Generalization of G^*_δ-diagonals and $w\Delta$-spaces

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Abstract

In this paper we introduce the concepts of a quasi-G^*_δ-diagonal and quasi-$w\Delta$-space as generalizations of the concepts of G^*_δ-diagonal and $w\Delta$-space respectively. It is shown that a quasi-Moore space may be characterised in terms of these concepts. As a consequence we obtain the following metrization theorems: every paracompact $w\Delta$-space with quasi-G_δ-diagonal is metrizable and every collectionwise normal σ quasi-$w\Delta$-space is metrizable.

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1 Introduction

The concept of G_δ-diagonal [6] has been generalized by several topologists to a concept called quasi-G_δ-diagonal [5].

A countable family $\{\, \mathcal{G}_n : n \in \mathbb{N} \,\}$ of open covers of a space X is called a ***G^*_δ*-diagonal**, if for each $x \in X$ we have $\bigcap_{n\in\mathbb{N}} \overline{st(x, \mathcal{G}_n)} = \{x\}$. A space X is said to be a ***$w\Delta$*-space** if there exists a countable family $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ of collections of open subsets of X such that for each x in X, each sequence $\{x_n\}_{n\in\mathbb{N}}$ for which $x_n \in st(x, \mathcal{G}_n)$ for each $n \in \mathbb{N}$ has a cluster point. A space X is **weakly *θ*-refinable** if every open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V} = \bigcup_{n\in\mathbb{N}}\{\mathcal{V}_n\}$ such that, given $x \in X$, some $\mathcal{V}_n$ has finite, positive order at x, which is the cardinality of the collection $\{ V \in \mathcal{V}_n : x \in V \}$. If, in addition, each $\mathcal{V}_n$ is required to cover X, then we have the familiar concept of **θ-refinability**. A space with a σ-locally finite net is called a **σ-space**. A **quasi-Moore space (Moore space)** is a regular, quasi-developable space (developable space). Our main theorem characterises quasi-Moore space as those regular quasi-$w\Delta$-spaces with quasi-G^*_δ-diagonal [see Definition2.1]. From this we can characterise developable spaces as those perfect quasi-$w\Delta$-spaces with quasi-G^*_δ-diagonal.

In section2 we define a quasi-G_δ-diagonal and related properties. We then give connections between these properties; in particular we show that for perfect spaces the concepts of having a G^*_δ-diagonal and having a quasi-G^*_δ-diagonal are the the same and quasi-$w\Delta$-spaces are $w\Delta$-spaces.

In section3 we state and prove the main results, including our characterisation of quasi-Moore spaces as those regular quasi-$w\Delta$-spaces with quasi-G^*_δ-diagonal, and our metrization theorems: every paracompact $w\Delta$-space with quasi-G_δ-diagonal is metrizable and every collectionwise normal σ quasi-$w\Delta$-space is metrizable.

We use the following notation: $\mathbb{N}$ = the set of natural numbers; $\overline{A}$ = the closure of a set A; $st(x, \mathcal{G}) = \bigcup\{ G : x \in G, G \in \mathcal{G} \}$, where $\mathcal{G}$ is a collection of open subsets of a space; $\mathcal{G}^* = \bigcup\{ G : G \in \mathcal{G} \}$;$ord(x, \mathcal{G})$ = the cardinality of the set of elements of $\mathcal{G}$ containing x.

2 Quasi-G_δ-diagonal, Quasi-G^*_δ-diagonal and Quasi-$w\Delta$-spaces

Let $\mathcal{G} = \{ \mathcal{G}_n : n \in \mathbb{N} \}$ be a countable family of collections of open subsets of a space X. Consider the following conditions on $\mathcal{G}$:

(a) For each $x \in X$, $\{ st(x, \mathcal{G}_n) : n \in \mathbb{N}, x \in \mathcal{G}^*_n \}$ is a local base at x;

(b) Each $\mathcal{G}_n$ is a covering of X;

(c) For any distinct $x, y \in X$, there exists $n \in \mathbb{N}$ such that

$$x \in st(x, \mathcal{G}_n) \subset X - \{y\};$$

(d) For any distinct $x, y \in X$, there exists $n \in \mathbb{N}$ such that

$$x \in st(x, \mathcal{G}_n) \subset X - \{y\} \text{ and } ord(x, \mathcal{G}_n) < \omega;$$

(e) For any distinct $x, y \in X$, there exists $n \in \mathbb{N}$ such that

$$x \in \overline{st(x, \mathcal{G}_n)} \subset X - \{y\};$$

(f) For any distinct $x, y \in X$, there exists $n \in \mathbb{N}$ such that

$$x \in \overline{st(x, \mathcal{G}_n)} \subset X - \{y\} \text{ and } ord(x, \mathcal{G}_n) < \omega.$$

