FINITE INTERVALS IN THE LATTICE OF TOPOLOGIES

C. GOOD AND D.W. MCINTYRE

Abstract. We prove two basic facts about finite intervals in the lattice of topologies on a set. One result states that a finite lattice is isomorphic to an interval of topologies if and only if it is isomorphic to an interval of topologies on a finite set, the other that not every finite lattice is an interval of topologies, although every finite lattice may be embedded into the lattice of topologies on a finite set.

1. Introduction

The collection $\Sigma(X)$ of all topologies on a set X, when ordered by inclusion, forms a complete, bounded lattice, the join of any two topologies being the topology generated by their union and the meet being their intersection. Such lattices were first studied by Birkhoff [1] and have been examined in some detail since (see for example the extensive bibliography in the survey article by Larson and Andima [4]). Recently there has been a revival of interest in the structure of these lattices and the relationship between topologies in a given lattice (see, for example, [2], [11]).

Let us say that a lattice L is realized as an interval of topologies on a set X, or simply realized on X, if there are two topologies σ and τ on X such that the subinterval $[\sigma, \tau]$ of $\Sigma(X)$ is isomorphic to L. Two very natural and obvious questions, which nevertheless appear to be unanswered in the literature, seem to us to be fundamental in the study of $\Sigma(X)$: namely, can every lattice be realized as an interval of topologies on a set and can every finite lattice that is realized be realized on a finite set?

In this note we answer both these questions. The second is shown to have a positive answer in Theorem 1, which is then used to provide a negative answer to the first question.

It is well known that any lattice is isomorphic to a sublattice of $\Sigma(X)$ for some X (see [4]). In an informal communication with the authors, Watson conjectured that every lattice can be realized as an interval—Theorem 7 shows that this is not the case. Results of Valent and Larson [10] and Rosický [9] show that a finite lattice is distributive if and only if it can be realized as an interval of T_1 topologies (a topology

1991 *Mathematics Subject Classification.* 06B15, 54A10.
Key words and phrases. finite intervals of topologies, lattice of topologies, M_3.

is T_1 if all singleton sets are closed), and recently Knight, Gartside and the second author have shown that one can strengthen T_1 to Hausdorff [5]. Given, then, the M_3-N_5 Theorem, which, in part, states that a lattice is not distributive if and only if either M_3 or N_5 embed as a sublattice (see [3]), the first place to look for a counter-example is either N_5 or M_3. Example 5 shows that N_5 can indeed be realized; Theorem 7 shows that M_3 cannot.

Given that not every finite lattice can be realized as an interval of topologies, an obvious question is to characterize those finite lattices that are realized as intervals of topologies. One might also ask for a (computable) test for realization and non-realization. However, such a test cannot depend on the containment or otherwise of particular lattices, as Theorem 8, which follows easily from a result of Pudlák and Tůma[8], states that any finite lattice can be embedded in the lattice of topologies on a finite set. In Corollary 4 we show that if L can be realized then it can be realized as an interval in $\Sigma(X)$ for some finite set X (which can be determined in terms of L): thus it is decidable whether or not L is realizable.

Both questions have parallels in group theory and universal algebra. Pálfy [7] asks which finite lattices can be represented as intervals in the subgroup lattice of a finite group and which can be represented as congruence lattices of a finite algebra. Every finite lattice can be represented as a congruence lattice of an (infinite) algebra and every finite distributive lattice, though not every lattice, can be represented as an interval in the subgroup lattice of some soluble group. It is interesting to note that, in reverse of the topological situation, M_3 is the subgroup lattice of the Klein-4 group, whilst N_5 is not the lattice of subgroups of any group.

2. Notation

We shall use the following notation. If $\mathcal{A}$ and $\mathcal{B}$ are collections of subsets of X, then $\langle \mathcal{A}, \mathcal{B} \rangle$ is the topology generated by the collection $\mathcal{A} \cup \mathcal{B}$ (typically $\mathcal{A}$ will be a topology and we may write $\langle \mathcal{A}, B_1, \ldots, B_n \rangle$ for $\langle \mathcal{A}, \{B_1, \ldots, B_n\} \rangle$). When talking about a fixed interval of topologies, we shall denote the least element by σ and the greatest by τ. Given any subsets A and B of X and any topology α on X, we denote $(A \setminus \operatorname{int}_\sigma A)$ by A^*, we write $A \sim_\alpha B$ if $\langle \alpha, A \rangle = \langle \alpha, B \rangle$, and $\sim$ for $\sim_\sigma$. Clearly $\sim$ is an equivalence relation on $\tau \setminus \sigma$ for any topology τ refining σ, and we denote the equivalence class containing A by $[A]$. An element b of a lattice is said to *cover* a if $a < b$ and there is no c such that $a < c < b$. If a topology β covers α, then for any $B \in \beta \setminus \alpha$ we have $\beta = \langle \alpha, B \rangle$.

