BASIC INTERVALS IN THE PARTIAL ORDER OF METRIZABLE TOPOLOGIES

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Abstract. For a set X, let $\Sigma_m(X)$ denote the set of metrizable topologies on X, partially ordered by inclusion. We investigate the nature of intervals in this partial order, with particular emphasis on basic intervals (in other words, intervals in which the topology changes at at most one point).

We show that there are no non-trivial finite intervals in $\Sigma_m(X)$ (indeed, every such interval contains a copy of $\mathbb{P}(\omega)/\text{fin}$). We show that although not all intervals in $\Sigma_m(X)$ are lattices, all basic intervals in Σ_m are lattices. In the case where X is countable, we show that there are at least two isomorphism classes of basic intervals in $\Sigma_m(X)$, and assuming the Continuum Hypothesis there are exactly two such isomorphism classes.

1. Introduction

For a set X, Let $\Sigma(X)$ denote the collection of all topologies on X, partially ordered by inclusion. Then $\Sigma(X)$ is a complete, bounded lattice in which the meet of a collection of topologies is their intersection, while the join is the topology with their union as a subbasis. This lattice has been the subject of study since it was first defined by Birkhoff in [1]. Given $\sigma, \tau \in \Sigma(X)$, one can form the interval $[\sigma, \tau]$, defined by

$$[\sigma, \tau] = \{\, \mu \in \Sigma(X) \mid \sigma \leqslant \mu \leqslant \tau \,\}.$$

Our current research was motivated by the problem of identifying the finite lattices which can occur as such an interval, with various restrictions on the topologies σ and τ (for example that they be T_1, Hausdorff or metrizable). This problem was solved for σ and τ T_1 by Valent and

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Larson and Rosický: Valent and Larson showed in [7] that every finite distributive lattice occurs as an interval between T_1 topologies, and Rosický showed in [6] that every finite interval between T_1 topologies is distributive. More recently, Knight, Gartside and the first author extended this result to intervals between T_2 topologies [4], and Good and the authors extended this to intervals between T_3 (indeed, Hausdorff and zero-dimensional) topologies, assuming the existence of infinitely many measurable cardinals. Of course, if σ is T_1 (resp. Hausdorff) then every topology in $[\sigma, \tau]$ must be T_1 (resp. Hausdorff). However, if we refine a metrizable topology, it may not remain metrizable. This leads us to the problem of determining the structure of the set of *metrizable* topologies between two metrizable topologies σ and τ.

Let $\Sigma_m(X)$ and $\Sigma_i(X)$ denote, respectively, the sets of metrizable topologies and of T_i topologies ($i = 1, 2, 3$) in $\Sigma(X)$. When the underlying set X is clear from the context, we will omit mention of it, and simply write Σ, Σ_m or Σ_i. Σ_m, Σ_3 and Σ_2 are not lattices, as the following Example shows:

Example 1. *There exist a set X and zero-dimensional metrizable topologies σ and τ on X such that $\sigma \wedge \tau$ is not Hausdorff.*

Proof. Let $X = \omega \cup \{p, q\}$, where $p \neq q$ and $p, q \notin \omega$. Let μ be the topology $\mathbb{P}(\omega) \cup \{\, X \smallsetminus F \mid F \text{ is finite} \,\}$ on X. Let σ and τ be the topologies obtained from μ by isolating p and by isolating q respectively. Then $\sigma, \tau \in \Sigma_m$ and both are zero-dimensional. However, $\sigma \wedge \tau = \mu$, which is not Hausdorff. $\square$

Since Σ_m is not a lattice, it is possible that intervals in Σ_m are not lattices. Indeed, we will show that there are intervals in Σ_m which are not lattices. On the other hand, basic intervals in Σ_m (in other words intervals in which the topology changes at a most one point) are sublattices of Σ.

