Metrization and semimetrization theorems with applications to manifolds

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Abstract

This paper is a study of conditions under which a space with S_2 is metrizable, o-semimetrizable or semimetrizable. It is shown that: a wMN, $w\gamma$-space is metrizable if and only if it has S_2, a quasi-γ-space is metrizable if and only if it is a pseudo wN-space with S_2, a separable manifold is metrizable if and only if it has S_2 with property $(*)$, a perfectly normal manifold with quasi-G^*_δ-diagonal is metrizable and a separable manifold is a hereditarily separable metrizable if and only if it has θ-α_2.

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1 Introduction

The study of metrization theory motivated the introduction of a new topological space, called a generalized metric space, and many such spaces have been characterized in terms of the so-called g-function. R.W. Heath introduced a method of describing a generalized metric property of a topological space (X,τ) by means of a function $g\colon \mathbb{N} \times X \to \tau$ and R.E.Hodel, P.Fletcher, W.F.Lindgren and J.Nagata have modified this method to obtain important new classes of spaces. We discuss these and other definitions in Section **??**.

In Section **??** we prove some characterizations of these properties in relation to metrizability. Our main results in this section are:

- a wMN, $w\gamma$-space is a wM-space if and only if it has S_2;
- a wMN, $w\gamma$-space is metrizable if and only if it has S_2;
- the following are equivalent for a regular wN-space X.
  (1) X is a Nagata space;
  (2) X has S_2;
  (3) X has G_δ-diagonal and pt-st-open C_2;
  (4) X is an α-space;
  (5) X is a Symmetric space;
- a quasi-γ-space is metrizable if and only if it is a pseudo wN-space with S_2;
- the following are equivalent for a regular $w\theta$-space X.
  (1) X is a metrizable;
  (2) X is a semi-stratifiable collectionwise normal space;
  (3) X is a wN-space with S_2;

Section **??** considers o-semi-developable, semi-developable, o-semimetrizable and semimetrizable. Our main results are:

- every $w\Delta_1$-space with S_2 is an o-semidevelopable space;
- the following are equivalent for a regular $w\Delta_1$-space X:
  (1) X is o-semimetrizable;
  (2) X is semi-stratifiable;
  (3) X is θ-refinable and has a G_δ-diagonal;

(4) X has a G^*_δ-diagonal;

(5) X has S_2;

- the following are equivalent for a regular $w\Delta_2$-space X:

  (1) X is semimetrizable;

  (2) X is semi-stratifiable;

  (3) X is θ-refinable and has a G_δ-diagonal;

  (4) X has a G^*_δ-diagonal;

  (5) X has α_2.

In section **??** we show some results in metrizibility of manifolds with generalized diagonal properties as following.

- a perfectly normal manifold with quasi-G^*_δ-diagonal is metrizable.
- a separable manifold is metrizable if and only if it has S_2 with property $(*)$.
- a separable hereditarily normal manifold is metrizable if and only if it has S_2.
- a separable manifold is a hereditarily separable metrizable if and only if it has θ-α_2.

2 Preliminary definitions and results

Let (X, τ) be a space, let $g\colon \mathbb{N} \times X \to \mathcal{P}(X)$ be a function and $\mathcal{G} = \{g(n, x) : n \in \mathbb{N}, x \in X\}$. We call $\mathcal{G}$ a *graded system of open covers* for X if the following conditions are satisfied:

(a) $g(n, x)$ is an open subset of X for all $n \in \mathbb{N}$ and $x \in X$.

(b) $x \in \bigcap_{n\in\mathbb{N}} g(n, x)$ for all $x \in X$.

(c) $g(n + 1, x) \subset g(n, x)$ for all $n \in \mathbb{N}$ and $x \in X$.

If $\mathcal{G}$ satisfies (b), (c) and

(d) A subset U of X is open if and only if for every $x \in U$ there is an $n \in \mathbb{N}$ such that $g(n, x)$ is contained in U,

then $\mathcal{G}$ is called a *graded weak base* for X, and a space with a graded weak base is called *weakly first countable.* If $\mathcal{G}$ satisfies (c) and

(e) $x \in Intg(n, x)$ for all $n \in \mathbb{N}$ and $x \in X$,

then $\mathcal{G}$ will be called a *semi-graded system.*

Lemma 2.1 *[?] Let $\{f(n, x) : x \in X, n \in \mathbb{N}\}$ be a graded system of open covers for a space X and let $\{g(n, x) : x \in X, n \in \mathbb{N}\}$ be a graded weak base for X. If $h(n, x) = f(n, x) \cap g(n, x)$, then $\{h(n, x) : x \in X, n \in \mathbb{N}\}$ is a graded weak base for X.*

Consider the following conditions on g.

