# Sharp error estimatess for multivariate positive linear operators which reproduce the linear polynomials

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**ABSTRACT**

A *sharp* pointwise error estimate is given for multivariate positive linear operators which reproduce the linear polynomials. This quantitative Korovkin–type theorem generalises a known univariate result. It is applied to a number of operators including the multivariate Bernstein operators, and the recently introduced Bernstein–Schoenberg type operators of Dahmen, Micchelli and Seidel.

**Key Words:** Korovkin–type theorem, positive linear operator, Bernstein operator, variation diminishing splines, multivariate Bernstein–Schoenberg type operators, radial basis function approximation

**AMS (MOS) Subject Classifications:** primary 41A10, 41A65, 41A80, secondary 41A05, 41A55, 46E99, 65J05

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## 1. Introduction and the main result

If $L : C[a,b] \to C[a,b]$ is a positive linear operator which reproduces the linear polynomials, then it is known (see, e.g., DeVore [D72:Th.2.5,p.39]) that there is the sharp error estimate

$$|f(x) - Lf(x)| \leq \frac{1}{2} L\big((\cdot - x)^2\big)(x)\, \|D^2 f\|_\infty, \qquad \forall f \in C^2[a,b]. \tag{1.1}$$

In this paper we give the multivariate generalisation of (1.1). The author was suprised that this straightforward generalisation gave *sharp* multivariate bounds. This perhaps explains why it was not considered till now.

In the remainder of this section we present the main result. In Section 2, some examples are considered. These include multivariate Bernstein operators (like linear interpolation at the vertices of a simplex and bilinear interpolation at the vertices of a rectangle), and the recently introduced Bernstein–Schoenberg operators based on blossoming. In Section 3, we give a more general Korovkin–type theorem which also covers the case of multivariate positive linear interpolation operators which don't reproduce the linear polynomials.

For simplicity, we let $K$ be a compact subset of $\mathbb{R}^s$ and consider maps $L$ on $C(K)$ and $C^2(K)$. In the case of locally defined maps (for unbounded regions $K$), or other function spaces (like Sobolev spaces) appropriate modifications of the results below can be made. As usual, $L$ is *positive* means $Lf \geq 0$ whenever $f \geq 0$, and $L$ *reproduces* the linear polynomials means that $Lp = p$ whenever $p$ belongs to $\Pi_1$ (the linear polynomials).

To determine the sign of one of the constants occuring in Theorem 1.4, we need the following lemma, which is of independent interest.

**Lemma 1.2.** *Suppose that $\Omega$ is a convex subset of $\mathbb{R}^s$, and $L : C(\Omega) \to C(\Omega)$ a positive linear operator which reproduces the linear polynomials. Then*

$$f \text{ is convex} \quad \Longrightarrow \quad f \leq Lf. \tag{1.3}$$

**Proof:** Suppose that $f$ is convex and $x \in \Omega$. Since $f$ is convex it is possible to choose a linear polynomial $p$ with $p(x) = f(x)$ and $p \leq f$. When $f$ is $C^1$ at $x$ this is simply the tangent plane to $f$ at $x$. Since $L$ is positive and reproduces $p$,

$$p \leq f \quad \Longrightarrow \quad p = Lp \leq Lf,$$

and so

$$f(x) = p(x) \leq Lf(x),$$

as supposed. $\square$

This 'shape property' is given for the *Bernstein–Schoenberg operator* in Goodman [G95:Th.1,p.442], and it is presumably known for other operators also.

Now the main result. For $f \in C^2(K)$ we define the seminorm

$$\|D^2 f\|_{\infty,K} := \sup_{x\in K} \sup_{\substack{u_1,u_2\in\mathbb{R}^n\\ \|u_i\|=1}} |D_{u_1} D_{u_2} f(x)|,$$

where $D_y f$ is the derivative of $f$ in the direction $y$. This measures the maximum size of the second derivative of $f$ over $K$.