For subsets A and B of ω we write $A \subseteq^* B$ if $A \smallsetminus B$ is finite, and $A =^* B$ if $A \subseteq^* B$ and $B \subseteq^* A$. Then $=^*$ is an equivalence relation on $\mathbb{P}(\omega)$, and $\subseteq^*$ induces a partial order on the quotient with respect to $=^*$. This partially ordered set is a Boolean algebra, denoted by $\mathbb{P}(\omega)/\text{fin}$.

We will see in Section 2 that no interval in Σ_m is finite, or even countable, by showing that we can embed $\mathbb{P}(\omega)/\text{fin}$ in any such interval. In Section 3 we will consider basic intervals in $\Sigma_m(X)$ where X is countable, and, under the assumption of the Continuum Hypothesis (CH), show that there are up to isomorphism exactly two such intervals.

For $\sigma, \tau \in \Sigma_m$, we will denote $[\sigma, \tau] \cap \Sigma_m$ by $[\sigma, \tau]_m$. If $\mathcal{F}$ is a family of subsets of X, then $\langle \mathcal{F} \rangle$ denotes the topology on X with $\mathcal{F}$ as a subbasis. We abbreviate $\langle \sigma \cup \{A_1, \ldots, A_n\} \rangle$ by $\langle \sigma, A_1, \ldots, A_n \rangle$.

If d is a metric on X, then ρ_d denotes the topology generated by d, and, for $x \in X$ and $\varepsilon > 0$, $B(x, d, \varepsilon)$ denotes the ε-ball about x with respect to the metric d. If $f : X \to \mathbb{R}$ is a function, then ψ_f is the pseudometrizable topology on X with pseudometric e, where $e(x, y) = |f(x) - f(y)|$. Notice that ψ_f is the coarsest topology on X which makes f continuous, and that a topology τ on X is completely regular if and only if there is some family $\mathcal{F}$ of functions from X to $\mathbb{R}$ such that $\tau = \langle \bigcup_{f \in \mathcal{F}} \psi_f \rangle$.

We will abbreviate phrases like "open with respect to the topology σ" and "continuous with respect to the topology σ" by σ-open and σ-continuous respectively. To avoid ambiguity, we will refer to a countable union of locally finite sets as being sigma-locally finite.

2. Intervals in Σ_m

It is easy to show that the join of two metrizable topologies is metrizable:

Proposition 1. *Let d, e be metrics on a set X. Then $\max(d, e)$ is a metric on X, and $\rho_{\max(d,e)} = \rho_d \vee \rho_e$ (where the join refers to the lattice Σ, not just the subset Σ_m).*

Corollary 1. *Let $\sigma, \tau \in \Sigma_m$. Then $\sigma \leqslant \tau$ if and only if there are metrics d and e such that $\sigma = \rho_d$, $\tau = \rho_e$ and, for every $x, y \in X$, $d(x, y) \leqslant e(x, y)$*

On the other hand, the meet of two metrizable topologies need not be metrizable, as shown by Example 1. Even if there is a metrizable topology coarser than both topologies, they might have no meet in Σ_m, as shown by the following examples. In the light of Example 3, Example 2 is redundant: however we have included it to clarify the argument in the latter Example.

Example 2. *There exist metrizable topologies σ, μ and ν on a set X such that $\sigma \leqslant \mu \wedge \nu$, but $\{\mu, \nu\}$ has no greatest lower bound in Σ_m.*

Proof. Let R denote the set $\mathbb{R}$ of real numbers with its usual topology, and let R_d denote $\mathbb{R}$ with the discrete topology. Let $X = \mathbb{R} \times \mathbb{R}$. Let $(X, \sigma) = R \times R$, $(X, \mu) = R \times R_d$ and $(X, \nu) = R_d \times R$. Then $\sigma, \mu, \nu \in \Sigma_m$ and $\sigma \leqslant \mu \wedge \nu$.