An element of a lattice is *meet-irreducible* if it is not the largest element of the lattice and it cannot be expressed as a meet of two strictly larger elements. The set of meet-irreducible elements of a lattice L is denoted by $M(L)$.

We shall use the notion of a 'Boolean' partition of a set X generated by a collection of subsets. Let $\mathcal{C}$ be such a collection and let χ_C be the characteristic function of the set C in $\mathcal{C}$. For any x and y in X, let $x \approx y$ if $\chi_C(x) = \chi_C(y)$ for every C in $\mathcal{C}$. Clearly $\approx$ is an equivalence relation.

M_3 is the lattice consisting of the set $\{s, a, b, c, t\}$ in which a, b and c cover s and t covers a, b and c. N_5 is the lattice consisting of the set $\{s, a, b, c, t\}$ in which a and c cover s, b covers a and t covers b and c.

Given a topology μ on a finite set X, one can form the *specialization order* $\leqslant_\mu$ on X, given by

$$x \leqslant_\mu y \qquad \Leftrightarrow \qquad x \in \overline{\{y\}}$$

(so $U \subseteq X$ is μ-open if and only if for every $x \in U$ and every $y \in X$ with $x \leqslant_\mu y$ we have $y \in U$). The specialization order is a preorder (in other words, a reflexive, transitive relation), and is a partial order if and only if μ is T_0 (i.e. for every pair of distinct points in X there is an open set containing one and not the other). The collection of all preorders on X, partially ordered by extension, is isomorphic to $\Sigma(X)$ via the isomorphism $\mu \mapsto \leqslant_\mu$.

If R is a preorder on X and $e_1, e_2, \ldots, e_n$ are ordered pairs of elements of X, then $R_{e_1, e_2, \ldots, e_n}$ denotes the preorder obtained by adding the pairs e_i to R for $1 \leqslant i \leqslant n$ (in other words, the transitive closure of $R \cup \{e_1, e_2, \ldots, e_n\}$).

3. Realizing finite lattices on finite sets

Theorem 1. *A finite lattice can be realized as a lattice of topologies if and only if it can be realized as a lattice of topologies on a finite set.*

To this end, let L be a finite lattice realized as the interval of topologies $[\sigma, \tau]$ on the set X. Since L is finite, $\sim$ induces a finite partition $\mathcal{P}$ on $\tau \setminus \sigma$. Choose sets A_i such that $\mathcal{A} = \{\, [A_i] : i \in n \,\}$ lists $\{\, [A] : A \in \tau \setminus \sigma \,\}$.

Let $X^* = \bigcup_{i \in n} A_i^*$. Clearly the Boolean partition of X^* generated by $\{\, A_i^* : i \in n \,\}$ is finite. Let X' contain exactly one point from each equivalence class of this partition and let $Y = (X \setminus X^*) \cup X'$. It is easy to see that the lattice of topologies $\{\, \alpha \restriction Y : \alpha \in [\overline{\sigma}, \overline{\tau}] \,\}$ is isomorphic to L, so we may assume that X^* is finite.

List the finite subsets of n as $\{\, F_r : r \in s \,\}$. Obviously, if $A \sim B$ then $A \sim A \cap B$, so we may assume that A_j is a subset of $\bigcap\{\, A_i : i \in F_r \,\}$ whenever $A_j \sim \bigcap\{\, A_i : i \in F_r \,\}$. If j is not in F_r, but A_j is in $\langle \sigma, \{\, A_i : i \in F_r \,\} \rangle$, then there is a finite subset $\mathcal{U}(j, r)$ of σ such that A_j is in $\langle \mathcal{U}(j, r), \{\, A_i : i \in F_r \,\} \rangle$. Let $\mathcal{U} = \bigcup\{\, \mathcal{U}(j, r) : j \in n \setminus F_r, r \in s \,\}$.

Suppose that $\mathcal{V}$ is any collection of sets from σ and that $\overline{\sigma}$ is the topology generated by $\{\, \mathrm{int}_\sigma(B) : B \in \langle \mathcal{U}, \mathcal{V}, \mathcal{A} \rangle \,\}$. Let $\overline{\tau}$ be the topology $\langle \overline{\sigma}, \mathcal{A} \rangle$ and let ${}^- : [\sigma, \tau] \to [\overline{\sigma}, \overline{\tau}]$ be the map defined by ${}^-(\alpha) = \overline{\alpha} = \langle \overline{\sigma}, \{\, A_i : A_i \in \alpha \setminus \sigma \,\} \rangle$.

Clearly, as $\mathcal{U}$ and $\{A_i : i \in \omega\}$ are both finite, we would be done were we able to choose a finite $\mathcal{V}$ for which ${}^-$ is a lattice isomorphism. This, however, does not seem to be possible. Instead we shall first determine conditions under which ${}^-$ is an isomorphism and then show that one can find a finite collection of decreasing nests of open sets which satisfy these conditions. With this $\mathcal{V}$ it is then simple to show that L can be realized on a finite set.