(A) If $x \in g(n, x_n)$ for every $n \in \mathbb{N}$, then x is a cluster point of the sequence $\{x_n\}$.

(B) If for each $n \in \mathbb{N}$, $x \in g(n, y_n)$ and $y_n \in g(n, x_n)$, then x is a cluster point of the sequence $\{x_n\}$.

(C) If for each $n \in \mathbb{N}$, $\{x, x_n\} \subset g(n, y_n)$, then x is a cluster point of the sequence $\{x_n\}$.

(D) If for each $n \in \mathbb{N}$, $\{x, x_n\} \subset g(n, y_n)$ and $y_n \in g(n, x)$, then x is a cluster point of the sequence $\{x_n\}$.

(E) If for each $n \in \mathbb{N}$, $\{x, x_n\} \subset g(n, y_n)$ and the sequence $\{y_n\}$ has a cluster point, then x is a cluster point of the sequence $\{x_n\}$.

(F) If for each $n \in \mathbb{N}$, $x_n \in g(n, y_n)$ and the sequence $\{y_n\}$ converges in X, then the sequence $\{x_n\}$ has a cluster point.

(G) If for each $n \in \mathbb{N}$, $y_n \in g(n, x_n)$ and the sequence $\{y_n\}$ converges to x in X, then x is a cluster point of the sequence $\{x_n\}$.

(H) If for each $n \in \mathbb{N}$, $\{x, x_n\} \subset g(n, y_n)$ and $g(n,x) \cap g(n, x_n) \neq \emptyset$ then x is a cluster point of the sequence $\{x_n\}$.

(I) For each $n \in \mathbb{N}$, $x \in g(n, z_n)$, $g(n, z_n) \cap g(n, y_n) \neq \emptyset$, and $x_n \in g(n, y_n)$, then x is a cluster point of the sequence $\{x_n\}$.

(J) If for each $n \in \mathbb{N}$ and for all x, y in X, $y \in g(n,x)$ if and only if $x \in g(n,y)$ and $\bigcap_{n\in\mathbb{N}} \overline{g(n,x)} = \{x\}$.

(K) For each $n \in \mathbb{N}$ and for all x, y in X, $y \in g(n,x)$ if and only if $x \in g(n,y)$ and $\bigcap_{n\in\mathbb{N}} \; g(n,x) = \{x\}$.

Let S be any of the conditions (A), (B), (C), (D), (E), (F), (G), (H) or (I) and S^{-1} be the statement obtained by formally interchanging all memberships (e.g., C^{-1} is the condition: If for each $n \in \mathbb{N}$, $y_n \in g(n,x) \cap g(n, x_n)$, then x is a cluster point of the sequence $\{x_n\}$). If the graded system $\mathcal{G} = \{g(n,x) : n \in \mathbb{N}, x \in X\}$ of open covers satisfies condition S(resp. S^{-1}) for $S = (A)$, (B), (C), (D), (E), (F), (G), (H) or I, we say that g is an S-function (resp. S^{-1}-function) and that (X,τ) is an S-space (resp. S^{-1}-space). Corresponding to each of the conditions S above except (F) is the weaker condition, denoted wS, in which 'then x is a cluster point of the sequence $\{x_n\}$' is replaced by 'then the sequence $\{x_n\}$ has a cluster point'. If g satisfies wS, we say that g is a wS-function and that (X,τ) is a wS-space. wS^{-1}-functions and S^{-1}-spaces are defined analogously. The following are known, A =semi-stratifiable space, $B = \sigma$-space, C =developable space, $D = \theta$-space, $E = \Theta$-space, F =quasi-γ-space, G =strongly quasi Nagata space, $H = MN$-space, $I = M$-space, A^{-1} =first-countable space, $B^{-1} = \gamma$-space, C^{-1} =Nagata space, $wA = \beta$-space, $wB = w\sigma$-space, wC =$w\Delta$-space, $wD = w\theta$-space, $wE = w\Theta$-space, $wI = wM$-space, $wA^{-1} = q$-space, $wB^{-1} = w\gamma$-space, $wC^{-1} = wN$-space. For obvious reasons wH-spaces are called wMN-spaces.