**Theorem 1.4.** *Suppose that $K$ is compact convex subset of $\mathbb{R}^s$, and $L : C(K) \to C(K)$ is a positive linear operator which reproduces the linear polynomials. Then, for every $x \in K$ there is the sharp pointwise error estimate*

$$E(f,x) := |f(x) - Lf(x)| \le \frac{1}{2} S(x)\, \|D^2 f\|_{\infty,K}, \qquad \forall f \in C^2(K), \tag{1.5}$$

*where the nonnegative function $S := S_L : K \to \mathbb{R}^+$ is defined by*

$$S(x) := L(\|\cdot - x\|^2)(x) = E(\|\cdot - x\|^2, x) = E(\|\cdot\|^2, x) = L(\|\cdot\|^2)(x) - \|x\|^2. \tag{1.6}$$

*There is equality in (1.5) for $f$ from the space of quadratics*

$$Q := \Pi_1 \oplus \operatorname{span}\{\|\cdot\|^2\} = \operatorname{span}\{\|\cdot - c\|^2 : c \in \mathbb{R}^n\}. \tag{1.7}$$

**Proof:** Let $T_{1,x}f$ be the linear Taylor interpolant to $f$ at $x$, i.e., the linear polynomial which matches the value and first order derivatives of $f$ at $x$. From the integral error formula for Taylor interpolation

$$R_{1,x}f(y) := f(y) - T_{1,x}f(y) = \int_0^1 (1-t) D^2_{y-x} f(x + t(y-x))\, dt, \qquad y \in K,$$

it follows that

$$|R_{1,x}f| \le \frac{1}{2}\|\cdot - x\|^2 \|D^2 f\|_{\infty,K}. \tag{1.8}$$

Since $L$ reproduces $p = T_{1,x}f$, applying it to $T_{1,x}f - f = -R_{1,x}f$ gives

$$T_{1,x}f - Lf = -L(R_{1,x}f). \tag{1.9}$$

Since $L$ is positive, using (1.9) and (1.8) we obtain

$$|T_{1,x}f - Lf| \le L(|R_{1,x}f|) \le L(\frac{1}{2}\|\cdot - x\|^2 \|D^2 f\|_{\infty,K}) = \frac{1}{2}\|D^2 f\|_{\infty,K} L(\|\cdot - x\|^2). \tag{1.10}$$

Evaluating (1.10) at $x$ gives

$$|f(x) - Lf(x)| \le \frac{1}{2} L(\|\cdot - x\|^2)(x)\, \|D^2 f\|_{\infty,K} = \frac{1}{2} S(x)\, \|D^2 f\|_{\infty,K},$$

which is (1.5). This is sharp for $f := \|\cdot - x\|^2$, and hence for any quadratic polynomial from $Q$ (since $L$ reproduces $\Pi_1$). Since $\|x - x\|^2 = 0$ and $L$ reproduces $\Pi_1$, the first two equalities in (1.6) follow immediately. The third, that

$$E(\|\cdot\|^2, x) := \big|\, \|x\|^2 - L(\|\cdot\|^2)(x) \,\big| = L(\|\cdot\|^2)(x) - \|x\|^2,$$

follows from Lemma 1.2 and the fact that $\|\cdot\|^2$ is convex. $\square$

It is an immediate consequence of (1.5) that there is the sharp error estimate

$$\|f - Lf\|_{L_\infty(K)} \le \frac{1}{2} C_L \, \|D^2 f\|_{\infty,K}, \qquad \forall f \in C^2(K), \tag{1.11}$$

where

$$C_L := \max_{x \in K} S(x). \tag{1.12}$$

## 2. Examples of sharp estimates

In this section the function $S := S_L$ of Theorem 1.4 is computed for several operators $L$, and hence the sharp error estimate (1.5) is obtained.

**Example 1.** Let $T$ be a (nondegenerate) simplex in $\mathbb{R}^s$, with vertices $V$, and corresponding barycentric coordinate functions $(\lambda_v)_{v\in V}$. The *multivariate Bernstein operator* of degree $n$ on this simplex, $B_n := B_{n,T} : C(T) \to C(T)$, $n = 1, 2, \ldots$, is defined by

$$B_n f(x) := \sum_{v_1 \in V} \sum_{v_2 \in V} \cdots \sum_{v_n \in V} f\Bigl(\frac{v_1 + \cdots + v_n}{n}\Bigr) \, \lambda_{v_1}(x) \cdots \lambda_{v_n}(x). \tag{2.1}$$