Now ${}^{-}(\sigma) = \overline{\sigma}$, ${}^{-}(\tau) = \overline{\tau}$ and $[\overline{\sigma}, \overline{\tau}]$ is a finite lattice. Moreover, the choice of $\mathcal{U}$ ensures that ${}^{-}$ is one-to-one and order-preserving. Also note that $\langle \overline{\sigma}, A_i : i \in F_r \cup F_t \rangle = \langle \overline{\sigma}, A_i : i \in F_r \rangle \vee \langle \overline{\sigma}, A_i : i \in F_t \rangle$ so ${}^{-}$ is join-preserving, i.e. $\overline{\alpha \vee \beta} = \overline{\alpha} \vee \overline{\beta}$.

Claim. *If ${}^{-}$ is not onto then there are α and $\beta = \langle \alpha, A_j \rangle$ in $[\sigma, \tau]$ and D such that either*

1. *β covers α and $D = U_0 \cup (A_j \cap U_1) \sim A_i$ for some A_i, U_0 and U_1 in $\overline{\alpha}$, but $D \not\sim_{\overline{\alpha}} A_i$, or*
2. *$\langle \alpha, D \rangle = \langle \alpha, A_j \rangle = \beta$ and $D = U_0 \cup (A_j \cap U_1) \sim A_j$ for some U_0 and U_1 in $\overline{\alpha}$, but $D \not\sim_{\overline{\alpha}} A_j$.*

If ${}^{-}$ is not onto, then there are α and β in $[\sigma, \tau]$ and δ in $([\overline{\sigma}, \overline{\tau}] \smallsetminus \mathrm{ran}({}^{-}))$ such that $\alpha < \beta$ and $\overline{\alpha} < \delta < \overline{\beta}$. Let $\{D_1, \ldots, D_m\}$ be such that $\delta = \langle \overline{\alpha}, \{D_1, \ldots, D_m\} \rangle = \bigvee_{j \leqslant m} \langle \overline{\alpha}, D_j \rangle$. Since δ is not in $\mathrm{ran}({}^{-})$ and ${}^{-}$ is join-preserving, $\langle \overline{\alpha}, D_j \rangle$ is not in $\mathrm{ran}({}^{-})$ for some $j \leqslant m$. So, without loss of generality, we may assume that δ covers $\overline{\alpha}$ and $\delta = \langle \overline{\alpha}, D \rangle$ for some set D. Note that, by the definition of $\overline{\sigma}$, D is not in σ and $D \sim A_i$, for some $i \in n$.

Now either β covers α or it does not. If it does cover α, then $\beta = \langle \alpha, A_j \rangle$ for some $j \in n$, $D = U_0 \cup (A_j \cap U_1)$ for some U_0 and U_1 in $\overline{\alpha}$ (since D is in $\overline{\beta}$, which is generated by adding A_j to $\overline{\alpha}$), and either $\langle \alpha, D \rangle = \alpha$ or $\langle \alpha, D \rangle = \beta$. If $\langle \alpha, D \rangle = \alpha$, then $D \sim A_i$ for some A_i in $\overline{\alpha}$ and, if $\langle \alpha, D \rangle = \beta$, then $D \sim A_j$. If β does not cover α, then, without loss of generality, we may assume that $\delta \vee \overline{\gamma} = \overline{\beta}$ and $\delta \wedge \overline{\gamma} = \overline{\alpha}$ for any γ in (α, β). (Otherwise, if $\delta \vee \overline{\gamma} = \overline{\epsilon}$, consider $[\alpha, \epsilon]$; if $\delta \vee \overline{\gamma}$ is not in $\mathrm{ran}({}^{-})$, consider $[\gamma, \beta]$; and similarly for $\delta \wedge \overline{\gamma}$.) Since $\delta \vee \overline{\gamma} = \overline{\beta}$ and $\delta = \langle \overline{\alpha}, D \rangle > \overline{\alpha}$, $\langle \overline{\gamma}, D \rangle = \overline{\beta}$. Hence D is not in α and $\langle \gamma, D \rangle = \beta$ for every $\gamma > \alpha$ and, in particular, $\langle \alpha, D \rangle = \beta$. Clearly, in this case $D \sim A_j$ for some $j \in n$, and, again, $D = U_0 \cup (A_j \cap U_1)$ for some U_0 and U_1 in $\overline{\alpha}$. This leaves us with the two possibilities listed above and proves the claim.

