ON A PROJECTION FROM ONE CO-INVARIANT SUBSPACE ONTO ANOTHER IN CHARACTER-AUTOMORPHIC HARDY SPACE ON A MULTIPLY CONNECTED DOMAIN

Sergei Fedorov

Contents

1. Introduction

Let B and θ be the inner functions in the unit disk D and $K_B = H^2_+ \ominus BH^2_+$ and $K_\theta = H^2_+ \ominus \theta H^2_+$ be the corresponding coinvariant subspaces. Then the orthogonal projection $P_{K_\theta} K_B$ from K_B onto K_θ is an isomorphism if and only if (see, for example [1, 2, 3, 4]) the ratio B/θ can be represented in a following form

$$\frac{B}{\theta} = \frac{\bar{h}}{h},$$

where $h \in H^2_+$ is the so-called Helson-Segö outer function, that is for which the angle in $L_2(|h|^2 d\varphi)$ on the unit circle between $\bigvee_{n<0}\{z^n\}$ and $\bigvee_{n\geq 0}\{z^n\}$ is nonzero. The equivalent conditions on the function h are

(1) The norm of Hankel operator $\mathbf{H}_{\frac{\bar{h}}{h}} = P_- \frac{\bar{h}}{h} | H^2_+$ is strictly less then one.

(2) Function h satisfies $A_2(\mathbb{T})$ Muckenhoupt condition on the unit circle $\mathbb{T}$

$$\sup_{\lambda\in D} \int_0^{2\pi} \frac{1-|\lambda|^2}{|1-\bar{\lambda}e^{i\varphi}|^2} |h|^2 d\varphi \cdot \int_0^{2\pi} \frac{1-|\lambda|^2}{|1-\bar{\lambda}e^{i\varphi}|^2} \frac{1}{|h|^2} d\varphi < \infty. \qquad (A_2(\mathbb{T}))$$

If, in particular, B is a Blaschke product with simple zeroes, the above condition on h together with the fact that zeroes of B satisfy Carleson condition gives the answer to the question when the projections on K_θ of reproducing kernels in the zeroes of B form the Riesz basis in K_θ.

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In the present paper we continue to study the harmonic analysis on a multiply connected domain, started in [5, 6, 7, 8] and are dealing with the similar problems in the character-automorphic Hardy spaces on finitely ($g+1$) connected domain Ω. The simplest problem similar to those in the unit disk is the following:

When do the projections of the normalized reproducing kernels in $H^2_+(\Omega)$ at the points $z_j \in \Omega,\ j = 1, 2 \dots$, form a Riesz basis in the linear hull of reproducing kernels corresponding to another sequence of points $t_1, t_2, \dots \in \Omega$?

The first and essential difference of the case under consideration from that one in the unit disk is that as opposed to the case of the unit disk, where $L_2(\mathbb{T}) = H^2_+ \oplus H^2_-$ in our case we have the whole g-dimensional torus of similar decompositions: $L_2(\partial\Omega) = H^2_{+,\kappa} \oplus H^2_{-,\kappa} \oplus \mathfrak{M}_\kappa$, where $H^2_{+,\kappa}, H^2_{-,\kappa}$ are so-called character-automorphic Hardy spaces on Ω, corresponding to character $\kappa \in \mathbb{R}^g/\mathbb{Z}^g$ and $\mathfrak{M}_\kappa$ is g-dimensional κ-automorphic defect space. All this spaces for different characters are of the equal importance and for each particular character the above problem on the projection from one system of reproducing kernels onto another can be posed.

More general "character-automorphic" setting of this problem is as follows:

Let B_μ and Θ_λ be the character-automorphic inner functions[1)] in the domain Ω. and $K^\kappa_{B_\mu} = H^2_{+,\kappa} \ominus B_\mu H^2_{+,\kappa-\mu}$ and $K^\kappa_{\Theta_\lambda} = H^2_{+,\kappa} \ominus \Theta_\lambda H^2_{+,\kappa-\lambda}$ be the corresponding co-invariant subspaces of $H^2_{+,\kappa}$.

When is the orthogonal projection $P_{K^\kappa_{\Theta_\lambda}} K^\kappa_{B_\mu}$ from $K^\kappa_{B_\mu}$ onto $K^\kappa_{\Theta_\lambda}$ an an isomorphism?

It was natural to suspect that just as above, in this case as well particular results on the angles between subspaces in weighted L_2 space on $\Gamma = \partial\Omega$, would play an essential role.

The first steps towards the solution of this problem were given by the author in [7], where all the positive weights w on Γ, for which the angle in $L_2(\Gamma, w)$ between $L_2(\Gamma, w) \cap \{H^2_{-,\kappa} + \mathfrak{M}_\kappa\}$ and $L_2(\Gamma, w) \cap H^2_{+,\kappa}$ was nonzero, were described. These weights should satisfy the Muckenhoupt condition, which in this case has the same form as the classical one and the corresponding angles are nonzero for all characters simultaneously.

But it so happens that this description is not enough to to answer the question of the invertibility of the corresponding projection, namely,as it will be shown below the problem can be reduced to the problem of the description (in terms of necessary and sufficient conditions) of those weights w on Γ for which the angle in $L_2(\Gamma, w)$ between $L_2(\Gamma, w) \cap H^2_{-,\kappa}$ and $L_2(\Gamma, w) \cap H^2_{+,\kappa}$ is nonzero. The last mentioned description was obtained in [8] and the answer has a form similar to the Muckenhoupt condition, but essentially depends on a character. Namely it may happen that for one and the same weight w on Γ for one character the corresponding angle is zero and for another character it is nonzero. Our solution is essentially based on the results of the papers [6, 7, 8].

2. Preliminaries

In what follows we consider a planar domain as a half of its Schottky double - the compact Riemann surface, and essentially use the function theory on it.

1) For precise definitions see next section

2.1. We consider our $g+1$ connected domain $\Omega = \Omega_+$ with nondegenerate boundary components $\Gamma_0, \dots, \Gamma_g$ as a bordered Riemann surface and denote by $\hat{\Omega}$ its Schottky double - the compact Riemann surface of genus g, which is obtained topologically from two copies Ω_+ and Ω_- of the domain Ω by the identification of the points of $\partial\Omega_\pm \stackrel{def}{=} \Gamma$ (gluing a second copy Ω_- of Ω to Ω_+ along $\partial\Omega_+$) with the complex structure obtained by "reflecting" the complex structure from $\Omega_+ \cup \partial\Omega_+$ to Ω_-. The involution $J : \hat{\Omega} \longrightarrow \hat{\Omega}$ which fixes the points of Γ and interchanges the same points on Ω_+ and Ω_- is antiholomorphic.

Fix a point $P_0 \in \Gamma_0$ and let $\gamma_1, \dots, \gamma_g$ be oriented crosscuts from P_0 to the boundary components $\Gamma_1, \dots, \Gamma_g$ respectively, which except for their end points lie in Ω_+ and intersect each other only in the point P_0. We define the homology basis on the compact Riemann surface $\hat{\Omega}$ of genus g in the following way: the b-cycles are the boundary components, $b_j = \Gamma_j,\ j = 1, \dots, g$ and the a-cycles are constructed from the crosscuts γ_j and their antiholomorphic reflections $J(\gamma_j)$ on Ω_-, $a_j = \gamma_j - J(\gamma_j),\ j = 1, \dots, g$.

We consider the g-dimensional vector space $\mathfrak{N}$ of holomorphic differentials on $\hat{\Omega}$ and the normalized basis $\omega_1, \dots, \omega_g$ in this space which is dual to the homology basis

$$\int_{a_j} \omega_k = \delta_{j,k}$$

where $\delta_{j,k}$ is the Kronecker symbol. The differentials ω_j are purely imaginary on Γ and are antisymmetric with respect to the antiholomorphic involution J, $J^*\omega \stackrel{def}{=} \overline{\omega_j(J(z))} = -\omega_j(z)$. Consequently the matrix of b-periods $\tau = \left\{ \int_{b_j} \omega_k \right\}_{j,k=1}^{g}$ has the form $\tau = iP$ where P is a real positive definite matrix.

We denote by $d\eta_{P,Q},\ P, Q \in \hat{\Omega},\ P \neq Q$, the normalized differential of the third kind with only simple poles in the points P, Q with residues $\frac{1}{2\pi i}$ and $-\frac{1}{2\pi i}$ respectively and $\int_{a_j} d\eta_{P,Q} = 0,\ j = 1, \dots, g$.

Note that for the point $P \in \Omega_+$ the restriction of the differential $d\eta_{P,J(P)}$ on Γ determines exactly the harmonic measure $d\eta_P$.