This operator is positive and reproduces the linear polynomials, and so Theorem 1.4 can be applied . It can be shown (see comments below) that

$$S(x) := S_{B_n}(x) := B_n(\|\cdot - x\|^2)(x) = \frac{1}{n}(R^2 - \|x - c\|^2), \tag{2.2}$$

where $c$ is the centre and $R$ the radius of the (unique) sphere containing $V$. Hence $B_n$ satisfies the sharp error estimate, for $x \in T$, that

$$|f(x) - B_n f(x)| \le \frac{1}{2n}(R^2 - \|x - c\|^2) \, \|D^2 f\|_{\infty,T}, \qquad \forall f \in C^2(T), \tag{2.3}$$

and in particular the sharp error estimate, of the form (1.11), that

$$\|f - B_n f\|_{L_\infty(T)} \le \frac{1}{2n}(R^2 - d^2) \, \|D^2 f\|_{\infty,T}, \qquad \forall f \in C^2(T), \tag{2.4}$$

where

$$d := \text{the distance of } c \text{ from } T = \min_{x \in T} \|x - c\|.$$

The operator $B_1 = B_{1,T}$ is the map of linear interpolation at the vertices $V$ of $T$ (interpolation by linear polynomials). For it the estimates (2.3) and (2.4) were recently proved in Waldron [W97] by using an integral representation of the error. At the time, these were the only known sharp pointwise error estimates for a multivariate interpolation operator, and the role of the positivity of $B_1$ in obtaining them was not fully appreciated.

By taking the result proved in [W97] that $S_{B_1}(x) = R^2 - \|x-c\|^2$, and the formula for $S_{B_n}(x)$, when $T$ is a standard simplex, given in Altomare and Campiti [AC94:p.315], one can conclude (2.2). This formula for $S_{B_n}(x)$, there denoted by $\sigma^2_{n,x}$, is of the form $1/n$ multiplying a quadratic polynomial (namely $R^2 - \|x-c\|^2$ for the standard simplex), and is obtained through a probablistic interpretation of $S_{B_n}$ (see [AC94] for further details).

**Example 2.** Let $B_n$, $B_m$ be the univariate Bernstein operators of degrees $n$, $m$ defined on the intervals $[a,b]$, $[c,d]$ respectively, i.e., cf (2.1),

$$B_n f(x) := \sum_{k=0}^{n} f(v_k)\, p_k(x), \quad x \in [a,b], \qquad B_m f(y) := \sum_{j=0}^{m} f(w_j)\, q_j(y), \quad y \in [a,b],$$

where

$$v_k := \frac{ka + (n-k)b}{n}, \quad p_k := \lambda_a^k \lambda_b^{n-k}, \qquad w_j := \frac{jc + (m-j)d}{m}, \quad q_j := \lambda_c^j \lambda_d^{m-j}.$$

Then the bivariate *tensor product Bernstein operator* of coordinate degree $(n,m)$ on the rectangle $R := [a,b] \times [c,d]$, $B_{n,m} := B_{n,m,R} = B_n \otimes B_m : C(R) \to C(R)$, is defined by

$$B_{n,m} f(x,y) := \sum_{k=0}^{n} \sum_{j=0}^{m} f(v_k, w_j)\, p_k(x) q_j(y), \qquad (x,y) \in R. \tag{2.5}$$

This operator is positive, and it reproduces the bilinear polynomials (which contain the linear polynomials). Using the fact that the univariate Bernstein operator reproduces constants, we compute that

$$\begin{aligned} S_{B_{n,m}}(x,y) &:= B_{n,m}(\|\cdot - (x,y)\|^2)(x,y) \\ &= \sum_{k=0}^{n} \sum_{j=0}^{m} \left\{ (v_k - x)^2 + (w_j - y)^2 \right\} p_k(x) q_j(y) \\ &= \sum_{k=0}^{n} (v_k - x)^2 p_k(x) + \sum_{j=0}^{m} (w_j - y)^2 q_j(y) \\ &= B_n(|\cdot - x|^2)(x) + B_m(|\cdot - y|^2)(y) \\ &= S_{B_n}(x) + S_{B_m}(y) \\ &= \frac{1}{n}(x-a)(b-x) + \frac{1}{m}(y-c)(d-y). \end{aligned} \tag{2.6}$$

From (2.6) sharp pointwise error estimates for $B_{n,m}$ can be obtained. For example, if $L$ is the map of *bilinear interpolation* at the vertices of the unit square, i.e., $B_{1,1}$ with $R := [0,1]^2$, then for $(x,y) \in [0,1]^2$ we have the sharp estimate

$$|f(x,y) - Lf(x,y)| \le \frac{1}{2} \left\{ x(1-x) + y(1-y) \right\} \|D^2 f\|_{\infty,[0,1]^2}, \qquad \forall f \in C^2(R). \tag{2.7}$$

**Example 3.** It is possible to define the *tensor product of positive linear operators* (see [AC94:p.32]). This tensor product is a positive operator, and it reproduces the linear polynomials if each of its factors does so. The construction, an abstract version of (2.5) which relies on associated families of Radon measures, is technical. Hence we provide only a brief outline of it and the corresponding general form of (2.6). The reader should consult [AC94:p.32] for full details.