We shall choose the collection $\mathcal{V}$ so that neither of these cases occurs and $[\overline{\sigma}, \overline{\tau}]$ is isomorphic to L. Let $V_{i,0} = X$ for every $i \in n$ and suppose that we have chosen subsets $\mathcal{V}_m = \{\, V_{i,m} : i \in n \,\}$ of σ such that, for each $m \leqslant k$, $V_{i,m-1}$ contains A_i^* and is a subset of $V_{i,m}$. Let σ_k be the topology generated by $\{\, \mathrm{int}_\sigma(B) : B \in \langle \mathcal{U}, \bigcup_{m \leqslant k} \mathcal{V}_k, \mathcal{A} \rangle \,\}$ and let $\alpha_k = \langle \sigma_k, \{\, A_i : A_i \in \alpha \smallsetminus \sigma \,\} \rangle$. For each $i \in n$, let $\mathcal{U}_{i,k}$ be the collection $\{\, U \in \sigma_k : (U \cup A_i) \sim A_i \,\}$ and let $\mathcal{D}_{i,k}$ be the collection $\{\, D \in \langle \alpha_k, A_j \rangle : D = U_0 \cup (A_j \cap U_1) \sim A_i \in \alpha_k \subseteq \alpha, A_j \notin \alpha \,\}$. Since σ_k is finite, both $\mathcal{U}_{i,k}$ and $\mathcal{D}_{i,k}$ are finite. Hence there is some subset $V_{i,k+1}$ of $V_{i,k}$ in σ, containing A_i^*, such that $V_{i,k+1} \cap (U \cup A_i)$ is a subset of A_i and $V_{i,k+1} \cap A_i$ is a subset of D, for every U in $\mathcal{U}_{i,k}$ and D in $\mathcal{D}_{i,k}$. Let $\mathcal{V} = \{\, V_{i,k} : i \in n, k \in \omega \,\}$. Notice that by taking intersections and subsets if necessary, we may assume that $V_{i,k}$ is a (proper, if possible) subset of $V_{j,k}$, whenever A_i^* is a subset of $V_{j,k}$. Also note that we may assume that $V_{i,0} \cap X^*$ is a small as possible for each i.

Claim. $\overline{\sigma} = \bigcup_{k\in\omega} \sigma_k$.

Since $\{\, [A_i] : i \in n \,\}$ exhausts $\{\, [A] : A \in \tau \setminus \sigma \,\}$, for any $r \in s$, $\bigcap_{i\in F_r} A_i$ is either in σ or equivalent to, and a superset of, some A_j. Hence by our choice of the A_is and the definition of σ_0, either $\bigcap_{i\in F_r} A_i$ is in σ_0 or is equal to $A_j \cup U$ for some U in σ_0. This implies that any B in $\langle \mathcal{U}, \mathcal{V}, \mathcal{A} \rangle$ takes the form $U \cup \bigcup_{i\in n}(A_i \cap U_i)$, where each U is in $\langle \mathcal{U}, \mathcal{V} \rangle$ (and may be empty). Now every U in $\langle \mathcal{U}, \mathcal{V} \rangle$ takes the form $W \cup \bigcup_{t\in\omega} \bigcap_{i\in F_t} (V_{i,k_{i,t}} \cap W_{i,t})$, where the Ws are in $\mathcal{U}$ and each F_t is a subset of n. However, $\mathcal{U}$ is finite, each F_t is listed in $\{\, F_r : r \in s \,\}$, and $\{\, V_{i,k} : k \in \omega \,\}$ is a decreasing nest for each $i \in n$, so, in fact, U takes the form $W \cup \bigcup_{t\in t'} (\bigcup_{r\in s} \bigcap_{i\in F_r} (V_{i,k_{i,t}} \cap W_{t,r}))$ for some finite t'. Putting $k = \max\{\, k_{i,t} : i \in n, t \in t' \,\}$, we see that U is in $\langle \mathcal{U}, \mathcal{V}_k \rangle$ and hence that $\overline{\sigma} = \bigcup_{k\in\omega} \sigma_k$.

Claim. *For the collection $\mathcal{V}$ defined above, the map $^-$: $[\sigma, \tau] \to [\overline{\sigma}, \overline{\tau}]$ defined above is a lattice isomorphism.*

From the above, it suffices to show that $^-$ is onto; so suppose, for a contradiction, that $^-$ is not onto.

If case 2 holds, then $D = U_0 \cup (A_j \cap U_1) \sim A_j$, where U_0 and U_1 are in $\overline{\alpha}$ and $\langle \alpha, A_j \rangle = \beta$. There is some k such that U_0 and U_1 are in α_k. Now $U_0 \cup A_j = A_j \cup D \sim A_j$ and, by definition, there is some U in σ_{k+1} such that $U_0 \cup A_j = U \cup A_j$. But then

$$A_j^* \subseteq V_{i,k+1} \cap D = V_{i,k+1} \cap (U_0 \cup (A_j \cap U_1) \subseteq V_{i,k+1} \cap (U \cup A_j) \subset A_j.$$

Hence $A_j = (D \cap V_{i,k+1}) \cup \operatorname{int}_\sigma A_j$. However, this means that A_j is in $\langle \sigma_k, D \rangle$, contradicting the fact that $D \not\sim_{\overline{\alpha}} A_j$.