For $\mathbf{x} = (x, y) \in X$, $\varepsilon > 0$, let $P_\varepsilon(\mathbf{x}) = (\{x\} \times B_\varepsilon(y)) \cup (B_\varepsilon(x) \times \{y\})$, where $B_\varepsilon(x)$ denotes the ε-ball about x in the usual metric. Observe that the sets $P_\varepsilon(\mathbf{x})$ for $\varepsilon > 0$ and $\mathbf{x} \in X$ form a weak neighbourhood base for $\mu \wedge \nu$ (in other words, although the sets $P_\varepsilon(\mathbf{x})$ are not open in $\mu \wedge \nu$, a subset U of X is open in $\mu \wedge \nu$ if and only if for every $\mathbf{x} \in U$, there is some $\varepsilon > 0$ such that $P_\varepsilon(\mathbf{x}) \subseteq U$).

By a pigeonhole principle argument, one can easily see that any disjoint collection of sets $P_\varepsilon(\mathbf{x})$ must be countable, so $\mu \wedge \nu$ has the countable chain condition (CCC). However, the diagonal $\Delta = \{\, (x,x) \mid x \in \mathbb{R} \,\}$ is discrete and uncountable in this topology. So $\mu \wedge \nu$ is CCC but not hereditarily CCC, and this topology is therefore not metrizable.

Of course, the fact that $\mu \wedge \nu \notin \Sigma_m$ does not preclude the possibility that μ and ν have a greatest lower bound in Σ_m. So suppose that $\theta \in \Sigma_m$ with $\sigma \leqslant \theta \leqslant \mu \wedge \nu$. Choose some $x_0 \in \mathbb{R}$ such that (x_0, x_0) is not isolated in $(\Delta, \theta \restriction \Delta)$. Let

$$U_n = \{(x_0,x_0)\} \cup \{\, (x,y) \mid |x - x_0| < \frac{|y-x_0|}{2^n} \text{ or } |y - y_0| < \frac{|y-x_0|}{2^n} \,\},$$

and let $\rho = \langle \theta \cup \{\, U_n \mid n \in \omega \,\}\rangle$. Then $\rho \in \Sigma_m$ and $\theta < \rho \leqslant \mu \wedge \nu$. Hence θ cannot be a greatest lower bound for μ and ν in Σ_m. $\square$

One might ask whether such an example can be obtained in which the underlying set is countable. Of course, we cannot use CCC versus hereditary CCC to identify non-metrizability in a countable space, but a similar example will still work.

Example 3. *There exist metrizable topologies σ, μ and ν on a countable set X such that $\sigma \leqslant \mu \wedge \nu$, but $\{\mu, \nu\}$ has no greatest lower bound in Σ_m.*

Proof. Let Q denote the set $\mathbb{Q}$ of rational numbers, with its usual topology, and let Q_d denote $\mathbb{Q}$ with the discrete topology. Let $X = \mathbb{Q} \times \mathbb{Q}$. Let $\sigma = Q \times Q$, $\mu = Q \times Q_d$ and $\nu = Q_d \times Q$. Then $\sigma, \mu, \nu \in \Sigma_m$ and $\sigma \leqslant \mu \wedge \nu$.

For $\mathbf{x} = (x,y) \in X$, $n \in \omega$, let $P_n(\mathbf{x}) = (\{x\} \times B_{2^{-n}}(y)) \cup (B_{2^{-n}}(x) \times \{y\})$, where $B_\varepsilon(x)$ denotes the ε-ball about x in the usual metric. Again, the sets $P_n(\mathbf{x})$ for $n \in \omega$ and $\mathbf{x} \in X$ form a weak neighbourhood base for $\mu \wedge \nu$.

For $A \subseteq X$, we define a function $f_A : \omega \to \omega$ as follows: if $(2^{-n}, 0) \notin A$ then $f_A(n) = 0$. Otherwise, $f_A(n)$ is the least $m > 0$ such that $\{(2^{-n}, 0)\} \times B_{2^{-(m+n)}}(0) \subseteq A$.

Let θ be a metrizable topology with $\sigma \leqslant \theta \leqslant \mu \wedge \nu$, and let $(B_n)_{n\in\omega}$ be a neighbourhood basis for θ at $(0,0)$. Since $\{\, f_{B_n} \mid n \in \omega \,\}$ is countable, there is a function $g : \omega \to \omega \setminus \{0\}$ such that, for every $n \in \omega$, $f_{B_n}(m) < g(m)$ for all but finitely many $m \in \omega$.