Let $L_i : C(K_i) \to C(K_i)$, $i = 1, \ldots, p$, where $K_i \subset \mathbb{R}^{s_i}$ is compact and convex, be a (finite) collection of positive linear operators which reproduce $\Pi_1(K_i)$ (the linear polynomials on $K_i$). The tensor product

$$L := \bigotimes_{i=1}^{p} L_i : C(K) \to C(K), \qquad K := \prod_{i=1}^{p} K_i \subset \mathbb{R}^{s_1} \times \cdots \times \mathbb{R}^{s_p}$$

is a positive linear operator ($K$ is a compact convex region). It reproduces $\Pi_1(K_1) \otimes \cdots \otimes \Pi_1(K_p)$ which contains $\Pi_1(K)$. Using properties of the tensor product and the fact that each $L_i$ reproduces constants one can argue, as in (2.6), that

$$\begin{aligned} S_L(x_1, \ldots, x_p) &:= L(\| \cdot - (x_1, \ldots, x_p)\|^2)(x_1, \ldots, x_p) \\ &= L_1(\| \cdot - x_1\|^2)(x_1) + \cdots + L_p(\| \cdot - x_p\|^2)(x_p) \\ &= S_{L_1}(x_1) + \cdots + S_{L_p}(x_p), \end{aligned} \tag{2.8}$$

which is the general form of (2.6).

Here is a specific example. Suppose that $L : C(K) \to C(K)$ is the map of interpolation from $\Pi_1(\mathbb{R}^2) \otimes \Pi_1(\mathbb{R})$ at

$$V := \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,0,1), (0,1,1)\}$$

the vertices of the triangular prism $K$ in $\mathbb{R}^3$. With $K =: T \times I \subset \mathbb{R}^2 \times \mathbb{R}$, this map is of the form $L = B_{1,T} \otimes B_{1,I}$, and hence it is positive (and reproduces the linear polynomials). Thus, by (2.8) and (2.2), we obtain the sharp estimate

$$|f(x,y,z) - Lf(x,y,z)| \le \frac{1}{2} \{x(1-x) + y(1-y) + z(1-z)\} \|D^2 f\|_{\infty,K}, \qquad \forall f \in C^2(K). \tag{2.9}$$

**Example 4.** Here we briefly consider the *Bernstein–Schoenberg operators* recently introduced by Dahmen, Micchelli and Seidel [DMS92] (also see Goodman [G95]). These multivariate operators, which are based on blossoming, are locally defined, positive, and reproduce the linear polynomials. Hence (with appropriate modifications) we can apply Theorem 1.4 to them. They generalise the Bernstein operators of Example 1, and certain variation diminishing spline operators of Schoenberg. Following the notation of [G95:p.442] we define the Bernstein–Schoenberg operator

$$\mathcal{S}_n f(x) := \sum_{I \in J} \sum_{\alpha \in \Gamma_n} f(\zeta_\alpha^I) B_\alpha^I(x), \qquad x \in \mathbb{R}^s.$$

The proof of Theorem 3 of [G95] shows that

$$
\begin{aligned}
S(x) &:= S_{\mathcal{S}_n}(x) := \mathcal{S}_n(\|\cdot - x\|^2)(x) \\
&= \sum_{I\in J}\sum_{\alpha\in\Gamma_n} \|\zeta^I_\alpha - x\|^2 B^I_\alpha(x) \\
&= \sum_{I\in J}\sum_{\alpha\in\Gamma_n} \|\frac{1}{n}\sum_{j=0}^{s}\sum_{l=0}^{\alpha_j-1} x^{i_j,l} - x\|^2 B^I_\alpha(x) \\
&= O(1/n), \qquad n\to\infty,
\end{aligned}
\tag{2.10}
$$

where the 'constant' in the big $O$ depends on the geometry of the triangulation and clouds defining $\mathcal{S}_n$ near the point $x$, and it is possible to choose a constant which works for all $x$ from a given compact subset of $\mathbb{R}^s$. From (2.10) and (1.5) the convergence results of [G95] can be obtained. In light of the special case (2.2), finer estimates of (2.10) might be possible.