If case 1 holds, then $D = U_0 \cup (A_j \cap U_1) \sim A_i$, where A_i, U_0 and U_1 are in $\overline{\alpha}$ and $\langle \alpha, A_j \rangle = \beta$. There is some k such that U_0 and U_1 are in α_k so that D is in $\mathcal{D}_{i,k}$. In this case $V_{i,k+1} \cap A_i$ contains A_j^* and is a subset of D. As $\operatorname{int}_\sigma D$ is in σ_{k+1}, $D = (V_{i,k+1} \cap A_i) \cup \operatorname{int}_\sigma D$ is in α_{k+1}, again contradicting the fact that $D \not\sim_{\overline{\alpha}} A_i$ and proving the claim.

As $\mathcal{U} \cup \mathcal{A}$ is finite, it induces a finite Boolean partition on $X \setminus X^*$. Since $\mathcal{V}$ is a finite union of nests, there is some k for which the size of the partition of $X \setminus X^*$ generated by $\mathcal{V}_k$ is maximal but finite. Hence the size of the partition $\mathcal{P}$ generated by $\mathcal{U} \cup \mathcal{A} \cup \mathcal{V}_k$ is maximal but finite.

Let $X' = X^* \cup \mathcal{P}$ and let $q : X \to X'$ be the quotient map defined by $q(x) = x$, whenever x is in X^*, and $q(x) = P$ when $x \in P \in \mathcal{P}$. For each α in $[\overline{\sigma}, \overline{\tau}]$, let $Q(\alpha)$ be the quotient topology induced by α on the set X'. Clearly X' is a finite set, so we shall be done once we have established the following claim.

Claim. $Q : [\overline{\sigma}, \overline{\tau}] \to [Q(\overline{\sigma}), Q(\overline{\tau})]$ *is a lattice isomorphism.*

Q is certainly 1–1 and order-preserving. If B is in $Q(\overline{\beta}) \setminus Q(\overline{\alpha})$ then there is some B' in $\overline{\beta} \setminus \overline{\alpha}$. Conversely, if $\overline{\beta} \neq \overline{\alpha}$ then there is some A_i in $\overline{\beta} \setminus \overline{\alpha}$, but $\{\, q(x) : x \in A_i \,\}$

cannot lie in $Q(\overline{\alpha})$, since $\mathcal{P}$ distinguishes $A_i \smallsetminus X^*$ from all other sets in $\{ B \smallsetminus X^* : B \in \langle \mathcal{U}, \mathcal{A}, \mathcal{V}_k \rangle \}$.

To see that Q is onto, we first show that, for any $\overline{\alpha}$ in $[\overline{\sigma}, \overline{\tau}]$ and any D in $Q(\overline{\tau})$, $Q(\langle \overline{\alpha}, q^{-1}(D) \rangle) = \langle Q(\overline{\alpha}), D \rangle$ $(*)$. Since q is a quotient map, A is in $Q(\overline{\gamma})$ if and only if $q^{-1}(A)$ is in $\overline{\gamma}$. Now, if U is in $Q(\langle \overline{\alpha}, q^{-1}(D) \rangle)$ then $q^{-1}(U)$ takes the form $U_0 \cup (q^{-1}(D \cap U_1))$, where U_0 and U_1 are in $\overline{\alpha}$, so

$$U = q(U_0 \cup (q^{-1}(D) \cap U_1)) = q(U_0) \cup q(q^{-1}(D) \cap U_1).$$

However, $q^{-1}(D)$ is a union of equivalence classes, so U is equal to $q(U_0) \cup (D \cap q(U_1))$, which is in $\langle Q(\overline{\alpha}), D \rangle$. On the other hand, if V is in $\langle Q(\overline{\alpha}), D \rangle$ then it takes the form $V_0 \cup (D \cap V_1)$, where V_0 and V_1 are in $Q(\overline{\alpha})$. In this case V is equal to $q(q^{-1}(V_0) \cup (q^{-1}(D) \cap q^{-1}(V_1)))$, which is in $Q(\langle \overline{\alpha}, q^{-1}(D) \rangle)$.

Now suppose for a contradiction that Q is not onto. Then there must be some $\alpha, \beta \in [Q(\overline{\sigma}), Q(\overline{\tau})]$ such that β covers α, $\alpha \in \operatorname{ran}(Q)$ and $\beta \notin \operatorname{ran}(Q)$. Let $\overline{\alpha} \in [\overline{\sigma}, \overline{\tau}]$ with $Q(\overline{\alpha}) = \alpha$. Now $\beta = \langle \alpha, B \rangle$ for any $B \in \beta \smallsetminus \alpha$. Thus $\beta = \langle Q(\overline{\alpha}), B \rangle = Q(\langle \overline{\alpha}, q^{-1}(B) \rangle)$, so $\beta \in \operatorname{ran}(Q)$, a contradiction.