There is a close relation between the harmonic measure of Ω_+ with respect to the point P and the Green function $G(z, P)$ of the domain Ω_+ with singularity at the point P,

$$2\pi\Im \int_{P_0}^{z} d\eta_{P,J(P)} = G(z, P) \quad \text{or} \quad 2\pi d\eta_{P,J(P)} = 2i\frac{\partial G(z, P)}{\partial z} dz.$$

2.2. The divisor on $\hat{\Omega}$ is the finite formal sum

$$\mathfrak{a} = \sum_{j=1}^{k} n_j P_j,\ n_j \in \mathbb{Z},\ P_j \in \hat{\Omega}.$$

The divisor $\mathfrak{a}$ is called positive ($\mathfrak{a} \geq 0$) if $n_j \geq 0$ for all j.The order of a divisor is defined as

$$ord(\mathfrak{a}) = \sum_{j=1}^{k} n_j$$

The divisor of a meromorphic on $\hat{\Omega}$ function f is called a principal divisor and is denoted by (f). Obviously $ord(f) = 0$. Let us denote by $L(\hat{\Omega})$ the field of meromorphic functions on $\hat{\Omega}$ and by $L(\mathfrak{a})$ the vector space $\left\{f \in L(\hat{\Omega}) : (f) \geq \mathfrak{a}\right\}, r(\mathfrak{a}) \stackrel{def}{=} \dim L(\mathfrak{a})$.

The divisors $\mathfrak{a}, \mathfrak{b}$ are equivalent ($\mathfrak{a} \sim \mathfrak{b}$) if their difference is the principal divisor.

The divisor of a meromorphic differential ω on $\hat{\Omega}$ is called a canonical divisor and is also denoted by (ω), the corresponding vector space of meromorphic differentials on $\hat{\Omega}$ is denoted by $\mathfrak{N}(\hat{\Omega})$ and $\mathfrak{N}(\mathfrak{a}) \stackrel{def}{=} \left\{\omega \in \mathfrak{N}(\hat{\Omega}) : (\omega) \geq \mathfrak{a}\right\}, i(\mathfrak{a}) \stackrel{def}{=} \dim \mathfrak{N}(\mathfrak{a})$. Any two canonical divisors are equivalent and the order of any canonical divisor is equal to $2g-2$. The action of the antiholomorphic involution J can be obviously extended to divisors.

The Riemann-Roch theorem asserts that

$$r(-\mathfrak{a}) = ord(\mathfrak{a}) - g + 1 + i(\mathfrak{a}).$$

The period lattice of the holomorphic differentials is the lattice $\mathbb{Z}^g + \tau\mathbb{Z}^g$ in $\mathbb{C}^g$. The complex torus $Jac(\hat{\Omega}) = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ is called the Jacobian variety of the Riemann surface $\hat{\Omega}$. The Abel-Jacobi mapping based at the point P_0 (recall that we chose P_0 on $\Gamma_0 = b_0$) is defined by

$$\varphi(P) = \int_{P0}^{P} \vec{\omega},\ P \in \hat{\Omega}.$$

The mapping $\varphi : \hat{\Omega} \longrightarrow \mathbb{C}^g$ is multi-valued and, hence, is not correctly defined, but it is a correctly defined mapping from $\hat{\Omega}$ to $J(\hat{\Omega})$. φ is a one to one conformal map of $\hat{\Omega}$ onto its image in $Jac(\hat{\Omega})$. The mapping φ can be extended on $D_0(\hat{\Omega})$ - the divisors of order 0 on $\hat{\Omega}$. If

$$\mathfrak{a} = \sum_{j=1}^{k} n_j P_j,\ \sum_{j=1}^{k} n_j = 0$$

we define

$$\varphi(\mathfrak{a}) = \sum_{j=1}^{k} n_j \varphi(P_j).$$

Abel's theorem asserts that for $\mathfrak{a} \in D_0(\hat{\Omega})$ $\varphi(\mathfrak{a}) = 0$ if and only if $\mathfrak{a}$ is a principal divisor.

The divisor of zeros of $\eta_{P,J(P)}$ has an order 2g and is symmetric with respect to J and has the form $Z^* + J(Z^*)$ where $Z^* \subset \Omega_+,\ Z^* = z_1^* + \cdots + z_g^*, z_1^*, \ldots, z_g^*$ are the critical points of the Green function $G(z, P)$. That's why the divisor $Z^* = Z^*(P)$ is often referred to as the critical Green's divisor. The φ-image of the divisor $Z^* - gP_0$ in $Jac(\hat{\Omega})$ is well determined in terms of Riemann theta functions on $Jac(\hat{\Omega})$ (see [9,10]).

2.3. For any divisor $\mathfrak{a}$ of order 0, $\mathfrak{a} = \sum_{j=1}^{k} Q_j - \sum_{j=1}^{k} P_j$ and any vector $\vec{c} = (c_1, \dots, c_g) \in \mathbb{C}^g$ the function

$$f_{\mathfrak{a},\vec{c}} = e^{2\pi i \int\limits_{P0}^{P} \omega(\mathfrak{a},\vec{c})}, \quad \omega(\mathfrak{a},\vec{c}) = \sum_{j=1}^{k} \eta_{Q_j,P_j} + \sum_{l=1}^{g} c_l \omega_l \tag{1}$$

is the so-called multiplicative meromorphic function on $\hat{\Omega}$ corresponding to the divisor $\mathfrak{a}$ and vector $\vec{c} \in \mathbb{C}^g$. That is, $f_{\mathfrak{a},\vec{c}}(P)$ is a locally meromorphic multivalued function on $\hat{\Omega}$, whose multivalued behavior is determined by the $2g$-dimensional vector

$$\chi = (\int_{a_1} \omega(\mathfrak{a},\vec{c}), \dots, \int_{a_g} \omega(\mathfrak{a},\vec{c}) : \int_{b_1} \omega(\mathfrak{a},\vec{c}), \dots, \int_{b_g} \omega(\mathfrak{a},\vec{c}))$$

of a and b periods of the differential $\omega(\mathfrak{a},\vec{c})$ which due to (1) is equal to $(\vec{c} : \tau\vec{c} + \varphi(\mathfrak{a}))$ modulo $\mathbb{Z}^{2g}$. The values f_1, f_2 of $f_{\mathfrak{a},\vec{c}}$ at the point P corresponding to two different paths of integration l_1, l_2 from P_0 to P, for which the cycle $l = l_2 - l_1$ is homologous to $\sum_{j=1}^{g} n_j a_j + \sum_{j=1}^{g} m_j b_j$, are connected by the equality $f_2 = e^{2\pi i < \chi, \vec{\mathbf{n}_l} >} f_1$ where $< \chi, \vec{\mathbf{n}_l} > = \sum_{j=1}^{g} n_j \chi_j + \sum_{j=1}^{g} m_j \chi_{g+j}$. The inverse statement is also true (see for example [11]), any meromorphic multiplicative function on $\hat{\Omega}$ has the form (1).

We will consider the vector space $\mathcal{H}_\kappa$, $\kappa \in \mathbb{R}^g/\mathbb{Z}^g$ of functions f, locally analytic on Ω_+, with single-valued modulus and such that the analytic continuation of any functional element of f along a closed curve homologous to $\sum_{j=1}^{g} m_j b_j$ leads to multiplication of the initial value by the unimodular factor (character)

$$e^{2\pi i \sum_{j=1}^{g} m_j \kappa_j}, \ \kappa = (\kappa_1, \dots, \kappa_g).$$

From now on we fix the term character for the elements $\kappa \in \mathbb{R}^g/\mathbb{Z}^g$, this means that in place of usual multiplicative representation of the fundamental group of $\hat{\Omega}$ in $\mathbb{T}^g$ we use the (equivalent) additive one.

The functions from the space $\mathcal{H}_\kappa$ are called modulus automorphic or character automorphic corresponding to the character κ, or simply κ-automorphic. These functions can be considered as single-valued analytic functions f in the simply connected domain $\Omega'_+ = \Omega_+ \setminus \bigcup_{j=1}^{g} \gamma_j$ with $|f|$ continuous in Ω_+ and such that the limits $f(P\pm) = \lim_{z \to P\pm} f(z)$ exist on $\bigcup_{j=1}^{g} \gamma_j \cap \Omega_+$and satisfy $f(P+) = e^{2\pi i \lambda_j} f(P-)$, $P \in \gamma_j$, $j = 1, \dots, g$. Here the limits as $z \to P+$, $z \to P-$ on γ_j are respectively from the left and from the right side of γ_j.

It should be noted that the unit disk D is the universal covering $\pi : D \longrightarrow \Omega_+$ for the domain Ω_+. The group of covering transformations Σ (the Fuchsian group of the second kind) is isomorphic to the fundamental group $\pi_1(\Omega_+, P_0)$ of the domain and any element $f \in \mathcal{H}_\kappa$ is lifted to a single valued holomorphic function $\tilde{f}$ on D such that $|\tilde{f} \circ \sigma| = |\tilde{f}|$, $\sigma \in \Sigma$ and $\tilde{f} \circ \sigma_j = e^{2\pi i \lambda_j} \tilde{f}$, where $\sigma_j \in \Sigma$ corresponds to the element ${\gamma_j}^{-1} b_j \gamma_j$ in $\pi_1(\Omega_+, P_0)$. The corresponding lifted space $\mathcal{H}_\kappa = \mathcal{H}_\kappa(D)$ will be also called the space of character automorphic functions.

From now on we fix the universal covering mapping $\pi : D \longrightarrow \Omega_+$ by the condition $\pi(0) = a$ for some fixed point $a \in \Omega_+$ and consider the harmonic measure

$d\eta_a$ as a basic measure on Γ. By $L_p(\Gamma), p \geq 1$, we denote the the usual L_p space on Γ with respect to measure $d\eta_a$.