For each $k \in \omega$ let $g_k : (0,1] \to \mathbb{R}$ be the unique function which is linear on each interval $[2^{-(n+1)}, 2^{-n}]$ and takes the value $2^{-(g(n)+k)}$ at each 2^{-n}. Put

$$\begin{aligned} S_k &= \{\, (x,y) \mid x \leqslant 0 \text{ or } (k+1)x < |y| \,\} \\ T_k &= \{\, (x,y) \mid 0 < x \leqslant 1 \text{ and } y < g_k(x) \,\} \\ U_k &= \{\, (x,y) \mid x > 1 \text{ and } y < g_k(1) \,\} \\ V_k &= S_k \cup T_k \cup U_k \end{aligned}$$

Then V_k is σ-open except at $(0,0)$, $\overline{V_{k+1}} \subseteq V_k$ for each k and $f_{V_k}(n) = g(n) + k$ for each n. Since $0 < f_{B_n}(m) < g(m)$ for all but finitely many $m \in \omega$, $B_n \nsubseteq V_k$ for each n. Thus $V_k \notin \theta$. Let ρ be the topology with subbasis $\theta \cup \{\, V_k \mid k \in \omega \,\}$. Then ρ is regular and second countable, so it is metrizable, and $\theta < \rho \leqslant \mu \wedge \nu$. Thus μ and ν have no greatest lower bound in Σ_m. □

In considering finite intervals in Σ, one is led to the consideration of basic intervals. An interval $[\sigma, \tau]$ in Σ is *basic* if there is some point x (called the *base* of the interval) such that $\sigma \restriction X \setminus \{x\} = \tau \restriction X \setminus \{x\}$. It is easy to show that any finite interval in $\Sigma_1(X)$ is isomorphic to a basic interval in $\Sigma_1(Y)$ for some Y. We will frequently use without further comment the fact that if $[\sigma, \tau]$ is basic with base x and $x \in A$ then A is τ-closed if and only if it is σ-closed.

Lemma 1. *Let $\sigma, \tau \in \Sigma_m$ with $\sigma < \tau$. Then there exist $\sigma', \tau' \in [\sigma, \tau]_m$ with $\sigma' < \tau'$ such that $[\sigma', \tau']$ is basic.*

Proof. Let d, e be metrics such that $\rho_d = \sigma$, $\rho_e = \tau$ and $d(x,y) \leqslant e(x,y)$ for all $x, y \in X$, as guaranteed by Corollary 1.

Let $x \in X$ and $A \subseteq X$ with $x \in \overline{A}^{\sigma} \setminus \overline{A}^{\tau}$. Choose some sequence $(x_n)_{n\in\omega}$ of distinct elements of A which converges in σ to x but which does not converge in τ. Choose τ-neighbourhoods U_n of x_n for $n \in \omega$ and U of x such that $\{U\} \cup \{\, U_n \mid n \in \omega \,\}$ is a discrete collection in (X, τ), and U_n is contained in the d-ball about x_n whose radius is $\frac{1}{3}$ of the least d-distance from x_n to x or to any of the other x_i. Let $f_n : X \to [0,1]$ be τ-continuous with $f(x_n) = 1$ and $f(X \setminus U_n) = \{0\}$.

Let $\sigma' = \langle \sigma \cup \bigcup_{n\in\omega} \psi_{f_n} \rangle$. Then σ' is a join of topologies of the form ψ_f, so it is completely regular. Moreover, σ has a sigma-locally finite base, as does each ψ_{f_n}, so σ' has a sigma-locally finite subbasis. Thus σ' is metrizable. Similarly, $\tau' = \langle \sigma', \psi_f \rangle$ is metrizable, where $f = \sum_{n\in\omega} f_n$. Since each f_n is τ-continuous and $\{\, U_n \mid n \in \omega \,\}$ is discrete, f is τ-continuous. Thus we have $[\sigma', \tau'] \subseteq [\sigma, \tau]$.