This completes both the proof of the claim and of the theorem. □

Having reduced the realizability of L to the case of finite sets, it is fruitful to consider L not as a lattice of topologies but as a lattice of preorders. The following result was proved by Watson and the second author in [6]:

Lemma 2. *Let S and T be preorders on a finite set X with $S \supseteq T$. If R is meet-irreducible in $[S, T]$ then there exist $x, y \in X$ with $R = T_{\langle x,y \rangle}$. Moreover, if $U \in (R, T]$ then $R = U_{\langle x,y \rangle}$.*

Conversely, if $x, y \in X$ with $x\ S\ y$, $x \not{T}\ y$ and $y \not{T}\ x$ then $T_{\langle x,y \rangle}$ is meet-irreducible in $[S, T]$.

Theorem 3. *Let L be a lattice. If L is realizable on a finite set, then it is realizable on a set of at most $2|M(L)|$ elements.*

Proof. Let σ, τ be topologies on X with $L \cong [\sigma, \tau]$. Let $S = \leqslant_\sigma$, $T = \leqslant_\tau$. By Lemma 2 we know that each meet-irreducible element a of $[S, T]$ is of the form $T_{\langle x_a, y_a \rangle}$ for some $x_a, y_a \in X$. Let $Y = \{ x_a : a \in M(L) \} \cup \{ y_a : a \in M(L) \}$. We will show that L is realized on Y as $[\sigma \restriction Y, \tau \restriction Y]$

For each $R \in [S, T]$, let $\overline{R} = R \restriction Y$.

Claim. *For any $e_1, e_2, \ldots, e_n$ in $Y \times Y$,*

$$\overline{T_{e_1,e_2,\ldots,e_n}} = \overline{T}_{e_1,e_2,\ldots,e_n}.$$

Since $\overline{T} \cup \{e_1, e_2, \ldots, e_n\} \subseteq \overline{T_{e_1,e_2,\ldots,e_n}}$, we have $\overline{T}_{e_1,e_2,\ldots,e_n} \subseteq \overline{T_{e_1,e_2,\ldots,e_n}}$. Conversely, let $z, w \in Y$ with $\langle z, w \rangle \in T_{e_1,e_2,\ldots,e_n}$: we must show that $\langle z, w \rangle \in \overline{T}_{e_1,e_2,\ldots,e_n}$. Since $\langle z, w \rangle \in T_{e_1,e_2,\ldots,e_n}$, there is a sequence $z_1, z_2, \ldots, z_m$ with $z_1 = z$, $z_m = w$ and for each i either $z_i\ T\ z_{i+1}$ or $\langle z_i, z_{i+1} \rangle \in \{e_1, e_2, \ldots, e_n\}$ (we refer to such a sequence as a *calculation* in $T_{e_1,e_2,\ldots,e_n}$). We might have $z_i \notin Y$ for some i. However, if this is the case then we must have $z_{i-1}\ T\ z_i$ and $z_i\ T\ z_{i+1}$, so $z_{i-1}\ T\ z_{i+1}$. Hence we can delete z_i from the sequence and it will still be a calculation in $T_{e_1,e_2,\ldots,e_n}$. Repeating this process if necessary, we can assume that all the z_i are in Y. But then the calculation is in fact a calculation in $\overline{T}_{e_1,e_2,\ldots,e_n}$, so $\langle z, w\rangle \in \overline{T}_{e_1,e_2,\ldots,e_n}$ as required.

Since every element of $[\overline{S}, \overline{T}]$ is a meet of meet-irreducibe elements, every element is of the form $\overline{T}_{e_1,e_2,\ldots,e_n}$ for some $e_1, e_2, \ldots, e_n$ in $Y \times Y$. Thus the above claim shows that the map given by $R \mapsto \overline{R}$ is onto. To show that it is one-to-one, we will show that if $e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_m \in Y \times Y$ with $\overline{T}_{e_1,e_2,\ldots,e_n} = \overline{T}_{f_1,f_2,\ldots,f_m}$ then $T_{e_1,e_2,\ldots,e_n} = T_{f_1,f_2,\ldots,f_m}$. So suppose this is not the case. Then, without loss of generality, there exist $r, s \in X$ with $< r, s\rangle \in T_{e_1,e_2,\ldots,e_n}$ but $\langle r, s\rangle \notin T_{f_1,f_2,\ldots,f_m}$. Let $r_1, r_2, \ldots, r_l$ be a calculation in $T_{e_1,e_2,\ldots,e_n}$ with $r_1 = r$ and $r_l = s$. This calculation must use at least one of the pairs e_i. Let $< r_i, r_{i+1} >$ be the first such pair in the sequence and $\langle r_j, r_{j+1}\rangle$ the last such. Then $\langle r_i, r_{j+1}\rangle \in \overline{T_{e_1,e_2,\ldots,e_n}} = \overline{T_{f_1,f_2,\ldots,f_m}}$, so there is a calculation $t_1, t_2, \ldots, t_k$ in $\overline{T_{f_1,f_2,\ldots,f_m}}$ with $t_1 = r_i$ and $t_k = r_{j+1}$. But then

$$r_1, r_2, \ldots, r_{i-1}, t_1, t_2, \ldots, t_k, r_{j+2}, r_{j+3}, \ldots, r_l$$

is a calculation in $T_{f_1,f_2,\ldots,f_m}$, so $\langle r, s\rangle \in T_{f_1,f_2,\ldots,f_m}$, a contradiction.