The spaces $H^p_{+,\kappa}$, $1 \leq p \leq \infty$, mentioned in the previous section are the spaces of functions f from $\mathcal{H}_\kappa$ which have non-tangential limits on Γ a.e. $d\eta_a$ belonging to $L_p(\Gamma)$. The function f from $H^p_{+,\kappa}$ is lifted to the universal covering as a function $\tilde{f}$ from $\mathcal{H}_\kappa(D) \cap H^p_+(D)$, where $H^p_+(D)$ is the usual Hardy space in the unit disk. For the lifted space we will use the same abbreviation as for the space in the domain.

The multiplicative meromorphic functions on $\hat{\Omega}$ of the form (1) with $P_j = J(Q_j)$, $Q_j \in \Omega_+, \Re c_j = 0$, $j = 1, \dots, g$ for the point P and the path of integration from P_0 to P contained in $\overline{\Omega_+}$ give important examples of the character-automorphic functions. Thus, the function

$$b_Q \stackrel{def}{=} f_{Q-J(Q),\vec{\mathbf{o}}},\ Q \in \Omega_+$$

is a character-automorphic function with $\kappa = \kappa(Q) = \Re\varphi(Q - J(Q))$, contractive in Ω_+, unimodular on Γ, with only one simple zero at the point Q (and only one simple pole at $J(Q)$). We will call this function an elementary Blaschke factor. By character automorphic Blaschke product we will mean the product (finite or infinite) of elementary Blaschke factors (for conditions on the zero set of Blaschke factors which are necessary and sufficient for the uniform convergence of Blaschke product see for example [5]). The lifting of b_Q on the universal covering is the ordinary Blaschke product in the unit disk with the zero set $\pi^{-1}(Q)$.

In what follows we will denote by B_Z the Blaschke product with the divisor of zeros Z and the corresponding character by $\kappa(Z)$. Thus B_{Z^*} is the finite Blaschke product corresponding to the critical Green's divisor Z^* (with the zeros at the critical points of the Green function $G(z,a)$) with character $\kappa^* = \kappa_{Z^*} = \Re\varphi(Z^* - J(Z^*)) = 2\Re\varphi(Z^*)$.

By analogy with the theory in the unit disk we also consider the space

$$H^2_{-,\kappa} = \left\{\bar{f} : f \in H^2_{+,-\kappa}, f(a) = 0\right\}.$$

The spaces $H^2_{+,\kappa}$ and $H^2_{-,\kappa}$ are orthogonal since $< f, \bar{g} >= \int_\Gamma f g d\eta_a = f(a)g(a)$ for $f \in H^2_{+,\kappa}$, $g \in H^2_{+,-\kappa}$. The space $H^2_{-,\kappa}$ also can be treated as a Hardy space on the second "sheet" Ω_- of the double $\hat{\Omega}$.

The space $H^2_{+,\kappa}$ is a functional Hilbert space and for any $t \in \Omega_+$ there exists the reproducing kernel $k(\cdot, t, \kappa) = k^\kappa_t$, $< f, k^\kappa_t >= f(t)$, for any $f \in H^2_{+,\kappa}$ (for expression of k^κ_t in terms of Riemann theta-functions on $Jac(\hat{\Omega})$ see [12]).

For any $\kappa \in \mathbb{R}^g/\mathbb{Z}^g$ and any inner function B in Ω_+, with character κ_B let us consider the co-invariant subspaces

$$K^\kappa_B = H^2_{+,\kappa} \ominus B H^2_{+,\kappa-\kappa_B},\ \bar{K}^\kappa_B = H^2_{-,\kappa} \ominus \frac{1}{B} H^2_{,\kappa+\kappa_B}$$

For the critical Green's divisor $Z^* = z^*_1 + \cdots + z^*_g$ we consider the g-dimensional space $K^\kappa_{Z^*} = H^2_{+,\kappa} \ominus B_{Z^*} H^2_{+,\kappa-\kappa^*}$, $\kappa^* = \kappa(Z^*) = \Re\varphi(Z^* - J(Z^*))$, which is spanned by the reproducing kernels at the points z^*_j, $j = 1, \dots, g$. Then

$$L^2(\Gamma) = H^2_{-,\kappa} \oplus \mathfrak{M}_\kappa \oplus H^2_{+,\kappa},\ \mathfrak{M}_\kappa = \frac{1}{B_{Z^*}} K^{\kappa+\kappa^*}_{Z^*} = B_{Z^*} \bar{K}^{\kappa-\kappa^*}_{Z^*},$$

where $\mathfrak{M}_\kappa$ is the so-called κ-automorphic defect space, $dim\mathfrak{M}_\kappa = g$.

2.4. More generally by character automorphic meromorphic function on $\hat{\Omega}$ we mean the function of the form (1) with $\vec{c} \in \mathbb{Z}^g$ and $\varphi(\mathfrak{a}) = \kappa \in \mathbb{R}^g/\mathbb{Z}^g$ modulo $\mathbb{Z}^g + \tau\mathbb{Z}^g$. Clearly such a function corresponds exactly to the character κ. The analog of Abel's theorem (see for example [11]) asserts that a divisor $\mathfrak{a}$ of degree zero, is a divisor of a character-automorphic function if and only if $\varphi(\mathfrak{a}) = \kappa \in \mathbb{R}^g/\mathbb{Z}^g$, and κ is exactly the character of this function. The divisor of a character-automorphic function f will be also denoted by (f) (clearly $ord(f) = 0$). We denote by $L_\kappa(\hat{\Omega})$ the vector space of κ-automorphic meromorphic functions on $\hat{\Omega}$ and by $L_\kappa(\mathfrak{a})$ the vector space $\left\{ f \in L_\kappa(\hat{\Omega}) : (f) \geq \mathfrak{a} \right\}$, $r^\kappa[\mathfrak{a}] = dim L_\kappa(\mathfrak{a})$. We can also consider the character-automorphic meromorphic differentials on $\hat{\Omega}$. Actually all such differentials are obtained by multiplication of meromorphic differentials by character-automorphic functions on $\hat{\Omega}$. The vector space of κ-automorphic meromorphic differentials will be denoted by $\mathfrak{N}_\kappa$ and $\mathfrak{N}_\kappa(\mathfrak{a}) = \{\omega \in \mathfrak{N}_\kappa : (\omega) \geq \mathfrak{a}\}, i^\kappa[\mathfrak{a}] = dim \mathfrak{N}_\kappa(\mathfrak{a})$, where by (ω) we again denote the divisor of ω. The version of Riemann-Roch theorem for character-automorphic functions and differentials asserts that for any divisor $\mathfrak{a}$ on $\hat{\Omega}$

$$r^\kappa[\mathfrak{a}] = -ord(\mathfrak{a}) - g + 1 + i^{-\kappa}[-\mathfrak{a}].$$

In these notations we clearly have

$$\mathfrak{M}_\kappa = L_\kappa(J(a) - Z^* - J(Z^*)),$$

and consequently $r^\kappa[J(a) - J(Z^*)] = 0$, since otherwise the nonzero function from $L_\kappa(J(a) - J(Z^*))$ would have been from $\mathfrak{M}_\kappa$ and $H^2_{+,\kappa}$ simultaneously.

We will use below two following technical results prooved in [8].

Lemma 2.1. *For any positive divisor Z of order g on $\hat{\Omega}$, $\varphi(Z) = \mu + \tau\nu$, $\mu, \nu \in \mathbb{R}^g/\mathbb{Z}^g$, there exists the positive divisor T of order g, $T = t_1 + \cdots + t_g$, $t_1 \in \Gamma_1, \ldots, t_g \in \Gamma_g$, such that $\varphi(T) = (1, 1, \ldots, 1)^t + \tau\nu$. That is $\Im\varphi(T - Z) = 0$ and, by Abel's theorem for character-automorphic function, there exists the $((1, \ldots, 1) - \mu)$-automorphic function with the divisor $T - Z$.*

Lemma 2.2. *For any positive divisor Z of order g on $\hat{\Omega}$, $\varphi(Z) = \mu_Z + \tau\nu_Z$, $\mu_Z, \nu_Z \in \mathbb{R}^g/\mathbb{Z}^g$, there exists the positive divisor T of order g, which is contained totally inside Ω_+ and for which $\varphi(T) = \mu_T + \tau\nu_Z$. That is $\Im\varphi(T - Z) = 0$ and, by Abel's theorem for character-automorphic function, there exists the $(\mu_T - \mu_Z)$-automorphic function with the divisor $T - Z$.*

2.5. Let L, M be the subspaces of the Hilbert space H and $L \cap M = \{0\}$. Then on the sum $L + M$ we can define the operator

$$\mathcal{P}_{L\|M} : l + m \longrightarrow l,\ l \in L,\ m \in M,$$

Which is the so-called skew projection onto L parallel to M. By the closed graph theorem the boundedness of this operator is equivalent to the fact that $L + M = clos(L + M)$. The angle between L and M is defined as follows:

$$< (L, M) \in [0, \pi/2],\ \cos(L, M) = \sup_{x \in L, y \in M} \frac{< x, y >}{\|x\| \|y\|}.$$

It is easy to see that

$$\cos(L,M)=\sup_{x\in L}\frac{\|P_Mx\|}{\|x\|}=|P_MP_L|,\ \sin(L,M)=\inf_{x\in L}\frac{\|(I-P_M)x\|}{\|x\|}=|\mathcal{P}_{L\|M}|^{-1},$$

where P_L, P_M are orthogonal projections on L and M respectively.

