the space of interpolation conditions of this extension $\mathcal{H}$ is obtained.

## Applications

The extended mean value interpolation maps of Theorem 2.1 give rise to a family of finite elements. First we recall some definitions (see, e.g., Brenner and Scott [BS94]).

**Definition 2.9.** *Let*
(i) $K \subset \mathbb{R}^d$ *be a domain with piecewise smooth boundary (the* **element domain***),*
(ii) $P$ *be a finite-dimensional space of functions on* $K$ *(the* **shape functions***), and*
(iii) $N = \{N_i\}$ *be a basis for* $P'$ *(the* **nodal variables***).*
*Then* $(K, P, N)$ *is called a* **finite element**.

It is assumed that the nodal variables $\{N_i\}$ are defined on sufficiently smooth functions. For a finite element $(K, P, N)$, the basis for $P$ dual to $\{N_i\}$ is called the **nodal basis** for $P$. It will be denoted by $\{\phi_i\}$.

**Definition 2.10.** *Given a finite element* $(K, P, N)$*, its* **local interpolant** *(to $f$) is*

$$\mathcal{I}_K f := \mathcal{I}_{(K,P,N)} f := \sum_i N_i(f)\,\phi_i. \tag{2.11}$$

Observe that the map $\mathcal{I}_K$ defined by (2.11) is the linear projector onto $P$ with interpolation conditions $N$, i.e., for which

$$N_i(\mathcal{I}_K f) = N_i(f), \qquad \forall i,\ \forall f.$$

If $\Theta \subset \mathbb{R}^d$ is a sequence of $n \geq d+1$ points in general position, then $K := \operatorname{conv}\Theta$ is an element domain, and (2.5) is the local interpolant of a finite element with shape functions $P = \Pi_{n-1}$ and nodal variables given by the interpolation conditions (2.2). We now state this result and give some examples.

**Proposition 2.12 (Finite elements).** *Suppose that $\Theta \subset \mathbb{R}^d$ is a sequence of $n \geq d+1$ points in general position. Let $K := \operatorname{conv}\Theta$, $P := \Pi_{n-1}$, and $N$ be the nodal variables given by*

$$f \mapsto \int_{\tilde{\Theta}} g(D)f, \qquad \tilde{\Theta} \subset \Theta,\ m+1 \leq \#\tilde{\Theta} \leq d,\ g(D) \in B_{\tilde{\Theta}}, \tag{2.13}$$

*where $B_{\tilde{\Theta}}$ is some basis for $D^{\#\tilde{\Theta}-m-1} \perp \operatorname{aff}\tilde{\Theta}$. Then $(K,P,N)$ is a finite element with its local interpolant given by (2.5).*

The simplest bivariate example is $\Theta = (\theta_1,\theta_2,\theta_3)$ the vertices of a triangle $K = \operatorname{conv}\Theta$. Here $P = \Pi_2(\mathbb{R}^2)$ and the data matched by the nodal variables (2.13) is

$$f(\theta_1),\ f(\theta_2),\ f(\theta_3),\ \int_{\theta_1}^{\theta_2} D_{(\theta_2-\theta_1)^\perp} f,\ \int_{\theta_2}^{\theta_3} D_{(\theta_3-\theta_2)^\perp} f,\ \int_{\theta_1}^{\theta_3} D_{(\theta_3-\theta_1)^\perp} f, \tag{2.14}$$

with $\int_a^b$ denoting the line integral over the segment from $a$ to $b$. This finite element is given in Lorentz [L92:Th.13.2.2,Cor.13.2.3]. For $\Theta = (\theta_1,\theta_2,\theta_3,\theta_4)$ the vertices of a convex quadrilateral, $P = \Pi_3(\mathbb{R}^2)$ and the 10 nodal variables match the values

$$f(\theta_r), \quad 1 \leq r \leq 4, \qquad \int_{\theta_s}^{\theta_t} D_{(\theta_t-\theta_s)^\perp} f, \quad 1 \leq s < t \leq 4,$$

and so forth. See Bos and Calvi [BC97:Th.2.3] for a description of the local interpolant.

In $\mathbb{R}^3$, the first example is $\Theta = (\theta_1,\theta_2,\theta_3,\theta_4)$ the vertices of a simplex $K := \operatorname{conv}\Theta$. Here $P = \Pi_3(\mathbb{R}^3)$ and the data matched by the nodal variables is the 4 function values $f(\theta_r)$, $1 \leq r \leq 4$, 12 values of the form

$$\int_{\theta_s}^{\theta_t} D_{(\theta_t-\theta_s)^\perp} f, \quad 1 \leq s < t \leq 4,$$

where $D_{(\theta_t-\theta_s)^\perp} f$ now runs over some basis for the 2–dimensional space of derivatives in directions orthogonal to the line through $\theta_t$ and $\theta_s$, and the 4 values

$$\int_{\operatorname{conv}(\theta_r,\theta_s,\theta_t)} D_{\mathbf{n}}^2 f, \quad 1 \leq r < s < t \leq 4,$$

where the integration above is over the triangle with vertices $\{\theta_r,\theta_s,\theta_t\}$ and $\mathbf{n}$ is some vector orthogonal to this triangle. This finite element is given in [L92:Th.13.2.4,Cor.13.2.5]. In general case of $n \geq 4$ points, the nodal variables match additional values of the form above (simply replace the 4 by $n$).

It might be hoped to obtain more practicable finite elements by replacing the averages occuring in the nodal variables by averages over sets of smaller affine dimension (ideally point evaluations). For example, *Morely's finite element* can be obtained by replacing (2.14) by

$$f(\theta_1), f(\theta_2), f(\theta_3),\ D_{(\theta_2-\theta_1)^\perp} f\Big(\frac{\theta_1+\theta_2}{2}\Big),\ D_{(\theta_3-\theta_2)^\perp} f\Big(\frac{\theta_2+\theta_3}{2}\Big),\ D_{(\theta_3-\theta_1)^\perp} f\Big(\frac{\theta_1+\theta_3}{2}\Big). \tag{2.15}$$

The author is currently unaware of any general results in this direction.

## References


[BC97] L. Bos and J.–P. Calvi, Kergin interpolants at the roots of unity approximate $C^2$ functions, preprint, 1997.

[BS94] S. C. Brenner and L. R. Scott, "The mathematical theory of finite element methods", Springer-Verlag, New York, 1994.

[CGMS83] A. S. Cavaretta, T. N. T. Goodman, C. A. Micchelli, and A. Sharma, Multivariate interpolation and the Radon transform, part III: Lagrange representation, *Canad. Math. Soc. Conf. Proceedings* **3** (1983), 37–50.

[G83] T. N. T. Goodman, Interpolation in minimum semi-norm, and multivariate B-splines, *J. Approx. Theory* **37** (1983), 212–223.

[L92] R. A. Lorentz, "Multivariate Birkhoff Interpolation", Lecture Notes in Mathematics, No. 1516, Springer Verlag, Heidelberg, 1992.

[M80] C. A. Micchelli, A constructive approach to Kergin interpolation in $\mathbb{R}^k$: multivariate B-splines and Lagrange interpolation, *Rocky Mountain J. Math.* **10** (1980), 485–497.