Thus the map $R \mapsto \overline{R}$ is a bijection. It is clearly order-preserving, so it is an isomorphism, and L is isomorphic to $[\overline{S}, \overline{T}]$, which is an interval of preorders on Y which has at most $2|M(L)|$ elements, as required. □

Combining the results of Theorems 1 and 3 we obtain the following:

Corollary 4. *A finite lattice L is isomorphic to an interval of topologies if and only if it is isomorphic to a subinterval of the lattice of topologies $[\{\varnothing, X\}, \mathcal{P}X]$ on some set X with $|X| = 2|M(L)|$.*

4. Not every finite lattice is realizable

Example 5. N_5 can be realized as an interval of topologies.

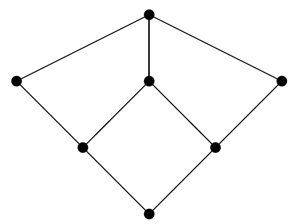
Proof. Let X be the three-point set $\{x, y, z\}$. Let σ be the topology generated by the sets X and $\{x, y\}$, and let τ be the topology generated by σ together with the sets $\{x\}$ and $\{z\}$. It is trivial to check that the interval $[\sigma, \tau]$ is isomorphic to N_5: let $\alpha = \langle\sigma, \{x\}\rangle$, $\beta = \langle\sigma, \{x, z\}\rangle$ and $\gamma = \langle\sigma, \{z\}\rangle$, then $\sigma < \alpha < \beta < \tau$ and $\sigma < \gamma < \tau$. □

As mentioned above, a finite interval of T_1 topologies is distributive, so it is not possible for N_5 to be realized as $[\sigma, \tau]$ with σ a T_1 topology. However, it is possible for σ to be T_0. To see this, let X be the set $\{a, b\} \cup \{\, c_n : n \in \omega\,\}$ and let C_k be the set $\{c_n : k \leqslant n\,\}$. Let σ be the topology generated by the collections $\{\, \{a\} \cup C_k : k \in \omega\,\}$ and $\{\, \{a, b\} \cup C_k : k \in \omega\,\}$, and let τ be the topology $\langle\sigma, \{a\}, \{b\}\rangle$. Again it is easy to check that $[\sigma, \tau]$ is isomorphic to N_5.

It is reasonable to ask whether every finite lattice that can be realized can be realized as an interval of T_0 topologies. Results due to Watson and the second author [6] show that N_5 cannot be realized as an interval of T_0 topologies on a finite set, so the requirement that σ be T_0 is incompatible with the requirement that the underlying set be finite and the reduction described in the proof of Theorem 1 destroys T_0 in general. On the other hand, we may always assume that the largest topology τ is T_0 (letting $x \approx y$ if and only if every τ-open set containing x contains y and vice versa and taking quotients of all topologies in $[\sigma, \tau]$ with respect to $\approx$ preserves the lattice).

Given that N_5 cannot occur as a interval between T_0 topologies on a finite set, it is tempting to conjecture that every interval between T_0 topologies is distributive. This is not the case, as is shown by the following example.

Example 6. The non-distributive lattice



is realizable as an interval between T_0 topologies on a finite set.

Proof. Let $X = \{1, 2, 3\}$, let $\sigma = \{\varnothing, \{3\}, \{2, 3\}, X\}$ and let τ be the discrete topology on X. The corresponding preorders are S, the usual linear order on X, and T, the trivial preorder. The preorders in $[S, T]$ are T, $T_{\langle 1,2\rangle}$, $T_{\langle 1,3\rangle}$, $T_{\langle 2,3\rangle}$, $T_{\langle 1,2\rangle,\langle 1,3\rangle}$, $T_{\langle 1,3\rangle,\langle 2,3\rangle}$ and $T_{\langle 1,2\rangle,\langle 2,3\rangle} = S$.

Notice that $\{S, T_{\langle 1,2\rangle,\langle 1,3\rangle}, T_{\langle 1,2\rangle}, T_{\langle 2,3\rangle}, T\}$ is a sublattice of $[S, T]$ which is isomorphic to N_5, so $[S, T]$ is non-distributive. $\square$

Notice that, by Theorem 1, any finite realizable lattice is isomorphic to a subinterval of $[\{\varnothing, X\}, \mathcal{P}(X)]$ for some finite set X. It is not hard to see that N_5 cannot be realized as $[\sigma, \tau]$ with either $\sigma = \{\varnothing, X\}$ or $\tau = \mathcal{P}(X)$.