3. Reduction to the estimation of the norms of Hankel operators

3.1. Let B and Θ be two character-automorphic inner functions on Ω_+, corresponding to the characters κ_B and κ_Θ respectively.

We fix character $\kappa\in\mathbb{R}^g/\mathbb{Z}^g$ and consider the coresponding to these inner functions co-invariant subspaces in $H^2_{+,\kappa}$, $K^\kappa_B=H^2_{+,\kappa}\ominus BH^2_{+,\kappa-\kappa_B}$ and $K^\kappa_\Theta=H^2_{+,\kappa}\ominus\Theta H^2_{+,\kappa-\kappa_\Theta}$.

The question we are interested in is when (in terms of necessary and sufficient conditions) the orthogonal projection $P_{K^\kappa_\Theta}|K^\kappa_B$ in $L_2(\Gamma)$ from K^κ_B onto K^κ_Θ is isomorphism?[(2)]

We use the following statement from Lemma on close subspaces from [2].

Lemma on Close Subspaces. *Let H be a Hilbert space and L, M be its closed subspaces. The following assertions are equivalent*

1. $P_L|M$ *is isomorphism from* M *onto* L.
2. $P_ML=M,\ L\cap M^\perp=\{0\}$.
3. $H=M^\perp+L,\ Clos(M+L^\perp)=H$.
4. $|P_{L^\perp}P_M|<1,\ |P_{M^\perp}P_L|<1$.

Therefore, by statement (4) of the last Lemma $P_{K^\kappa_\Theta}|K^\kappa_B$ is isomorphism from K^κ_B onto K^κ_Θ iff

$$|P_{(K^\kappa_\Theta)^\perp}P_{K^\kappa_B}|<1,\ |P_{(K^\kappa_B)^\perp}P_{K^\kappa_\Theta}|<1.$$

Now, since for any inner function S, corresponding to the character κ_S, we have the equality $P_{SH^2_{+,\kappa-\kappa_S}}=SP_+^{\kappa-\kappa_S}\overline{S}$, the last to inequalities are equivalent to

$$\begin{aligned}|P_{(K^\kappa_\Theta)^\perp}P_{K^\kappa_B}|&=|\Theta P_+^{\kappa-\kappa_\Theta}\overline{\Theta}|K^\kappa_B|=|P_+^{\kappa-\kappa_\Theta}\overline{\Theta}|K^\kappa_B|<1,\\ |P_{(K^\kappa_B)^\perp}P_{K^\kappa_\Theta}|&=|BP_+^{\kappa-\kappa_B}\overline{B}|K^\kappa_\Theta|=|P_+^{\kappa-\kappa_B}\overline{B}|K^\kappa_\Theta|<1.\end{aligned}\tag{2}$$

3.2. Following the ideas from [1, 2], we would like to reduce these two operator norm inequalities to equivalent pair of inequalities for the norms of corresponding character-automorphic Hankel operators.

To begin with let us recall the definitions of character-automorphic Hankel and Teoplitz operators. As above P_+^ν, $\nu\in\mathbb{R}^g/\mathbb{Z}^g$ is a "Riesz" orthogonal projection from $L_2(\Gamma)$ onto $H^2_{+,\nu}$ and let $P_-^\nu=I-P_+^\nu$, be the orthogonal projection from $L_2(\Gamma)$ onto $H^2_{-,\nu}+\mathfrak{M}_\nu$. Recall that (see for example [6]) for any $f\in L_\infty(\Gamma)$ and $\mu,\nu\in\mathbb{R}^g/\mathbb{Z}^g$ the Hankel operator $\mathbf{H}_f^{\nu,\mu}$ with a symbol is an operator acting from $H^2_{+,\nu}$ into $H^2_{-,\nu+\mu}+\mathfrak{M}_{\nu+\mu}$ according to the formula

$$\mathbf{H}_f^{\nu,\mu}g=P_-^{\nu+\mu}fg,\quad\forall g\in H^2_{+,\nu}.$$

(2) Here by P_M we denote the orthogonal projection in L_2 on a closed subspace M.

The character-automorphic Teoplitz operator $\mathbf{T}_f^{\nu,\mu}$ with a symbol f is acting from $H^2_{+,\nu}$ into $H^2_{+,\nu+\mu}$,

$$\mathbf{T}_f^{\nu,\mu} g = fg - \mathbf{H}_f^{\nu,\mu} g = P_+^{\nu+\mu} fg, \quad \forall g \in H^2_{+,\nu}.$$

Here character ν describes the Hardy space, on which the operators act, and character μ prescribes the character to the symbol $f \in L_\infty(\Gamma)$[3)]

If f is unimodular on Γ function then clearly for any $g \in H^2_{+,\nu}$

$$\|\mathbf{H}_f^{\nu,\mu} g\|_2^2 + \|\mathbf{T}_f^{\nu,\mu} g\|_2^2 = \|g\|_2^2, \tag{3}$$

and the adjoint operator $\left(\mathbf{T}_f^{\nu,\mu}\right)^*$ is a Teoplitz operator $\mathbf{T}_{\bar f}^{\nu+\mu,-\mu}$.

In what follows, to simplify the formulae, we will omit the upper indices for Hankel and Teoplitz operators and the reader should keep in mind that the characters of the symbols, which would be the products or quotients of character-automorphic functions, could be restored from these expressions, and the characters of the Hardy spaces could be restored from the context. For example, as above, the character of the symbol B/Θ is the difference of the characters of B and Θ, i.e. is equal to $\kappa_B - \kappa_\Theta$.

Lemma 3.1. *The inequalities (2) are equivalent to*

$$|\mathbf{H}_{\frac{B}{\Theta}}| < 1 \quad on \quad H^2_{+,\hat\kappa}, \qquad |\mathbf{H}_{\frac{\Theta}{B}}| < 1 \quad on \quad H^2_{+,\hat\kappa-\kappa_\Theta+\kappa_B}, \tag{4}$$

where $\hat\kappa = \kappa^* + \kappa_\Theta - \kappa - \kappa_a$.

Proof. By [6, Corollary 3.1], the space $\frac{K_\Theta^\kappa}{B_{Z^*}\Theta}$ is a co-invariant subspace of the space $H^2_{-,\kappa-\kappa^*-\kappa_\Theta}$, corresponding to an inner function $\overline{\Theta}$ and $\frac{K_B^\kappa}{B_{Z^*}B}$ is a co-invariant subspace of the space $H^2_{-,\kappa-\kappa^*-\kappa_B}$, corresponding to an inner function $\overline{B}$. Thus

$$\frac{K_\Theta^\kappa b_a}{B_{Z^*}\Theta} = \overline{K_\Theta^{\kappa^*+\kappa_\Theta-\kappa-\kappa_a}} \qquad and \qquad \frac{K_B^\kappa b_a}{B_{Z^*}B} = \overline{K_B^{\kappa^*+\kappa_B-\kappa-\kappa_a}}.$$

Using, say, the second inequality from (2) we get

$$1 > |P_+^{\kappa-\kappa_B}\overline{B}|K_\Theta^\kappa| = |P_+^{\kappa-\kappa_B}\frac{\Theta}{B}|\frac{K_\Theta^\kappa}{\Theta}| = |P_+^{\kappa-\kappa_B}\frac{\Theta}{B}|\overline{K_\Theta^{\kappa^*+\kappa_\Theta-\kappa-\kappa_a}}\frac{B_{Z^*}}{b_a}|$$

Now, since $\overline{H^2_{+,\kappa-\kappa_B}} = b_a H^2_{-,\kappa_B-\kappa-\kappa_a} = \frac{b_a}{B_{Z^*}}\{H^2_{-,\kappa_B-\kappa-\kappa_a+\kappa^*} \oplus \mathfrak{M}_{\kappa_B-\kappa-\kappa_a+\kappa^*}\}$, we can continue the last chain of equalities in a following way,

$$1 > |P_{b_a H^2_{-,\kappa_B-\kappa-\kappa_a}}\frac{Bb_a}{\Theta B_{Z^*}}|K_\Theta^{\kappa^*+\kappa_\Theta-\kappa-\kappa_a}| =$$

$$|P_{\frac{b_a}{B_{Z^*}}\{H^2_{+,\kappa_B-\kappa-\kappa_a+\kappa^*}\}^\perp}\frac{Bb_a}{\Theta B_{Z^*}}|K_\Theta^{\kappa^*+\kappa_\Theta-\kappa-\kappa_a}| = |P_-^{\kappa_B-\kappa-\kappa_a+\kappa^*}\frac{B}{\Theta}|K_\Theta^{\kappa^*+\kappa_\Theta-\kappa-\kappa_a}|$$

$$= |P_-^{\kappa_B-\kappa-\kappa_a+\kappa^*}\frac{B}{\Theta}|H^2_{+,\kappa^*+\kappa_\Theta-\kappa-\kappa_a}|$$

But the last expression is exactly equal to the norm of Hankel operator $\mathbf{H}_{\frac{B}{\Theta}}$ on a space $H^2_{+,\kappa^*+\kappa_\Theta-\kappa-\kappa_a}$. Finaly, applying the same procedure to the first inequality from (2) and denoting $\hat\kappa = \kappa^* + \kappa_\Theta - \kappa - \kappa_a$, we get

$$|\mathbf{H}_{\frac{B}{\Theta}}| < 1 \quad on \quad H^2_{+,\hat\kappa}, \qquad |\mathbf{H}_{\frac{\Theta}{B}}| < 1 \quad on \quad H^2_{+,\hat\kappa-\kappa_\Theta+\kappa_B}.$$

■

[3)] Note that (see [6]) $L^\infty(\Gamma)$ can be identified with $L^\infty_\mu(\Gamma)$ for all characters μ.