Remark 2.2 *There is no loss of generality in conditions A, B, C, D, E, G, H and I in assuming that the sequence $\{x_n\}$ converges to x. (See [**?**, Remark 2.1])*

¿From the definitions of the spaces above we have the following implications:

$wM \Rightarrow wN \Rightarrow wMN \Rightarrow w\sigma \Rightarrow \beta$

$wM \Rightarrow w\gamma$

$w\Delta \Rightarrow wMN$

In [**?**, Remark 4.9] R.E.Hodel has showed:

Proposition 2.3 *Every β, $w\gamma$-space is a $w\Delta$-space.*

G.R.Hiremath, in [**?**], introduced the classes of spaces with S_2 and with S_1: A space (X,τ) is said to have S_2 (resp. S_1) if the graded system $\mathcal{G} = \{g(n,x) : n \in \mathbb{N}, x \in X\}$ of open covers satisfies condition J (resp. K).

The concept of a pseudo wN-space has been introduced by Y.Inui and Y.Kotake [**?**]. A space X is called a pseudo wN-space if it has a graded weak base $\mathcal{G}$ satisfying wS^{-1} when $S = (C)$. [Here we are not assuming that each $g(n,x)$ is open.]

In [**?**], G.R.Hiremath, proved the following results:

Theorem 2.4 *A space X has S_2 (resp. S_1) if and only if it admits a sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ of point-star-open covers such that for each $x \in X$, $\bigcap_{n\in\mathbb{N}} \overline{St(x,\mathcal{G}_n)} = \{x\}$ (resp. $\bigcap_{n\in\mathbb{N}} St(x,\mathcal{G}_n) = \{x\}$).*

Theorem 2.5 *In a space X with S_2 the following are equivalent:*

(1) X is a β-space.

(2) X is a semi-stratifiable space.

Definition 2.6 *A countable family $\{\mathcal{G}_n : n \in \mathbb{N}\}$ of open covers of a space X is called a G^*_δ-diagonal (resp. G_δ-diagonal), if for each $x \in X$ we have $\bigcap_{n\in\mathbb{N}} \overline{st(x,\mathcal{G}_n)} = \{x\}$ (resp. $\bigcap_{n\in\mathbb{N}} st(x,\mathcal{G}_n) = \{x\}$).*

Theorem 2.7 *A space X with G_δ-diagonal and point-star-open C_2 has S_2. (refer to [**?**] for a discussion of the definition of point-star-open C_2)*

Theorem 2.8 *For a space X the following are equivalent:*

(1) X is a regular $w\Delta$-space with S_2;

(2) X is a Moore space;

(3) X is a $w\Delta$-space with a G_δ-diagonal and point-star-open C_2.

Theorem 2.9 *For a space X the following are equivalent:*

(1) X is a wM-space with S_2;

(2) X is a wM-space with a G_δ-diagonal and point-star-open C_2;

(3) X is a Hausdorff wM-space such that the Cartesian product space X^2 is perfectly normal;

(4) X is a Hausdorff wM-space such that the Cartesian product space X^3 is hereditarily normal;

(5) X is a Hausdorff wM-space such that the Cartesian product space X^3 is hereditarily countably paracompact;

(6) X is metrizable.

Y.Inui and Y. Kotake [**?**] proved the following result:

Theorem 2.10 *The following are equivalent for a first countable space X:*

(1) X is a wN-space;

(2) X is a quasi-N-space;

(3) X is a pseudo N-space;

(4) X is a pseudo wN-space.

3 Metrizability results

Theorem 3.1 *A wMN, $w\gamma$-space is wM-space if and only if it has S_2.*

Proof. Let (X, τ) be a wMN, $w\gamma$-space with S_2 and let f be a wMN-function and g be a $w\gamma$-function for a space X. Assume that $f(n+1, x) \subset f(n, x)$ and $g(n+1, x) \subset g(n, x)$ for all $n \in \mathbb{N}$ and $x \in X$. Let $h: \mathbb{N} \times X \to \tau$be defined by $h(n, x) = f(n, x) \cap g(n, x)$. Let $p \in h(n, z_n)$, $t_n \in h(n, z_n) \cap h(n, y_n)$, $x_n \in h(n, y_n)$ for all $n \in \mathbb{N}$ and let us show that $\{x_n\}$ has a cluster point. Now from Proposition **??**, h is a $w\Delta$-function and so $\{p, t_n\} \subset g(n, z_n)$ for all $n \in \mathbb{N}$, implies that $\{t_n\}$ has a cluster point, say q.