Definition 2.1 *X has a quasi-G^*_δ-diagonal (θ-quasi-G^*_δ-diagonal) if there exists such a family $\mathcal{G}$ satisfying (e)[(f)]. Recall that X has a quasi-G_δ-diagonal (θ-quasi-G_δ-diagonal) if there exists such a family $\mathcal{G}$ satisfying (c)[(d)]; X is a quasi-developable space if there exists such a family $\mathcal{G}$ satisfying (a); X is a developable space if there exists such a family $\mathcal{G}$ satisfying (a) and (b).*

¿From the definition it is clear that if the space X has a quasi-G^*_δ-diagonal (θ-quasi-G^*_δ-diagonal) then X has a quasi-G_δ-diagonal (θ-quasi-G_δ-diagonal). Relationships between quasi-G^*_δ-diagonal, θ-quasi-G^*_δ-diagonal and G^*_δ-diagonal are given by the following theorems.

Theorem 2.2 *A perfect space X has a G^*_δ-diagonal if and only if it has a quasi-G^*_δ-diagonal.*

Proof. Every G^*_δ-diagonal is a quasi-G^*_δ-diagonal. Conversely suppose that X has a quasi-G^*_δ-diagonal. Then there is a countable family $\mathcal{G} = \{\mathcal{G}_n\}_{n\in\mathbb{N}}$ of collections of open subsets of X such that, for any distinct points $x, y \in X$, there exists $n \in \mathbb{N}$ such that

$$x \in \overline{st(x, \mathcal{G}_n)} \subset X - \{y\}.$$

For each $n \in \mathbb{N}$, the set $\mathcal{G}^*_n$ is open and hence, since X is perfect, $\mathcal{G}^*_n$ is an F_σ-set. Thus $\mathcal{G}^*_n = \bigcup_j F_{nj}$, where each F_{nj} is a closed subset of X, for each $j \in N$. For each ordered pair (n, j) of natural numbers, let

$$\mathcal{U}_{nj} = \mathcal{G}_n \cup \{X - F_{nj}\}.$$

Then $\{\mathcal{U}_{nj} : n, j \in \mathbb{N}\}$ is a countable family of open covers of X. For each $x \in X$ there is $n \in \mathbb{N}$ for which $x \in \mathcal{G}^*_n$. Then there is $j \in \mathbb{N}$ such that $x \in F_{nj}$. Thus $st(x, \mathcal{U}_{nj}) = st(x, \mathcal{G}_n)$ and hence $\bigcap_{n,j} \overline{st(x, \mathcal{U}_{nj})} = \{x\}$ so that X has a G^*_δ-diagonal. $\square$

Theorem 2.3 *Every θ-refinable space with G^*_δ-diagonal has a θ-G^*_δ-diagonal.*

Proof. Let X be a θ-refinable space and $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ be a countable family of open covers of X exhibiting the G^*_δ-diagonal property for X. For each $n \in \mathbb{N}$, let $\{\mathcal{V}_{nk}\}_{n,k\in\mathbb{N}}$ be a θ-refinement of $\mathcal{U}_n$. Then $\{\mathcal{V}_{nk}\}_{n,k\in\mathbb{N}}$ is a θ-G^*_δ-diagonal of X. □

Theorem 2.4 *Every hereditarily weak θ-refinable space with quasi-G^*_δ-diagonal has a θ-quasi-G^*_δ-diagonal.*

Remark 2.5 It is easy to check that part (e) in Definition 2.1 is equivalent to: for each $x \in X$, $\bigcap_{n\in c(x)} \overline{st(x,\mathcal{G}_n)} = \{x\}$, where $c(x) = \{n \in \mathbb{N} : x \in \mathcal{G}^*_n\}$.

The following theorem gives a relationship between having a quasi-G_δ-diagonal and having a quasi-G^*_δ-diagonal (moreover θ-quasi-G^*_δ-diagonal).

Theorem 2.6 *Every regular hereditarily weakly θ-refinable space with quasi-G_δ-diagonal has a quasi-G^*_δ-diagonal (moreover has a θ-quasi-G^*_δ-diagonal).*