To show that $\sigma' < \tau'$, we note that $x \notin \overline{\{ x_n \mid n \in \omega \}}^{\tau'}$, since $f(x_n) = 1$ for every n, while $f(x) = 0$. On the other hand, each f_n is equal to 0 on a σ-neighbourhood of x, so every σ' neighbourhood of x is actually a σ-neighbourhood of x. Thus $x \in \overline{\{ x_n \mid n \in \omega \}}^{\sigma'}$.

Finally, to see that $[\sigma', \tau']$ is basic, we note that $\{ U_n \mid n \in \omega \}$ is discrete with respect to σ except at x, and thus f is σ-continuous except at x. □

Proposition 2. *Let $[\sigma, \tau]$ be a basic interval with $\sigma, \tau \in \Sigma_m$. Then $[\sigma, \tau]_m$ is a sublattice of $[\sigma, \tau]$.*

Proof. Let $\mu, \nu \in [\sigma, \tau]_m$. We have already seen that $\mu \vee \nu \in \Sigma_m$, so it is sufficient to show that $\mu \wedge \nu \in \Sigma_m$. So let $\mathcal{B}$ be a sigma-locally finite basis for σ, and let $\{ U_n \mid n \in \omega \}$ and $\{ V_n \mid n \in \omega \}$ be local bases for μ and ν respectively at x, the base of the interval $[\sigma, \tau]$. Let θ be the topology with basis $\mathcal{B} \cup \{ U_n \cup V_n \mid n \in \omega \}$. Then θ is regular and has a sigma-locally finite basis, so θ is metrizable. But it is easy to see that $\theta = \mu \wedge \nu$, so $\mu \wedge \nu \in \Sigma_m$ as required. □

Valent and Larson have shown that if σ is T_2 and first countable, and $\sigma < \tau$, then $[\sigma, \tau]$ has a subinterval which is isomorphic to the power lattice $\mathbb{P}(\mathfrak{c})$, where $\mathfrak{c} = 2^\omega$ and $\mathbb{P}(\mathfrak{c})$ is ordered by inclusion [7, Theorem 10]. As they observe [7, Corollaries 1 and 3], this implies that no T_2 first countable topology has a cover in Σ_1, and all the topologies in the subinterval isomorphic to $\mathbb{P}(\mathfrak{c})$ (except possibly the largest) do have covers, and are therefore not first countable. Thus Valent and Larson's result does not tell us anything about intervals in Σ_m. In particular, we cannot hope to show that any such interval has at least $2^\mathfrak{c}$ many elements, since if X is countable then there are only $\mathfrak{c}$ many metrics on X. The following result shows that there are always at least $\mathfrak{c}$ many elements in any interval in Σ_m.

Theorem 1. *Let $\sigma, \tau \in \Sigma_m$ with $\sigma < \tau$. Then there is an order-embedding $\Phi : \mathbb{P}(\omega)/fin \to [\sigma, \tau]_m$.*

Lemma 2. *Let d be a metric on X, and let $f : X \to \mathbb{R}$ be a function. Define $e : X \times X \to [0, \infty)$ by $e(x, y) = d(x, y) + |f(x) - f(y)|$. Then e is a metric, and e gives the same topology at x as d does if and only if f is continuous at x.*

Proof of Theorem 1. We can assume without loss of generality that $[\sigma, \tau]$ is a basic interval. Let d, e be metrics such that $\rho_d = \sigma$, $\rho_e = \tau$ and $d(x, y) \leqslant e(x, y)$ for all $x, y \in X$, as guaranteed by Corollary 1.