Let $\{t_{n_k}\}$ be a subsequence of $\{t_n\}$ such that $t_{n_k} \in h(k, q)$, for all $k \in \mathbb{N}$. Then $q \in h(k, t_{n_k})$, so $t_{n_k} \in f(k, q) \cap f(k, y_{n_k})$ and $\{q, y_{n_k}\} \subset f(k, t_{n_k})$ for all $k \in \mathbb{N}$. This implies that the sequence $\{y_{n_k}\}$ has a cluster point, say r. Let $\{y_{n_{k_j}}\}$ be a subsequence of $\{y_{n_k}\}$ such that $y_{n_{k_j}} \in g(j, r)$ and $x_{n_{k_j}} \in g(j, y_{n_{k_j}})$ for all $j \in \mathbb{N}$. Then $\{x_{n_{k_j}}\}$ has a cluster point, so $\{x_n\}$ has a cluster point.

The converse is obvious. □

The following corollary is proved from Theorem **??** and Theorem **??**.

Corollary 3.2 *A wMN, $w\gamma$-space is metrizable if and only if it has S_2.*

Proposition 3.3 *Every q-space with S_2 is a first countable.*

Proof. Let f be a q-function and g be an S_2-function for a space (X, τ). For each x in X and n in $\mathbb{N}$, let $h(n, x) = f(n, x) \cap g(n, x)$. Let $x_n \in h(n, x)$. Then $\{x_n\}$ has a cluster point, say y.

Suppose that $y \neq x$. Since $g(n, x)$ is an S_2-function, choose k in $\mathbb{N}$, $n > k$ such that $q \notin \overline{g(k, x)}$. Then there is a number m in $\mathbb{N}$ such that $x_m \in X - \overline{g(k, x)}$. This gives $x_m \notin g(m, x)$ and which contradicts $x_n \in h(n, x)$ for all n. □

Theorem 3.4 *The following are equivalent for a regular wN-space X.*

(1) X is a Nagata space;

(2) X has S_2;

(3) X has G_δ-diagonal and point-star-open C_2;

*(4) X is an α-space. (refer to [**?**] for α-spaces);*

*(5) X is a Symmetric space. (refer to [**?**] for Symmetric spaces)*

Proof. It is clear that (1) implies all the other conditions. To prove the converse, it suffices to prove that X is semi-stratifiable.

To prove (2) $\Rightarrow$ (1). Note that every wN-space is a β-space. The implication is proved from Theorem **??** and [**?**, Theorem 1.2].The implication (3) $\Rightarrow$ (1) is due to Heremath [**?**]. To prove (4) $\Rightarrow$ (1), note that X is an α and β-space, so X is a semi-stratifiable space [**?**, Theorem 5.2]. The implication (5) $\Rightarrow$ (1) is due to Kotake [**?**, Theorem 1.3]. $\square$

Theorem 3.5 *A quasi-γ-space is metrizable if and only if it is a pseudo wN-space with S_2.*

Proof. Let X be a quasi-γ, pseudo wN-space with S_2. By Proposition **??**, X is first countable. From Theorem **??** and [**?**, Proposition 3.2], X is countably paracompact, so by result [**?**], X is regular. From Theorem **??**, X is N-space which is K-semi-stratifiable (refer to [**?**] for the K-semi-stratifiable spaces). Applying [**?**, Theorem 2], completes the proof. $\square$

Theorem 3.6 *The following are equivalent for a regular $w\theta$-space X:*

(1) X is a metrizable;

(2) X is a semi-stratifiable collectionwise normal space;

(3) X is a wN-space with S_2.

Proof. It is clear that (1) implies (2) and (3). For the implication (2) $\Rightarrow$ (1), follows from [**?**, Corollary 4.6: A space X is a Moore space if and only if it is a regular semi-stratifiable $w\theta$-space] and Bing's result [**?**, a collectionwise normal developable space is a metrizable]. To prove that (3) $\Rightarrow$ (1), we need to apply Theorem **??**, [**?**, Corollary 4.6] and [**?**, Theorem 3.7: Every Hausdorff developable wN-space is metrizable]. $\square$

Theorem 3.7 *Every regular $w\Delta$, wN-space with S_2 is metrizable.*

Proof. Since a Hausdorff developable wN-space is metrizable [**?**, Theorem 3.7], from [**?**, Corollary 2.6.1], the proof is complete. □

Theorem 3.8 *A regular subparacompact, $w\Delta$, wN-space is metrizable if and only if it has a quasi-G^*_δ-diagonal.*

Proof. Let X be a regular subparacompact, $w\Delta$, wN-space with quasi-G^*_δ-diagonal. ¿From Hodel's result [**?**, Corollary 3.5: every subparacompact T_1 wN-space is metacompact], [**?**, Theorem 2], and [**?**, Theorem 2.2], X is a developable. Applying Theorem **??** gives X metrizable. The converse is obvious. □

4 Semimetrizability results

Definition 4.1 *[?] Suppose $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is a countable family of covers of a space X satisfying the following property: if $x_n \in St(x, \mathcal{G}_n)$, then the sequence $\{x_n\}$ has a cluster point.*

(A) If, for each $n \in \mathbb{N}$, $\mathcal{G}_n$ is an open cover of X, then X is called a $w\Delta$-space.