4. Reduction to the estimation of an angle between subspaces in a weighted L_2 on the boundary

Let us denote the unimodular on Γ function B/Θ by ψ and let $\kappa_\psi = \kappa_B - \kappa_\Theta$. Following the approach in [1, 2] we consider the properties of the corresponding Teoplitz operators

$$\mathbf{T}_\psi = \mathbf{T}_\psi^{\kappa_\psi,\hat\kappa} = P_+^{\hat\kappa+\kappa_\psi}\psi|H^2_{+,\hat\kappa}$$
$$\mathbf{T}_{\bar\psi} = (\mathbf{T}_\psi)^* = P_+^{\hat\kappa}\overline{\psi}|H^2_{+,\hat\kappa+\kappa_\psi}.$$

The main goal of this section is to prove the following

Proposition 4.1. *If (4) holds then*

(1) *There exist two character-automorphic outer functions h_0, χ_0, such that*

$$\frac{B}{\Theta} = \psi = \frac{\overline{h_0}}{h_0}\frac{B_{Z^*}}{v_0v_0'} = \frac{\chi_0}{\overline{\chi_0}}\frac{w_0w_0'}{B_{Z^*}},$$

where v_0, v_0', w_0, w_0' are finite character-automorphic Blaschke products with the divisors of zeros V_0, V_0', W_0, W_0' respectively, $ordV_0+ordV_0' \le g$, $ordW_0+ordW_0' \le g$.

(2) *The function $h_0\chi_0$ is a character-automorphic meromorphic on $\hat\Omega$ function from the space $L_{2\hat\kappa+\kappa_\psi-\kappa^*}(V_0 + J(V_0') + W_0 + J(W_0') - Z^* - J(Z^*))$ and there exist two nonnegative divisors U_H, U_X, contained totally in Γ, such that $ordV_0+ordV_0'+ordU_H = g, ordW_0+ordW_0'+ordU_X = g$ and $(h_0\chi_0) = V_0 + J(V_0') + U_H + W_0 + J(W_0') + U_X - Z^* - J(Z^*)$.*

(3) *For any point $u \in \Gamma$ the expressions*

$$\frac{(z-u)^{k_u}}{h_0}, \ \frac{h_0}{(z-u)^{k_u}}, \ \frac{(z-u)^{k_u'}}{\chi_0}, \frac{\chi_0}{(z-u)^{k_u'}}$$

are locally square summable at u, where $k_u = ordU_H|_u$, $k_u' = ordU_X|_u$.

Proof. From (3) and (4) we see that $Ker\mathbf{T}_\psi = Ker\mathbf{T}_{\bar\psi} = \{0\}$. But this means that

$$Ran\mathbf{T}_\psi = \mathbf{T}_\psi(H^2_{+,\hat\kappa}) = H^2_{+,\hat\kappa+\kappa_\psi}, \quad Ran\mathbf{T}_{\bar\psi} = \mathbf{T}_{\bar\psi}(H^2_{+,\hat\kappa+\kappa_\psi}) = H^2_{+,\hat\kappa}.$$

Therefore the operators $\mathbf{T}_\psi$ and $\mathbf{T}_{\bar\psi}$ are invertible and for any elementary Blaschke factor $b_t,\ t\in\Omega_+$,

$$\begin{aligned} dimKer\mathbf{T}_{\frac{\psi}{b_t}} = 1,\ Ker\mathbf{T}_{\frac{\psi}{b_t}} &= (\mathbf{T}_\psi)^{-1}k_t^{\hat\kappa+\kappa_\psi} = \{h_tv_t \in H^2_{+,\hat\kappa}|\ \mathbf{T}_\psi h_tv_t = k_t^{\hat\kappa+\kappa_\psi}\}\\ dimKer\mathbf{T}_{\frac{\bar\psi}{b_t}} = 1,\ Ker\mathbf{T}_{\frac{\bar\psi}{b_t}} &= (\mathbf{T}_{\bar\psi})^{-1}k_t^{\hat\kappa} = \{\chi_tw_t \in H^2_{+,\hat\kappa+\kappa_\psi}|\ \mathbf{T}_{\bar\psi}\chi_tw_t = k_t^{\hat\kappa}\}. \end{aligned} \tag{5}$$

Here h_tv_t and χ_tw_t are character-automorphic inner-outer factorizations of functions $(\mathbf{T}_\psi)^{-1}k_t^{\hat\kappa+\kappa_\psi}$ and $(\mathbf{T}_{\bar\psi})^{-1}k_t^{\hat\kappa}$ respectively, with outer functions h_t, χ_t and inner functions v_t, w_t. Now, since $P_+^{\hat\kappa+\kappa_\psi}\psi h_tv_t = 0$ and $P_+^{\hat\kappa}\bar\psi\chi_tw_t = 0$,

$$\psi h_tv_t \in H^2_{-,\hat\kappa+\kappa_\psi} \oplus \mathfrak{M}_{\hat\kappa+\kappa_\psi} = B_{Z^*}H^2_{-,\hat\kappa+\kappa_\psi-\kappa^*},$$
$$\bar\psi\chi_tw_t \in H^2_{-,\hat\kappa} \oplus \mathfrak{M}_{\hat\kappa} = B_{Z^*}H^2_{-,\hat\kappa-\kappa^*},$$

and the absolute values of these functions on Γ are equal to $|h_t|$ and $|\chi_t|$ respectively. Thus, using the inner-outer factorization for Hardy spaces on Ω_-, we see that there exist inner functions v'_t, w'_t, such that

$$\begin{aligned}\frac{B_{Z^*}}{b_a}\overline{h_t v'_t} &= \frac{\psi}{b_t} h_t v_t \\ \frac{B_{Z^*}}{b_a}\overline{\chi_t w'_t} &= \frac{\bar{\psi}}{b_t}\chi_t w_t.\end{aligned} \tag{6}$$

Let us use the above reasoning for t equal to $a, z_1^*, \ldots, z_g^*$ and denote $h_a = h_0, v_a = v_0, v'_a = v'_0, \chi_a = \chi_0, w_a = w_0, w'_a = w'_0, h_{z_j^*} = h_j, \chi_{z_j^*} = \chi_j, v_{z_j^*} = v_j, v'_{z_j^*} = v'_j, w_{z_j^*} = w_j, w'_{z_j^*} = w'_j$. Then for the finite Blaschke product $B_{Z^*} b_a$

$$dimKer\mathbf{T}_{\frac{\psi}{B_{Z^*}b_a}} = g+1, \; Ker\mathbf{T}_{\frac{\psi}{B_{Z^*}b_a}} = \bigvee_{j=0}^{g}\{h_j v_j\}$$

$$dimKer\mathbf{T}_{\frac{\bar\psi}{B_{Z^*}b_a}} = g+1, \; Ker\mathbf{T}_{\frac{\bar\psi}{B_{Z^*}b_a}} = \bigvee_{j=0}^{g}\{\chi_j w_j\}$$

and (6) may be rewriten in the following form

$$\frac{B_{Z^*}}{b_a}\overline{h_j v'_j} = \frac{\psi}{b_{z_j^*}} h_j v_j, \quad \frac{B_{Z^*}}{b_a}\overline{\chi_j w'_j} = \frac{\bar\psi}{b_{z_j^*}}\chi_j w_j, \; j = 0,\ldots,g \tag{6$'$}$$

here for $j = 0$ we set $z_0^* = a$. Multiplying the right-hand side and left-hand side expressions from $(6')$for j=0, we get

$$B_{Z^*}\overline{h_0\chi_0 v'_0 w'_0} = \frac{h_0\chi_0 v_0 w_0}{B_{Z^*}} = \varphi_{00}. \tag{7}$$

On one hand $\varphi_{00} \in \overline{B_{Z^*}} H^1_{+,2\hat\kappa+\kappa_\psi} = H^1_{+,2\hat\kappa+\kappa_\psi-\kappa^*} + \mathfrak{M}_{2\hat\kappa+\kappa_\psi-\kappa^*}$ and on the other hand $\varphi_{00} \in b_a B_{Z^*} H^1_{-,2\hat\kappa+\kappa_\psi-\kappa_a-2\kappa^*} = H^1_{-,2\hat\kappa+\kappa_\psi-\kappa^*} + \mathfrak{M}_{2\hat\kappa+\kappa_\psi-\kappa^*} + k_a^{2\hat\kappa+\kappa_\psi-\kappa^*}$. Thus, by the analog of Riesz brothers theorem (see for example [6]), φ_{00} is annihilated by all functions from $H^\infty_{-,2\hat\kappa+\kappa_\psi-\kappa^*} + b_a H^\infty_{+,2\hat\kappa+\kappa_\psi-\kappa^*-\kappa_a}$. That is, again by analog of Riesz brothers theorem, $\varphi_{00} \in \mathfrak{M}_{2\hat\kappa+\kappa_\psi-\kappa^*} + k_a^{2\hat\kappa+\kappa_\psi-\kappa^*}$. Therefore, φ_{00} is a character-automorphic meromorphic on $\hat\Omega$ function and since h_0 and χ_0 are outer functions, the divisor of φ_{00} on $\hat\Omega$ satisfy the condition $(\varphi_{00}) > V_0 + W_0 + J(V'_0) + J(W'_0) - Z^* - J(Z^*)$, where V_0, W_0, V'_0, W'_0 are respectively the divisors of zeros of inner functions v_0, w_0, v'_0, w'_0. therefore all this inner functions are finite Blashke products and $ordV_0 + ordW_0 + ordV'_0 + ordW'_0 \le 2g$. Thus, $\varphi_{00} \in L_{2\hat\kappa+\kappa_\psi-\kappa^*}(V_0 + W_0 + J(V'_0) + J(W'_0) - Z^* - J(Z^*))$.