Proof. Let X be a regular, hereditarily weakly θ-refinable space with a quasi-G_δ-diagonal. Then there is a countable family $\{\,\mathcal{G}_n : n \in \mathbb{N}\,\}$ of collections of open subsets of X such that for each distinct $x, y \in X$, there exists $n \in \mathbb{N}$ such that $x \in st(x,\mathcal{G}_n) \subset X - \{y\}$. For each $n \in \mathbb{N}, \mathcal{G}_n$ is an open cover of $\mathcal{G}^*_n$, a subspace of X. By regularity there is an open cover $\mathcal{U}_n$ of $\mathcal{G}^*_n$, such that $\overline{\mathcal{U}}_n = \{\,\overline{U} : U \in \mathcal{U}_n\,\}$ refines $\mathcal{G}_n$. Moreover, there exists an open refinement $\bigcup_j \mathcal{V}_{nj}$ of $\mathcal{U}_n$ such that for each $x \in \mathcal{G}^*_n$, there exists a $j \in \mathbb{N}$ such that $ord(x,\mathcal{V}_{nj}) < \omega$. Clearly $\bigcup_n \bigcup_j \mathcal{V}_{nj}$ is a σ-refinement of $\bigcup_n \mathcal{U}_n$, that is, $(\bigcup_j \mathcal{V}_{nj})^* = \mathcal{U}^*_n$ and $\mathcal{V}_{nj}$ is a refinement of $\mathcal{U}_n$. Since

$$\begin{aligned}\overline{st(x,\mathcal{V}_{nj})} = \overline{\bigcup\{\,V : V \in \mathcal{V}_{nj}, x \in V\,\}} &= \bigcup\{\,\overline{V} : V \in \mathcal{V}_{nj}, x \in V\,\}\\ &\subset st(x,\overline{\mathcal{U}_n}) \subset st(x,\mathcal{G}_n),\end{aligned}$$

it follows that $\{\,\mathcal{V}_{nj} : n, j \in \mathbb{N}\,\}$ is a quasi-G^*_δ-diagonal of X (moreover it is a θ-quasi-G^*_δ-diagonal of X).□

Bennett and Lutzer have shown that a quasi-developable space is hereditarily weakly θ-refinable [3, Proposition 7]. ¿From this and Theorem 2.2 we have:

Corollary 2.7 *Every regular quasi-developable space has a θ-quasi-G^*_δ-diagonal.*

Another interesting consequence of Theorem 2.6 is the following:

Corollary 2.8 *Every regular space which has a σ-point finite base, has a θ-quasi-G^*_δ-diagonal.*

Proof. Bennett and Lutzer showed that if a space has a σ-point finite base then it is quasi-developable [3, Corollary 10]. Our corollary follows immediately from this and Corollary 2.7. $\square$

Theorem 2.9 *A perfect manifold has a quasi-G^*_δ-diagonal if and only if it is a Moore manifold.*

Proof. Immediate from [6, Theorem 2.15] and Theorem 2.6. $\square$

Definition 2.10 *Let X be a space and $\mathcal{G} = \{\, \mathcal{G}_n : n \in \mathbb{N} \,\}$ a countable family of collections of open subsets of a space X. Then $\mathcal{G} = \{\, \mathcal{G}_n : n \in \mathbb{N} \,\}$ is called a quasi-$w\Delta$-sequence for X if and only if :*

(1) for all x, $c(x) = c_{\mathcal{G}_n}(x) = \{\, n \in \mathbb{N} : x \in \mathcal{G}^*_n \,\}$ is infinite.

(2) if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence with $x_n \in st(x, \mathcal{G}_n)$ for all $n \in c(x)$ then $\{x_n\}_{n\in\mathbb{N}}$ has a cluster point.

If the space X has a such countable family then it said to be a quasi-$w\Delta$-space.

It is clear that if X is a $w\Delta$-space then it is a quasi-$w\Delta$-space and every quasi-developable space is a quasi-$w\Delta$-space.

We now state and prove a relationship between quasi-$w\Delta$-spaces and $w\Delta$-spaces.

Theorem 2.11 *Every perfect quasi-$w\Delta$-space is a $w\Delta$-space.*

Proof. Let X be a perfect space and let $\mathcal{G} = \{\, \mathcal{G}_n : n \in \mathbb{N}\}$ be a countable family of collections of open subsets of a space X exhibiting the quasi-$w\Delta$-property for X. Then $\mathcal{G}_n^* = \bigcup_{j\in\mathbb{N}} F_{nj}$, where each F_{nj} is a closed subset of X. Let the countable family of open covers $\{\mathcal{U}_{nj}\}_{nj\in\mathbb{N}}$ be defined as in the proof of Theorem 2.2, so that $st(x, \mathcal{U}_{nj}) = st(x, \mathcal{G}_n)$. Suppose that $\{x_{nj}\}_{nj\in\mathbb{N}}$ is a sequence such that $x_{nj} \in st(x, \mathcal{U}_{nj})$ for each $n, j \in \mathbb{N}$. For each $n \in c(x)$ choose j_n such that $x \in F_{nj_n}$. Let $y_n = x_{nj_n} \in st(x, \mathcal{U}_{nj_n}) = st(x, \mathcal{G}_n)$. Then we have $y_n \in st(x, \mathcal{G}_n)$, for each $n \in c(x)$, so $\{y_n\}_{n\in c(x)}$ has a cluster point. Since $\{y_n\}_{n\in c(x)} \subset \{x_{nj}\}_{n,j\in\mathbb{N}}$ it follows that the sequence $\{x_{nj}\}_{n,j\in\mathbb{N}}$ has a cluster point.□

3 The main results

Hodel's theorem states that a space is developable if and only if it is a $w\Delta$-space with a G_δ^*-diagonal [4]. We now extend Hodel's theorem.