**Example 5.** The sharpness of (1.5) implies certain *saturation results*. For example, if $B_n$ is the multivariate Bernstein operator of Example 1, then (2.4) implies that

$$\|f - B_n f\|_{L_\infty(T)} = O(1/n), \quad n\to\infty, \qquad \forall f\in C^2(T),$$

while for $0\neq f\in Q$ (see (1.7)),

$$|f(x) - B_n f(x)| = \frac{1}{n}C_{x,f}, \qquad x\in T\setminus V,$$

where

$$C_{x,f} := \frac{1}{2}(R^2 - \|x-c\|^2)\|D^2 f\|_{\infty,T} > 0.$$

In other words, $B_n$ has *saturation order* $1/n$ (at every point $x\in T\setminus V$). Similarly, by (2.6), the bivariate tensor product Bernstein operator $B_{n,m}$ of Example 2 has saturation order $1/n + 1/m$.

The general result is that a family $(L_k)$ of multivariate positive linear operators that reproduces the linear polynomials has saturation order $S_{L_k}(x)$ at the point $x$.

**Example 6.** Univariate positive linear operators that reproduce the linear polynomials continue to be of interest in approximation theory, e.g., the *multiquadric B–splines* of Beatson and Dyn [BD96].

## 3. A Korovkin–type theorem

For positive linear operators which possibly don't reproduce the linear polynomials, the proof of Theorem 1.4 can be adapted to obtain the following quantitative Korovkin–type theorem for $C^2$–functions. Let $e_i$ denote the linear polynomial $x \mapsto x_i$.

**Theorem 3.1.** *Suppose that $K$ is a compact convex subset of $\mathbb{R}^n$, and $L : C(K) \to C(K)$ is a positive linear operator. Then, for every $x \in K$, there is the pointwise error estimate*

$$\begin{aligned} E(f,x) := |f(x) - Lf(x)| \le |f(x)|\, E(1,x) + \bigl|L\bigl(D_{\cdot - x} f(x)\bigr)(x)\bigr| \\ + \frac{1}{2} E(\|\cdot - x\|^2, x)\, \|D^2 f\|_{\infty,K}, \qquad \forall f \in C^2(K), \end{aligned} \tag{3.2}$$

*where the second term on the right can be estimated by either of*

$$\bigl|L\bigl(D_{\cdot - x} f(x)\bigr)(x)\bigr| \le \|\nabla f(x)\| \sqrt{\textstyle\sum_{i=1}^n E(e_i - x_i, x)^2} \tag{3.3}$$

*or*

$$\bigl|L\bigl(D_{\cdot - x} f(x)\bigr)(x)\bigr| \le \|\nabla f(x)\|\, E(\|\cdot - x\|, x). \tag{3.4}$$

**Proof:** Apply $L$ to $f = T_{1,x} f + R_{1,x} f = f(x) + D_{\cdot - x} f(x) + R_{1,x} f$ to obtain that (after rearrangement)

$$f(x) - Lf = f(x)\{1 - L(1)\} - L\bigl(D_{\cdot - x} f(x)\bigr) - L(R_{1,x} f). \tag{3.5}$$

Evaluating (3.5) at $x$, then using the triangle inequality and (1.10) gives

$$|f(x) - Lf(x)| \le |f(x)| E(1,x) + \bigl|L\bigl(D_{\cdot - x} f(x)\bigr)(x)\bigr| + \frac{1}{2} E(\|\cdot - x\|^2, x)\, \|D^2 f\|_{\infty,K},$$

which is (3.2). Finally, with $D_i$ denoting the derivative in the $i$–th coordinate direction,

$$\begin{aligned} \bigl|L\bigl(D_{\cdot - x} f(x)\bigr)(x)\bigr| &= \bigl|L\bigl(\textstyle\sum_i (e_i - x_i) D_i f(x)\bigr)(x)\bigr| = \bigl|\textstyle\sum_i D_i f(x) L(e_i - x_i)(x)\bigr| \\ &\le \|\nabla f(x)\| \sqrt{\textstyle\sum_i |L(e_i - x_i)(x)|^2} = \|\nabla f(x)\| \sqrt{\textstyle\sum_i E(e_i - x_i, x)^2}, \end{aligned}$$

and

$$\bigl|L\bigl(D_{\cdot - x} f(x)\bigr)(x)\bigr| \le L(\|\nabla f(x)\|\, \|\cdot - x\|) = \|\nabla f(x)\|\, E(\|\cdot - x\|, x).$$

$\square$

If, in addition, $L$ reproduces the linear polynomials, then (3.2) reduces to the sharp estimate of Theorem 1.4. Similar estimates to (3.2) involving moduli of continuity can be found in [AC94:§5.1,p.265].

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