Moreover we can definitely state that $(\varphi_{00})|_{\Omega_+} = V_0 + W_0 - Z^*$ and $(\varphi_{00})|_{\Omega_-} = J(V'_0) + J(W'_0) - J(Z^*)$.

In the same way from $(6')$ we obtain that for $j = 1,\ldots,g$, inner functions v_j, v'_j, w_j, w'_j are finite Blaschke products with the divisors of zeros respectively

V_j, V_j', W_j, W_j' and

$$\begin{aligned}
\varphi_{0j} &= \frac{B_{Z^*} b_{z_j^*}}{b_a}\overline{h_0\chi_j v_0' w_j'} = \frac{h_0\chi_j v_0 w_j}{B_{Z^*}} \\
&\in L_{2\hat\kappa+\kappa_\psi-\kappa^*}(V_0 + W_j + J(V_0') + J(W_j') + J(a) - Z^* - J(Z^*) - J(z_j^*)) \\
&\subset \mathfrak{M}_{2\hat\kappa+\kappa_\psi-\kappa^*} + k_{z_j^*}^{2\hat\kappa+\kappa_\psi-\kappa^*}, \\
\varphi_{j0} &= \frac{B_{Z^*} b_{z_j^*}}{b_a}\overline{h_j\chi_0 v_j' w_0'} = \frac{h_j\chi_0 v_j w_0}{B_{Z^*}} \\
&\in L_{2\hat\kappa+\kappa_\psi-\kappa^*}(V_j + W_0 + J(V_j') + J(W_0') + J(a) - Z^* - J(Z^*) - J(z_j^*)) \\
&\subset \mathfrak{M}_{2\hat\kappa+\kappa_\psi-\kappa^*} + k_{z_j^*}^{2\hat\kappa+\kappa_\psi-\kappa^*}.
\end{aligned} \tag{7'}$$

Now let us consider the finite dimensional spaces

$$\begin{aligned}
H &= \bigvee_{j=0}^{g}\{\varphi_{0j}\} = \frac{h_0 v_0}{B_{Z^*}} Ker\mathbf{T}_{\frac{\bar\psi}{B_{Z^*}b_a}} \\
&\subset L_{2\hat\kappa+\kappa_\psi-\kappa^*}(V_0 + J(V_0') - Z^* - 2J(Z^*)), \quad dimH = g+1, \\
X &= \bigvee_{j=0}^{g}\{\varphi_{j0}\} = \frac{\chi_0 w_0}{B_{Z^*}} Ker\mathbf{T}_{\frac{\psi}{B_{Z^*}b_a}} \\
&\subset L_{2\hat\kappa+\kappa_\psi-\kappa^*}(W_0 + J(W_0') - Z^* - 2J(Z^*)), \quad dimX = g+1.
\end{aligned} \tag{8}$$

Let us consider the greatest common divisors D_H and D_X of all functions from H and X respectively. Obviously, by (8), $D_H = V_0 + J(V_0') - Z^* - 2J(Z^*) + U_H, U_H \geq 0$ and $D_X = W_0 + J(W_0') - Z^* - 2J(Z^*) + U_X, U_X \geq 0$. Now for sure $H \subset L_{2\hat\kappa+\kappa_\psi-\kappa^*}(D_H)$, $X \subset L_{2\hat\kappa+\kappa_\psi-\kappa^*}(D_X)$.

Let us show that in place of set inclusions in the last two expressions we actually have the set equalities. Let take arbitrary element $k \in L_{2\hat\kappa+\kappa_\psi-\kappa^*}(D_H)$ and let us show that $k \in H$. Note that all the functions from $L_{2\hat\kappa+\kappa_\psi-\kappa^*}(D_H)$ are locally analytic on Γ and for any $\varphi \in H$ the function φ/h_0 is square summable on Γ. This means that at any point $u \in \Gamma$ the expression $(z-u)^{k_u}/h_0,\ k_u = ordD_H|_u = ordU_H|_u$, is locally square summable at u. But since for any $u \in \hat\Omega$, $ord(k)|_u \geq ordD_H|_u$, we conclude that function k/h_0 is square summable on Γ and moreover, by the same reason, $B_{Z^*}k/(h_0v_0) \in H^2_{+,\hat\kappa+\kappa_\psi}$ and $k/(B_{Z^*}b_a\overline{h_0v_0'}) \in H^2_{-,\hat\kappa-\kappa_a-\kappa^*} \oplus \mathfrak{M}_{\hat\kappa-\kappa_a-\kappa^*}$. But, by (6′), $\bar\psi = h_0v_0/(\overline{h_0v_0'}B_{Z^*})$. Hence,

$$\mathbf{T}_{\frac{\bar\psi}{B_{Z^*}b_a}}\frac{kB_{Z^*}}{h_0v_0} = P_+^{\hat\kappa-\kappa_a-\kappa^*}\frac{h_0v_0}{\overline{h_0v_0'}B_{Z^*}^2b_a}\frac{kB_{Z^*}}{h_0v_0} = P_+^{\hat\kappa-\kappa_a-\kappa^*}\frac{k}{\overline{h_0v_0'}B_{Z^*}b_a} = 0.$$

Thus $\frac{kB_{Z^*}}{h_0v_0} \in Ker\mathbf{T}_{\frac{\bar\psi}{B_{Z^*}b_a}}$ and, therefore $k \in H$, which means that

$$H = L_{2\hat\kappa+\kappa_\psi-\kappa^*}(D_H).$$

In the same way we can show that $X = L_{2\hat\kappa+\kappa_\psi-\kappa^*}(D_X)$.

Note that ($6'$) implies that the divisors U_H and U_X are contained in Γ. Let us show that

$$\begin{aligned} ordV_0 + ordV_0' + ordU_H \geq g, \\ ordW_0 + ordW_0' + ordU_X \geq g. \end{aligned} \tag{9}$$

Indeed, using the Riemann-Roch theorem we see that if $ordV_0 + ordV_0' + ordU_H < g$ then $ord(-D_H) > 2g$ and hence $i^{-2\hat{\kappa}-\kappa_\psi+\kappa^*}(-D_H) = 0$. Therefore, again by Riemann-Roch theorem,

$$\begin{gathered} g + 1 = dimH = r^{2\hat{\kappa}+\kappa_\psi-\kappa^*}(D_H) = \\ 3g - ord(V_0 + J(V_)') + U_H) - g + 1 + i^{-2\hat{\kappa}-\kappa_\psi+\kappa^*}(-D_H) = \\ 2g + 1 - ord(V_0 + J(V_0') + U_H) > g + 1. \end{gathered}$$

Similarly the second inequality from (9) also holds.

Let us show now that the both inequalities in (9) are actually equalities. Suppose, for example, that $ordV_0 + ordV_0' + ordU_H > g$ and let U' be a positive subdivisor of $V_0 + J(V_0') + U_H$ of order $g + 1$, i.e. $0 < U' \leq V_0 + J(V_0') + U_H, ordU' = g + 1$. Then $i^{-2\hat{\kappa}-\kappa_\psi+\kappa^*}(Z^* + 2J(Z^*) - U') = 0$ and, by Riemann Roch theorem

$$\begin{gathered} g + 1 = dimH = r^{2\hat{\kappa}+\kappa_\psi-\kappa^*}(D_H) \leq r^{2\hat{\kappa}+\kappa_\psi-\kappa^*}(U' - Z^* - 2J(Z^*)) = \\ g + i^{-2\hat{\kappa}-\kappa_\psi+\kappa^*}(Z^* + 2J(Z^*) - U') = g. \end{gathered}$$

This contradiction shows that $ordV_0 + ordV_0' + ordU_H = g$. Similarly $ordW_0 + ordW_0' + ordU_X = g$.