Theorem 3.1 *A space is a quasi-Moore space if and only if it is a regular quasi-$w\Delta$-space with quasi-G_δ^*-diagonal.*

Proof. Since the proof of the 'only if' part is evident, we prove the 'if' part. Let X be a space, $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ be a quasi-$w\Delta$-sequence for X, and $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ be a quasi-G_δ^*-diagonal sequence for X.

For each $n \in \mathbb{N}$, let $\mathcal{G}_n = \{\, G : G = U \cap V$, for some $U \in \mathcal{U}_n, V \in \mathcal{V}_m, m \geq n\}$. Then $\mathcal{G}_n^* \subseteq \mathcal{U}_n^*$ for all $n \in \mathbb{N}$. Suppose that $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is not a quasi-development for X. Then there is a point $x \in \mathcal{G}_n^*$, a neighborhood M of x and a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that, for all $n \in c(x)$, $x_n \in st(x, \mathcal{G}_n) - M$. Since $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ is a quasi-$w\Delta$-sequence, it follows that $\{\, x_n : n \in c_{\mathcal{U}}(x)\}$ has a cluster point p. Now $p \notin M$, since otherwise $x_n \in M$ for all but finitely many $n \in c_{\mathcal{U}}(x)$. Thus $p \neq x$. Choose n large enough such that $p \notin \overline{st(x, \mathcal{V}_n)}$; there is no loss of generality if we assume that $st(x, \mathcal{V}_m) \subset st(x, \mathcal{V}_n)$, for all $m \geq n$. This implies that $\bigcup_{m\geq n} st(x, \mathcal{G}_m) \subset \overline{st(x, \mathcal{G}_n)}$. For every $k \geq n$, we have $x_k \in st(x, \mathcal{G}_k)$ so $x_k \in \bigcup_{m\geq n} st(x, \mathcal{G}_m) \subset \overline{st(x, \mathcal{G}_n)}$, and $x_k \notin V = X - \overline{st(x, \mathcal{G}_n)}$ for all $k \geq n$. Now we have $p \in V$ and V open, which contradicts the fact that p is a cluster point of $\{x_n\}_{n\in\mathbb{N}}$. Thus $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is a quasi-development for X. □

Corollary 3.2 *A space is a developable if and only if it is a perfect, quasi-$w\Delta$-space with a quasi-G^*_δ-diagonal.*

The following theorem weakens the conditions of result 3.5 of [2]. By using Theorems 2.2 and 2.11 we obtain :

Theorem 3.3 *A regular hereditarily weakly θ-refinable space with quasi-G_δ-diagonal is a quasi-Moore space if it is a quasi-$w\Delta$-space.*

Corollary 3.4 *Every paracompact $w\Delta$-space with quasi-G_δ-diagonal is metrizable.*

From [2, Corollary 3.2] and Theorems 2.2 and 2.11 we have the following result:

Corollary 3.5 *A regular hereditarily θ-refinable $w\Delta$-space with quasi-G_δ-diagonal is a Moore space.*

Theorem 3.6 *Every collectionwise normal σ quasi-$w\Delta$-space is metrizable.*

Proof. Siwiec-Nagata [7], proved that a collectionwise normal σ $w\Delta$-space is metrizable. Since σ-spaces are perfect, the result follows using Theorem 2.11. □

Example 3.7

(1) Let M be the Michael line [5], the $LOTS$ M^* constructed from the generalized ordered (GO) space M has a quasi-G_δ-diagonal, but does not have a G_δ-diagonal, therefore has no G^*_δ-diagonal. From remark [5] and Theorem 2.2 M^* has a quasi-G^*_δ-diagonal.

(2) Since the space X in example 2.6 [1] is quasi-developable, θ-refinable, regular, but not developable, it follows that X is a quasi-$w\Delta$-space and from [2, Corollary 3.2], it is not a $w\Delta$-space.

The relationship between some of the classes of spaces considered in this paper can be summarized in a diagram as follows.

$$
\begin{tikzcd}
 & \text{developable} \arrow[dl] \arrow[dr] \arrow[dd] & \\
w\Delta\text{-space} \arrow[dr] & & \text{Quasi-developable} \arrow[dl] \\
\text{Perfect} + \arrow[u] & \text{Quasi-}w\Delta\text{-space} & + \text{ Quasi-}G^*_\delta\text{-diagonal} \arrow[u]
\end{tikzcd}
$$

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