(B) If, for each $x \in X$ and $n \in \mathbb{N}, St(x, \mathcal{G}_n)$ is an open subset of X, then X is called a $w\Delta_1$-space.

(C) If, for each $x \in X$ and $n \in \mathbb{N}, x \in IntSt(x, \mathcal{G}_n)$, then X is called a $w\Delta_2$-space.

Definition 4.2 *[?] Suppose $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is a countable family of covers of a space X. X has α_2 (resp. α_1), if for each $x \in X$ and $n \in \mathbb{N}, x \in IntSt(x, \mathcal{G}_n)$, and $\bigcap_{n\in\mathbb{N}} \overline{St(x, \mathcal{G}_n)} = \{x\}$ (resp. $\bigcap_{n\in\mathbb{N}} St(x, \mathcal{G}_n) = \{x\}$). If in addition, for each $x \in X, ord(x, \mathcal{G}_n) < \omega$ for all n in $\mathbb{N}$, Then X has θ-α_2 (resp. θ-α_1).*

We can use the same technique used in Theorem **??** to prove the following lemma.

Lemma 4.3 *A space X has α_2 (resp. α_1) if and only if there is a semi-graded system $\mathcal{G} = \{g(n, x) : n \in \mathbb{N}, x \in X\}$ such that for each $n \in \mathbb{N}$ and for all x, y in X, $y \in g(n, x)$ if and only if $x \in g(n, y)$ and $\bigcap_{n\in\mathbb{N}} \overline{g(n, x)} = \{x\}$(resp. $\bigcap_{n\in\mathbb{N}} g(n, x) = \{x\}$).*

Definition 4.4 *[?] A space X is semidevelopable if there exists a sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ of covers of X such that $\{St(x,\mathcal{G}_n) : n \in \mathbb{N}\}$ is a local system of neighborhoods at x. The sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is called a semidevelopment for X. If in addition, $\{St(x,\mathcal{G}_n) : n \in \mathbb{N}\}$ is an open base at x, then X is said to be o-semidevelopable. The sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is called an o-semidevelopment for X.*

Theorem 4.5 *Every $w\Delta_1$-space with S_2 is an o-semidevelopable space.*

Proof. Let $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be a countable family of covers of a space X illustrating that X is a $w\Delta_1$-space. Since X has S_2, there exists a sequence $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ of covers of a space X such that, for each $x \in X$ and $n \in \mathbb{N}$, $St(x,\mathcal{V}_n)$ is an open subset of X and $\bigcap_{n\in\mathbb{N}}\overline{St(x,\mathcal{V}_n)} = \{x\}$. For each $n \in \mathbb{N}$, let

$$\mathcal{U}_n = \{U : U = (\bigcap_{i=1}^{n} G_i) \cap (\bigcap_{i=1}^{n} V_i), G_i \in \mathcal{G}_n, V_i \in \mathcal{V}_n\}.$$

It is easy to see that $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$ for all $n \in \mathbb{N}$ and that, for each $x \in X$, $\bigcap_{n\in\mathbb{N}}\overline{St(x,\mathcal{U}_n)} = \{x\}$. Furthermore, for each $x \in X$ and $n \in \mathbb{N}$

$$St(x,\mathcal{U}_n) = (\bigcap_{i=1}^{n} St(x,\mathcal{G}_i)) \cap (\bigcap_{i=1}^{n} St(x,\mathcal{V}_i))$$

and thus $St(x,\mathcal{U}_n)$ is an open subset of X. Also it is easy to check that $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ is a $w\Delta_1$-sequence for X.

It remains to show that $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ is a semidevelopment for X. Suppose instead that $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ is not a semidevelopment for X. Then there is a point x, a neighborhood W of x, and a sequence $\{x_n\}$ such that for all $n, x_n \in St(x,\mathcal{U}_n)$ and $x_n \notin W$. Since $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ is a $w\Delta_1$-sequence for X, the sequence $\{x_n\}$ has a cluster point p. Clearly $p \notin W$ so $p \neq x$. Since $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ is an S_2-sequence for X, there is k in $\mathbb{N}$ and a neighborhood V of p such that $V \cap St(x,\mathcal{U}_k) = \emptyset$. Now for $n \geq k, x_n \in St(x,\mathcal{U}_n) \subset St(x,\mathcal{U}_k)$ and so $x_n \notin V$. This contradicts the fact that p is a cluster point of $\{x_n\}$. Thus $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ is a semidevelopment for X. □