Note now that $\varphi_{00} = \frac{h_0 v_0 \chi_0 w_0}{B_{Z^*}} \in H \cap X$ and therefore $\varphi_{00} \in L_{2\hat{\kappa}+\kappa_\psi-\kappa^*}(V_0 + V_0' + U_H - Z^* - J(Z^*)) \cap L_{2\hat{\kappa}+\kappa_\psi-\kappa^*}(W_0 + W_0' + U_X - Z^* - J(Z^*))$. Recall that we already know that $(\varphi_{00}) \geq V_0 + V_0' + W_0 + W_0' - Z^* - J(Z^*)$, that is $(\varphi_{00}) = V_0 + V_0' + W_0 + W_0' - Z^* - J(Z^*) + \tilde{U}$, where $\tilde{U}$ is nonnegative divisor contained totally in Γ, $\tilde{U} \geq U_H, \tilde{U} \geq U_X$. Thus if the greatest common divisor of U_H and U_X is zero then definitely

$$(\varphi_{00}) = V_0 + V_0' + U_H + W_0 + W_0' + U_X - Z^* - J(Z^*). \tag{10}$$

Let us show that (10) is still valid even if this greatest common divisor of U_H and U_X is greater then zero. We can represent (φ_{00}) in the following form

$$(\varphi_{00}) = V_0 + V_0' + W_0 + W_0' - Z^* - J(Z^*) + U_H + U_X - U_1 + U_2,$$

where U_1 is some nonnegative common divisor of U_H and U_X, and U_2 some nonnegative divisor, contained totally in Γ, $ordU_1 = ordU_2$. To prove (10) we need to show that $U_1 = U_2$.

Since, by ($6'$), for any $\varphi_H \in H$, $\varphi_H/h_0 \in L_2(\Gamma)$ and any $\varphi_X \in X$, $\varphi_X/\chi_0 \in L_2(\Gamma)$, for any measurable set $Y \subset \Gamma$

$$\int_Y \left| \frac{\varphi_H \varphi_X}{\varphi_{00}} \right| d\eta_a = \int_Y \left| \frac{\varphi_X}{\chi_0} \frac{\varphi_H}{h_0} \right| d\eta_a \leq \left\{ \int_Y \left| \frac{\varphi_X}{\chi_0} \right|^2 d\eta_a \right\}^{1/2} \cdot \left\{ \int_Y \left| \frac{\varphi_H}{h_0} \right|^2 d\eta_a \right\}^{1/2},$$

which means that at any $u \in \Gamma$

$$ord((\varphi_H)+(\varphi_X))|_u \geq ord(\varphi_{00})|_u = ord(U_H+U_X-U_1+U_2)|_u. \tag{11}$$

Since (11) holds for any $\varphi_H \in H, \varphi_X \in X$, we get

$$ord(U_H+U_X)|_u = ord(U_H+U_X-U_1+U_2)|_u, \tag{11'}$$

which means that $U_1 \geq U_2$. But since $ordU_1 = ordU_2$, we finally proved that $U_1 = U_2$. Therefore (10) is true.

From (10) we conclude that at any point $u \in \Gamma$

$$\frac{(z-u)^{k_u}}{h_0} \sim \frac{\chi_0}{(z-u)^{k'_u}},$$

where $k_u = ordU_H|_u, k'_u = ordU_X|_u$, and that the expressions

$$\frac{(z-u)^{k_u}}{h_0},\ \frac{h_0}{(z-u)^{k_u}},\ \frac{\chi_0}{(z-u)^{k'_u}},\ \frac{(z-u)^{k'_u}}{\chi_0}$$

are locally square summable at u. ■

Now we are able to reduce our problem to the problem from [8] on the estimation of an angle between subspaces in a weighted L_2 space on Γ.

Proposition 4.2. *If (4) holds then there exists a character-automorphic outer function $h^* \in H^2_{+,\nu^*}$ for some character ν^*, such that*

$$\frac{B}{\Theta} = \psi = \frac{\overline{h^*}}{h^*} B_{Z^*}.$$

That is

$$|\mathbf{H}_{\frac{\overline{h^*}}{h^*}B_{Z^*}}| < 1 \quad on \quad H^2_{+,\hat{\kappa}}$$

or, equivalently, the angle in the space $L_2(|h^*|^2 d\eta_a)$ on Γ between the subspaces $Clos_{L_2(|h^*|^2 d\eta_a)} H^\infty_{-,\hat{\kappa}-\nu^*}$ and $Clos_{L_2(|h^*|^2 d\eta_a)} H^\infty_{+,\hat{\kappa}-\nu^*}$ is nonzero.

Proof. By by Lemma 2.1, there exist the character-automorphic meromorphic on $\hat{\Omega}$ functions ψ_H and ψ_X with the divisors

$$(\psi_H) = J(V_0)+J(V'_0)+U_H-U^*, (\psi_X) = J(W_0)+J(W'_0)+U_X-U^{**},$$

where $U^* = u_1^*+\dots+u_g^*, U^{**} = u_1^{**}+\dots+u_g^{**}$ $u_j^*, u_j^{**} \in \Gamma_j$, $j = 1,\dots,g$. Therefore, $(\overline{\psi_H}/\psi_H) = V_0 - J(V_0)+V'_0-J(V'_0)$ and $(\overline{\psi_X}/\psi_X) = W_0-J(W_0)+W'_0-J(W'_0)$, which means that $\overline{\psi_H}/\psi_H = v_0 v'_0$ and $\overline{\psi_X}/\psi_X = w_0 w'_0$. Now the functions $h^* = h_0/\psi_H$ and $\chi^* = \chi_0/\psi_X$ are square summable on Γ and, moreover are the character-automorphic outer functions in Ω_+. By (7) we get

$$\begin{aligned} \frac{B}{\Theta} &= \psi = \frac{\overline{h_0}}{h_0}\frac{B_{Z^*}}{v_0 v'_0} = \frac{\overline{h^*}}{h^*}B_{Z^*}, \\ \frac{\Theta}{B} &= \overline{\psi} = \frac{\overline{\chi_0}}{\chi_0}\frac{B_{Z^*}}{w_0 w'_0} = \frac{\overline{\chi^*}}{\chi^*}B_{Z^*}. \end{aligned} \tag{12}$$

If we rewrite (4) using (12) we get

$$|\mathbf{H}_{\frac{\overline{h^*}}{h^*}B_{Z^*}}| < 1 \quad on \quad H^2_{+,\hat{\kappa}}, \qquad |\mathbf{H}_{\frac{\overline{\chi^*}}{\chi^*}B_{Z^*}}| < 1 \quad on \quad H^2_{+,\hat{\kappa}+\kappa_B-\kappa_\Theta}. \tag{13}$$

If h^* corresponds to the character ν^* then the first inequality from (13) is equivalent to the nonzero angle in $L_2(|h^*|^2 d\eta_a)$ between the subspaces $Clos_{L_2(|h^*|^2 d\eta_a} H^\infty_{-,\hat{\kappa}-\nu^*}$ and $Clos_{L_2(|h^*|^2 d\eta_a} H^\infty_{+,\hat{\kappa}-\nu^*}$. ■

5. Necessary and sufficient conditions

Note that, by construction of the function h^*, the expressions $(z-u)^k/h^*$ and $h^*/(z-u)^k, k = ordU^*|_u$, are locally square summable at $u \in \Gamma$, which means that the function $1/h^*$ may be not locally square summable only at points $u_1^*, \dots, u_g^*$ on Γ. Moreover, since for any measurable set Y on Γ, by Hölder inequality,

$$\int_Y \left|\frac{1}{z-u}\right| d\eta_a = \int_Y \left|\frac{h^*}{z-u}\frac{1}{h^*}\right| d\eta_a \leq \left\{\int_Y \left|\frac{h^*}{z-u}\right|^2 d\eta_a\right\}^{\frac{1}{2}} \cdot \left\{\int_Y \left|\frac{1}{h^*}\right|^2 d\eta_a\right\}^{\frac{1}{2}},$$

we see that $1/h^*$ definitely is not locally square summable at the points $u_1^*, \dots, u_g^*$. Thus the weight $w = |h^*|^2$ is exactly the weight from part (2) of [8, Proposition 4.2] with the divisor of "bad behaviour" on Γ exactly equal to U^*. By [8, Proposition 5.3],

$$|\mathbf{H}_{\frac{h^*}{\overline{h^*}}\overline{B_{Z^*}}}| < 1 \quad on \quad H^2_{+,\hat{\kappa}-2\nu^*+\kappa^*} = H^2_{+,\hat{\kappa}+\kappa_B-\kappa_\Theta}.$$

But since

$$\frac{h^*}{\overline{h^*}}\overline{B_{Z^*}} = \frac{\Theta}{B} = \frac{\overline{\chi^*}}{\chi^*}B_{Z^*},$$

we see that the first inequality from (13) implies the second one.