Theorem 7. *Not every lattice is a lattice of topologies.*

Proof. Suppose, for a contradiction, that the interval of topologies $[\sigma, \tau]$ on the set X is isomorphic to M_3. By Theorem 1, we can assume that X is finite. Denote the three incompatible elements of the interval (σ, τ) by α, β and γ. There are sets A, B and C such that $\alpha = \langle\sigma, A\rangle$, $\beta = \langle\sigma, B\rangle$ and $\gamma = \langle\sigma, C\rangle$ and, since X is finite, we can assume that A, B and C are as small as possible.

We claim first that $A^* \cup B^* \cup C^*$ is a subset of $A \cap B \cap C$. Obviously, it is enough to show that $C^* \subseteq A \cap B$, $B^* \subseteq C \cap A$ and $A^* \subseteq B \cap C$, so let us suppose that this is not the case and that, without loss of generality, there is some point p in $C^* \setminus B$. C is a τ-open neighbourhood of p, $\tau = \langle\sigma, A, B\rangle$, and p is not in B, so there is some U

in σ such that $A \cap U$ contains p and is a subset of C. Since $A \cap U$ is in α and α covers σ, either $A \cap U$ is in σ or $A \cap U \sim A$. The first is impossible, since $p \in C^* \smallsetminus B$, so A, which is as small as possible, is equal to $A \cap U$ and is a subset of C. Now, if $C^* \smallsetminus A$ were empty, C^* would be a subset of A, which is in turn a subset of C, placing C in α. Hence $C^* \smallsetminus A$ is non-empty and, as for A, B is also a subset of C. The same argument shows that, if $A^* \smallsetminus B$ is non-empty, $A = C$, which is impossible. Hence $A^* \smallsetminus B$ is empty and A^* is a subset of B. As A is in $\langle \sigma, B, C\rangle$ and B is a subset of C, $A = U_0 \cup (B \cap U_1) \cup (C \cap U_2) \cup (B \cap C \cap U_3) = U_0 \cup (B \cap U_1) \cup ((B \cup \operatorname{int}_\sigma A) \cap U_2) \cup (B \cap U_3)$, which is in β—a contradiction. This proves that $A^* \cup B^* \cup C^*$ is a subset of $A \cap B \cap C$.

But now $A^* \subseteq A \cap B \subseteq A$ so $A \cap B$ generates α and, similarly, it generates β. Hence τ is generated from σ by each of $A \cap B$, $B \cap C$ and $C \cap A$. It is also the case that $A \cup B$, $B \cup C$ and $C \cup A$ are all σ-open. To see this, suppose that p is in $(A \cup B)^*$. Clearly $\operatorname{int}_\sigma A$ is a subset of $\operatorname{int}_\sigma (A \cup B)$, so if p is in A, it must be in A^*. As A^* is a subset of B, p is in $\operatorname{int}_\beta (A \cup B)$, but obviously B is a subset of $\operatorname{int}_\beta (A \cup B)$, so $A \cup B$ is in β. Similarly it is in α, so $A \cup B$ is in σ.

As C is in τ, which is generated by $A \cap B$, we have $C^* \subseteq A \cap B \cap U \subseteq C$ for some U in σ. Consider the σ-open set $W = U \cap ((A \cup C) \cap (B \cup C))$.

$$\begin{aligned} W &= U \cap ((A \cap B) \cup (C \cap B) \cup (C \cap A) \cup C) \\ &= U \cap ((A \cap B) \cup C) = (A \cap B \cap U) \cup (C \cap U). \end{aligned}$$

Clearly then, W is a subset of C, but as C^* is a subset of W, C is σ-open. This proves that M_3 (and, indeed, M_k for any $k \geqslant 2$) cannot occur as an interval of topologies. □

Given Theorem 7 and the fact that every lattice embeds as a subspace into the lattice of topologies on some set, it is natural to ask whether every finite lattice can be embedded into the lattice of topologies on a finite set. Pudlák and Tůma[8] have shown that every finite lattice can be embedded into the lattice of partitions (ordered by refinement). A partition generates a 'partition' topology (in the obvious way) and, given two partitions, it is easy to see that the join (meet) of their respective partition topologies is equal to the partition topologies of their join (meet). Hence:

Theorem 8. *Every finite lattice embeds as a sublattice of the lattice of topologies on a finite set.*

The proof in [8] does not explicitly describe the lattice of partitions into which a given finite lattice can be embedded; however, for M_3 one can do so simply. Let X be the set $\{1, 2, 3, 4\}$, σ the trivial topology, τ the discrete topology, and

$$\begin{aligned} \alpha &= \langle \sigma, \{1, 2\}, \{3, 4\}\rangle \\ \beta &= \langle \sigma, \{1, 3\}, \{2, 4\}\rangle \\ \gamma &= \langle \sigma, \{1, 4\}, \{2, 3\}\rangle \end{aligned}$$

To embed M_k, one can take a similar approach, using a finite affine plane with at least k sets of parallel lines.

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