Theorem 4.6 *The following are equivalent for a regular $w\Delta_1$-space X:*

*(1) X is o-semimetrizable. (refer to [**?**] for a discussion of the definition of o-semimetrizable spaces);*

(2) X is semi-stratifiable;

(3) X is θ-refinable and has a G_δ-diagonal;

(4) X has a G^*_δ-diagonal;

(5) X has S_2.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) follows from[**?**, Theorem 2.5]. It is easy to see that (4) $\Rightarrow$ (5). The implication (5) $\Rightarrow$ (1) follows from Theorem **??** and the fact every o-semidevelopable space is o-semimetrizable. □

Theorem 4.7 *The following are equivalent for a regular $w\Delta_2$-space X:*

*(1) X is semimetrizable. (refer to [**?**] for a discussion of the definition of semimetrizable spaces);*

(2) X is semi-stratifiable;

(3) X is θ-refinable and has a G_δ-diagonal;

(4) X has a G^*_δ-diagonal;

(5) X has α_2.

Proof. The only implication requiring comment is (5) $\Rightarrow$ (1). To prove (5) $\Rightarrow$ (1), let $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be a countable family of covers of a space X illustrating that X is a $w\Delta_2$-space. Let $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ be an α_2-sequence for X. Let the sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ be defined as in the proof of Theorem **??**. Since for each $x \in X$ and $n \in \mathbb{N}$,

$$IntSt(x,\mathcal{U}_n) = (\bigcap_{i=1}^{n} IntSt(x,\mathcal{G}_i)) \cap (\bigcap_{i=1}^{n} IntSt(x,\mathcal{V}_i)),$$

we have $x \in IntSt(x,\mathcal{U}_n)$. It follows, exactly as before, that $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ is a semidevelopment for X and thus X is semimetrizable by [**?**, Theorem 1.3]. □

5 Applications to Manifolds

In 1935 Alexandroff, then in 1949 Wilder, asked whether every perfectly normal manifold is metrizable. Rudin and Zenor [?], [?] showed that the answer depends on set theory. We show that by adding the condition that the manifold has a quasi-G^*_δ-diagonal suffices to ensure metrizability.

Definition 5.1 *[?] A countable family $\{\mathcal{G}_n : n \in \mathbb{N}\}$ of collections of open subsets of a space X is called a quasi-G^*_δ-diagonal, if for each $x \in X$ we have $\bigcap_{n \in C(x)} \overline{st(x, \mathcal{G}_n)} = \{x\}$ where $C(x) = \{n : x \in$ some $G \in \mathcal{G}_n\}$.*

Theorem 5.2 *[?] A perfect space X has a G^*_δ-diagonal if and only if it has a quasi-G^*_δ-diagonal.*

Proof. Every G^*_δ-diagonal is a quasi-G^*_δ-diagonal. Conversely suppose that X has a quasi-G^*_δ-diagonal. Then there is a countable family $\mathcal{G} = \{\mathcal{G}_n\}_{n \in \mathbb{N}}$ of collections of open subsets of X such that, for any distinct points $x, y \in X$, there exists $n \in \mathbb{N}$ such that

$$x \in \overline{st(x, \mathcal{G}_n)} \subset X - \{y\}.$$

For each $n \in \mathbb{N}$, the set $\mathcal{G}^*_n$ is open and hence, since X is perfect, $\mathcal{G}^*_n$ is an F_σ-set. Thus $\mathcal{G}^*_n = \bigcup_j F_{nj}$, where each F_{nj} is a closed subset of X, for each $j \in N$. For each ordered pair (n, j) of natural numbers, let

$$\mathcal{U}_{nj} = \mathcal{G}_n \cup \{X - F_{nj}\}.$$

Then $\{\mathcal{U}_{nj} : n, j \in \mathbb{N}\}$ is a countable family of open covers of X. For each $x \in X$ there is $n \in \mathbb{N}$ for which $x \in \mathcal{G}^*_n$. Then there is $j \in \mathbb{N}$ such that $x \in F_{nj}$. Thus $st(x, \mathcal{U}_{nj}) = st(x, \mathcal{G}_n)$ and hence $\bigcap_{n,j} \overline{st(x, \mathcal{U}_{nj})} = \{x\}$ so that X has a G^*_δ-diagonal. □

Theorem 5.3 *A perfectly normal manifold with quasi-G^*_δ-diagonal is metrizable.*

Proof. Immediate from Reed and Zenor's metrization theorem [?], [?, Theorem 2.15] and Theorem **??**. □