By [8, Theorem 6.1] the first inequality from (13) holds iff for any character-automorphic meromorphic on $\hat{\Omega}$ function ϕ with $(\phi)|_\Gamma = U^*$,

$$\sup_{\lambda\in\Omega_+} \int_\Gamma \left|\frac{h^*}{\phi}\right|^2 d\eta_\lambda \cdot \int_\Gamma \left|\frac{\phi}{h^*}\right|^2 d\eta_\lambda < \infty.$$

and

$$r^{\hat{\kappa}-\nu^*}[J(a) - \sum_{j=1}^g u_j^*] = dim L_{\hat{\kappa}-\nu^*}(J(a) - \sum_{j=1}^g u_j^*) = 0.$$

Thus we proved the following

Proposition 5.1. *If the orthogonal projection $P_{K^\kappa_\Theta}|K^\kappa_B$ from the the coinvariant subspace $K^\kappa_B = H^2_{+,\kappa} \ominus BH^2_{+,\kappa-\kappa_B}$ onto the coinvariant subspace $K^\kappa_\Theta = H^2_{+,\kappa} \ominus \Theta H^2_{+,\kappa-\kappa_\Theta}$ is an isomorphism then*

$$\frac{B}{\Theta} = \frac{\overline{h^*}}{h^*}B_{Z^*},$$

where h^* is an outer function from H^2_{+,ν^*} for some character $\nu^* \in \mathbb{R}^g/\mathbb{Z}^g$, which satisfy the following conditions

1. *Function $1/h^*$ is locally square summable at any point on Γ except of g points $u_1^* \in \Gamma_1, \dots, u_g^* \in \Gamma_g$, and at each of this points u_j^* the expressions $(z-u_j^*)/h^*$ and $h^*/(z-u_j^*)$ are locally square summable.*
2. *For any character-automorphic meromorphic on $\hat{\Omega}$ function ϕ with $(\phi)|_\Gamma = u_1^* + \dots + u_g^*$,*

$$\sup_{\lambda\in\Omega_+} \int_\Gamma \left|\frac{h^*}{\phi}\right|^2 d\eta_\lambda \cdot \int_\Gamma \left|\frac{\phi}{h^*}\right|^2 d\eta_\lambda < \infty.$$

(3)
$$r^{\hat{\kappa}-\nu^*}[J(a)-\sum_{j=1}^{g}u_j^*] = dimL_{\hat{\kappa}-\nu^*}(J(a)-\sum_{j=1}^{g}u_j^*) = 0,$$
where $\hat{\kappa} = k^* - \kappa_\Theta - \kappa - \kappa_a$.

Let us show that the statement of the last proposition could be reversed, i.e. that the following necessary and sufficient conditions are true

Theorem 5.1. *Let B and Θ be the inner functions corresponding respectively to the characters κ_B and κ_Θ. Then the orthogonal projection $P_{K^\kappa_\Theta}|K^\kappa_B$ from the the coinvariant subspace $K^\kappa_B = H^2_{+,\kappa}\ominus BH^2_{+,\kappa-\kappa_B}$ onto the coinvariant subspace $K^\kappa_\Theta = H^2_{+,\kappa}\ominus \Theta H^2_{+,\kappa-\kappa_\Theta}$ is an isomorphism if and only if*

(1)
$$\frac{B}{\Theta} = \frac{\overline{h^*}}{h^*}B_{Z^*},$$
where h^* is an outer function from H^2_{+,ν^*} for some character $\nu^* \in \mathbb{R}^g/\mathbb{Z}^g$.

(2) *Function $1/h^*$ is locally square summable at any point on Γ except of g points $u_1^* \in \Gamma_1, \dots, u_g^* \in \Gamma_g$, and at each of this points u_j^* the expressions $(z-u_j^*)/h^*$ and $h^*/(z-u_j^*)$ are locally square summable.*

(3) *For any character-automorphic meromorphic on $\hat{\Omega}$ function ϕ with $(\phi)|_\Gamma = u_1^* + \dots + u_g^*$,*
$$\sup_{\lambda\in\Omega_+}\int_\Gamma\left|\frac{h^*}{\phi}\right|^2 d\eta_\lambda\cdot\int_\Gamma\left|\frac{\phi}{h^*}\right|^2 d\eta_\lambda < \infty.$$

(4)
$$r^{\hat{\kappa}-\nu^*}[J(a)-\sum_{j=1}^{g}u_j^*] = dimL_{\hat{\kappa}-\nu^*}(J(a)-\sum_{j=1}^{g}u_j^*) = 0,$$
where $\hat{\kappa} = k^* - \kappa_\Theta - \kappa - \kappa_a$,

Proof. The implication $\Longrightarrow$ is contained in Proposition 5.1. Let us prove the implication $\Longleftarrow$.

By [8, Theorem 6.1] the angle in the space $L_2(|h^*|^2d\eta_a)$ between the subspaces $Clos_{L_2(|h^*|^2d\eta_a)}H^\infty_{-,\hat{\kappa}-\nu^*}$ and $Clos_{L_2(|h^*|^2d\eta_a)}H^\infty_{+,\hat{\kappa}-\nu^*}$ is nonzero. Thus, by [8, Proposition 5.3], not only
$$|\mathbf{H}_{\frac{\overline{h^*}}{h^*}B_{Z^*}}| < 1 \quad \text{on} \quad H^2_{+,\hat{\kappa}},$$
but also
$$|\mathbf{H}_{\frac{h^*}{\overline{h^*}}\overline{B_{Z^*}}}| < 1 \quad on \quad H^2_{+,\hat{\kappa}-2\nu^*+\kappa^*} = H^2_{+,\hat{\kappa}+\kappa_B-\kappa_\Theta}.$$

Now by reversing the way we used to get (4) from (2), we see that conditions (2) hold, which means that the desired projection is an isomorphism. ∎

Remark 5.1. By Lemma 2.1 we have reduced our problem to that one from [8] for the weight $|h^*|^2$, where the divisor $U^* =$ of "bed behaviour" on Γ of the outer function

h^* has the maximal possible order g. Therefore the condition (4) of Theorem 5.1 could be replaced by condition

$$r^{\hat{\kappa}-\nu^*}[\sum_{j=1}^{g} u_j^* + J(a) - Z^* - J(Z^*)] = 0, \tag{14}$$

which means that in $\mathfrak{M}_{\hat{\kappa}-\nu^*}$ there are no functions with the divisor of zeros greater then U^*. To obtain (14) we should first observe that

$$0 = r^{\hat{\kappa}-\nu^*}[J(a)-\sum_{j=1}^{g} u_j^*] = r^{-\hat{\kappa}+\nu^*}[a-\sum_{j=1}^{g} u_j^*] = i^{-\hat{\kappa}+\nu^*}[Z^*+J(Z^*)-J(a)-\sum_{j=1}^{g} u_j^*].$$

The second equality in the last chain of equalities is obtained by using the reflection by means of antiholomorphic involution J. The third one is valid since the multiplication of any function from $L_{-\hat{\kappa}+\nu^*}(a - \sum_{j=1}^{g} u_j^*)$ by the harmonic measure $d\eta_a$, which is a meromorphic ($\mathbf{0}$-automorphic) differential on $\hat{\Omega}$ with the divisor $(d\eta_a) = Z^* + J(Z^*) - a - J(a)$, will give us a differential from $\mathfrak{N}{-\hat{\kappa}+\nu^*}(Z^* + J(Z^*) - \sum_{j=1}^{g} u_j^* - J(a))$ and vice versa any such differential is obtained in this way. After that one should simply use the charcter-automorphic version of Riemann-Roch theorem.

On the other hand, now we can use Lemma 2.2 to find the character-automorphic meromorphic on $\hat{\Omega}$ function ϕ with the divisor $(\phi) = U^* - J(Q)$, where Q is positive divisor of order g contained totally in Ω_+. Since $\phi/\overline{\phi} = B_Q$, where as above B_Q is a finite Blaschke product with divisor of zeros equal to Q, by introducing the function $h = h^*/\phi$, we can rewrite Theorem 5.1 in the following way.

Theorem 5.2. *Let B and Θ be the inner functions corresponding respectively to the characters κ_B and κ_Θ. Then the orthogonal projection $P_{K_\Theta^\kappa}|K_B^\kappa$ from the the coinvariant subspace $K_B^\kappa = H^2_{+,\kappa} \ominus BH^2_{+,\kappa-\kappa_B}$ onto the coinvariant subspace $K_\Theta^\kappa = H^2_{+,\kappa} \ominus \Theta H^2_{+,\kappa-\kappa_\Theta}$ is an isomorphism if and only if*

(1)
$$\frac{B}{\Theta} = \frac{\overline{h}}{h}\frac{B_{Z^*}}{B_Q},$$
where h is an outer function from $H^2_{+,\nu}$ for some character $\nu \in \mathbb{R}^g/\mathbb{Z}^g$, and B_Q is a finite Blaschke product with divisor of zeros equal to Q, $ordQ = g$.

(2) *Function h satisfies the Muckenhoupt condition*
$$\sup_{\lambda\in\Omega_+} \int_\Gamma |h|^2\, d\eta_\lambda \cdot \int_\Gamma \left|\frac{1}{h}\right|^2 d\eta_\lambda < \infty.$$

(3)
$$r^{\hat{\kappa}-\nu}[Q + J(a) - Z^* - J(Z^*)] = dimL_{\hat{\kappa}-\nu}(Q + J(a) - Z^* - J(Z^*)) = 0,$$
that is there are no functions from $\mathfrak{M}_{\hat{\kappa}-\nu}$ with the divisor greater or equal to $Q + J(a) - Z^* - J(Z^*)$. where $\hat{\kappa} = k^* - \kappa_\Theta - \kappa - \kappa_a$,

Remark 5.2. It is possible to give an equivalent statement even when the order of divisor Q from Theorem 5.2 is an integer between 0 and g, but then the divisor of "bad behaviour" on Γ of function h will have the order $g - ordQ$, and statement would be a combination of statements of Theorems 5.1 and 5.2. We leave this simple exercise to the reader.

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