Definition 5.4 *A space X has S_2 with property $(*)$ if X has an S_2-sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ satisfying the following condition: for each $x, y \in X, y \in \overline{St(x, \mathcal{G}_n)}$ if and only if $x \in \overline{St(y, \mathcal{G}_n)}$, for all $n \in \mathbb{N}$.*

Definition 5.5 *A countable open cover $\mathcal{G}$ of a space X is said to be a countable separating (resp., strongly separating) open cover if, for every pair, $x, y \in X, x \neq y$, of points, there is a $G \in \mathcal{G}$ with $x \in G, y \notin G$ (resp., $x \in G, y \notin \overline{G}$).*

Z.Balogh and H.Bennett [?] proved that a locally compact, locally connected space with a point countable strongly separating open cover is metrizable. We give here an application of this result.

Theorem 5.6 *A separable, locally compact, locally connected space with S_2 with property $(*)$ is metrizable.*

Proof. Applying Z.Balogh and H.Bennett's result, we need just to prove that this space has a strongly separating open cover. Let D be a countable dense subset of X and let $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be an S_2-sequence. Let $\mathcal{G} = \{St(d, \mathcal{G}_n) : n \in \mathbb{N}, d \in D\}$. We show that $\mathcal{G}$ is a point countable strongly separating open cover. Let $x \in X$, then there is a $d \in D$ such that $d \in St(x, \mathcal{G}_1)$ which implies that $x \in St(d, \mathcal{G}_1)$ and $St(d, \mathcal{G}_1) \in \mathcal{G}$, so $\mathcal{G}$ is an open cover of X. Let $x, y \in X$ and $x \neq y$. Let m, k be such that $x \notin \overline{St(y, \mathcal{G}_m)}$ and $y \notin \overline{St(x, \mathcal{G}_k)}$. Let $n \geq m, k$, then $St(x, \mathcal{G}_n) \subset St(x, \mathcal{G}_k)$ and $St(y, \mathcal{G}_n) \subset St(y, \mathcal{G}_m)$, so $x \notin \overline{St(y, \mathcal{G}_n)}$ and $y \notin \overline{St(x, \mathcal{G}_n)}$. Since $St(x, \mathcal{G}_n) - \overline{St(y, \mathcal{G}_n)} \neq \emptyset$, there is a $d \in D$ such that $d \in St(x, \mathcal{G}_n) - \overline{St(y, \mathcal{G}_n)}$, so $d \in St(x, \mathcal{G}_n)$ and $d \notin \overline{St(y, \mathcal{G}_n)}$ which implies $x \in St(d, \mathcal{G}_n)$ and $y \notin \overline{St(d, \mathcal{G}_n)}$ and this completes the proof. □

Corollary 5.7 *A separable manifold is metrizable if and only if it has S_2 with property $(*)$.*

Remark 5.8 *In 1984 P.Nyikos [?] have constructed an example of a separable Moore manifold which is not metrizable.*

By using the same technique in the Theorem **??** and apply [?, Corollary 1.5], we have the following theorem.

Theorem 5.9 *A separable hereditarily normal manifold is metrizable if and only if it has S_2.*

In 1993 D. Gauld [**?**] constructed in CH a strongly hereditarily separable, nonmetrizable manifold. We show that by adding the condition that the manifold has θ-α_2 to separable suffices to ensure metrizability.

Lemma 5.10 *If $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is an α_2-sequence in a space X and $x_n \in St(x, \mathcal{G}_n)$ for each n and fixed $x \in X$, then either x is a cluster point of $\{x_n\}_{n\in\mathbb{N}}$ or $\{x_n\}_{n\in\mathbb{N}}$ does not cluster at all.*

Proof. There is no loss of generality if we assume that $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is a decreasing sequence. Then, $\overline{\{x_m : m \geq n\}} \subset \overline{St(x, \mathcal{G}_n)}$. Since $\bigcap_{n\in\mathbb{N}} \overline{St(x, \mathcal{G}_n)} = \{x\}$, either x is a cluster point of $\{x_n\}_{n\in\mathbb{N}}$ or $\{x_n\}_{n\in\mathbb{N}}$ does not cluster at all. □

In the following theorem we use the same technique as in [**?**, Theorem 2.15].

Theorem 5.11 *A separable manifold is hereditarily separable and metrizable if and only if it has θ-α_2.*

Proof. Let $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be θ-α_2-sequence for X. We show that $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is a point finite semi-development. We may assume that $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$. By passing to components, we may assume that each member of $\mathcal{G}_n$ is connected. Suppose that $x_n \in St(x, \mathcal{G}_n)$ for each n and fixed $x \in X$. By lemma **??** x is a cluster point of $\{x_n\}_{n\in\mathbb{N}}$ or $\{x_n\}_{n\in\mathbb{N}}$ does not cluster at all. Suppose it does not cluster at all. So there is a compact neighborhood U of x and $U \cap \{x_n\} = \emptyset$, so $St(x, \mathcal{G}_n)$ is not subset of U for each n. Then since $\overline{St(x, \mathcal{G}_n)}$ is connected, $\overline{St(x, \mathcal{G}_n)} \cap \partial U \neq \emptyset$ for each n. Since ∂U is compact, $\partial U \cap \bigcap_n \overline{St(x, \mathcal{G}_n)} \neq \emptyset$, a contradiction. Since every separable regular space with a point finite semi-development is hereditarily separable metrizable [**?**, Theorem 1.7, Proposition 1.12], the proof is done. □

We can prove the following theorem using the same technique.

Theorem 5.12 *Every manifold with S_2 is a o-semi-developable.*

The following open problems will be formulated for the manifold case, but can easily be generalized to the context of locally connected, locally compact spaces.

(1) Is every normal manifold with S_2 metrizable ?

(2) Is every hereditarily normal manifold with S_2 metrizable ?

(3) Is every separable normal manifold with S_2 metrizable ?

(4) Is every hereditarily normal manifold with a quasi-G_δ^*-diagonal metrizable ?

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References

[1] C.C.Alexander, Semi-developable spaces and quotient images of metric spaces, Pacific J. Math. 37, (1971), 277-293.

[2] C.E.Aull, A note on countably paracompact spaces and metrization, Proc. Amer. Math. Soc. 16, (1965), 1316-1317.

[3] Z.Balogh and H.Bennett, Conditions which imply metrizability in manifolds, Houston J. Math. 15, No. 2, (1989), 153-162.

[4] D. Gauld, A strongly hereditarily separable, nonmetrizable manifold, General Topology and its Appl. 51, (1993), 221-228.

[5] R.F.Gittings, On o-semimetrizable spaces, Studies in Topology, Academic Press, New York, (1975), 179-188.

[6] G.Gruenhage, Generalized metric spaces, Handbook of Set-theoretic Topology (1984), 423-501.

[7] G.R.Hiremath, Some characterizations of semimetrizability, metrizability, and generalized diagonal properties in terms of point -star-open covers, Far East J. Math. Sci. 4(1), (1996), 39-57.

[8] G.R.Hiremath, Generalization of G_δ Diagonal and Metrization, Proc. Japan Acad. 64 Ser. A, (1988), 288-291.

[9] R.E.Hodel, Spaces defined by sequences of open covers which guarantee certain sequences have cluster points, Duke Math. J. 39, (1972), 253-263.

[10] R.E.Hodel, Moore spaces and $w\Delta$-spaces, Pacific J. Math. 38, (1971), 641-652.

[11] Y.Inui and Y.Kotake, Metrization theorems for some generalized metric spaces, Q and A in General Topology 2, (1984), 147-155.

[12] Y.Kotake, On Nagata spaces and wN-spaces, Sc. Rep. T.K.D. Sect. A 12, (1973), 46-48.

[13] D.J.Lutzer, Semimetrizable and stratifiable spaces, General Topology and its Appl. 1, (1971), 43-48.

[14] H.W.Martin, Remarks on the Nagata-Smirnov metrization theorem, Topology (Proc. Conf. Memphis, Tennessee, 1975) Dekker, New York, (1976), 217-224.

[15] H.W.Martin, A note on the metrization of γ-spaces, Proc. Amer. Math. Soc. 57, (1976), 332-336.

[16] A.M.Mohamad, Generalization of G^*_δ-diagonals and $w\Delta$-spaces (to appear).

[17] P.Nyikos, The theory of nonmetrizable manifolds, Handbook of Set-theoretic Topology (1984), 633-684.

[18] M.Reed and P.Zenor, Metrization of Moore spaces and generalized manifolds, Fund. Math., 91, (1976), 203-210.

[19] M.Rudin, The undecidability of the existence of a perfectly normal nonmetrizable manifold, Houston J. Math. 5, No.2, (1979), 249-252.

[20] M.Rudin and P. Zenor, A perfectly normal nonmetrizable manifold, Houston J. Math. 2, No.1, (1976), 129-134.

[21] J.Sakong and M.Woo, On o-semidevelopable spaces, J. Korean Math. Soc. 17, (1981), 235-238.

[22] T.Shiraki, A note on spaces with a uniform base, Proc. Japan Acad. 47, (1971), 1036-1041.