Let $x \in X$ and $A \subseteq X$ with $x \in \overline{A}^{\sigma} \setminus \overline{A}^{\tau}$. Choose some sequence $(x_n)_{n\in\omega}$ of distinct elements of A which converges in σ to x but which does not converge in τ. Choose τ-neighbourhoods U_n of x_n for $n \in \omega$ and U of x such that $\{U\} \cup \{ U_n \mid n \in \omega \}$ is a discrete collection in (X, τ). Note that U_n is actually a σ-neighbourhood of x_n for each n: we can choose these sets in such a way that U_n is contained in the d-ball about x_n whose radius is $\frac{1}{3}$ of the least d-distance from x_n to x or to any of the other points x_i. Let $f_n : X \to [0,1]$ be σ-continuous with $f(x_n) = 1$ and $f(X \setminus U_n) = \{0\}$. For $A \in \mathbb{P}(\omega)$ let F_A be the function $\sum_{n\in A} f_n$. Then, by Lemma 2, the functions $d_A, e_A : X \times X \to [0,\infty)$ given by

$$\begin{aligned} d_A(y,z) &= d(y,z) + |F_A(y) - F_A(z)| \\ e_A(y,z) &= e(y,z) + |F_A(y) - F_A(z)| \end{aligned}$$

are both metrics. Now F_A is σ-continuous except (for infinte A) at x, and is τ-continuous at x (being constant on the τ-open set U), so F_A is τ-continuous. Thus e_A gives the same topology as e, namely τ. Moreover, we have $d \leqslant d_A \leqslant e_A$, so by Corollary 1 we have $\sigma \leqslant \rho_{d_A} \leqslant \tau$.

By the choice of the sets U_n, if $A, B \subseteq \omega$ with $A \subseteq^* B$ then there is a σ-neighbourhood of x on which $F_A \leqslant F_B$, and therefore $d_A(x,y) \leqslant d_B(x,y)$ for every y in this neighbourhood. Since x is the only place at which the topologies change, this implies that $\rho_{d_A} \leqslant \rho_{d_B}$. Thus the function $\Phi : \mathbb{P}(\omega)/\text{fin} \to [\sigma,\tau]_m$ given by $\Phi([A]) = \rho_{d_A}$ is well-defined and order-preserving. Conversely, suppose that $A, B \subseteq \omega$ with $A \not\subseteq^* B$. Then $C = A \setminus B$ is infinite. For each $x_n \in C$, we have $F_B(x_n) = F_B(x) = 0$, $F_A(x_n) = 1$. Thus $d_B(x,x_n) = d(x,x_n)$ for each $x_n \in C$, so $x \in \overline{C}^{\rho_{d_B}}$. On the other hand, since $d_A(x,x_n) > 1$ for each $x_n \in C$, $x \notin \overline{C}^{\rho_{d_A}}$. Thus $\Phi([A]) \not\leqslant \Phi([B])$. Hence Φ is an order-embedding. □

Although the previous result shows that any interval in Σ_m contains a copy of $\mathbb{P}(\omega)/\text{fin}$, no such interval can be isomorphic to $\mathbb{P}(\omega)/\text{fin}$. This follows from the following result. Recall that a bounded lattice L with least element 0 and greatest element 1 is *complemented* if every element x has a complement x' with $x \wedge x' = 0$ and $x \vee x' = 1$. It is easy so show that every subinterval of $\mathbb{P}(\omega)/\text{fin}$ is a complemented lattice, and by Lemma 1 every interval in Σ_m contains a basic interval in Σ_m.

Theorem 2. *Let $\sigma, \tau \in \Sigma_m$ with $\sigma < \tau$ and $[\sigma,\tau]$ basic. Then $[\sigma,\tau]_m$ is not complemented.*

Proof. Let d, e, x, $(x_n)_{n\in\omega}$, U, $(U_n)_{n\in\omega}$ and $(f_n)_{n\in\omega}$ be as in the proof of Theorem 1. Let $\omega = \bigcup_{n\in\omega} A_n$, where the sets A_n are infinite and pairwise disjoint. Define $F : X \to \mathbb{R}$ by

$$F(y) = \sum_{n\in\omega} \sum_{m\in A_n} 2^{-n} f_m(y).$$

Then F is σ-continuous except at x, and is τ-continuous. Let $\mu = \rho_{d'}$, where $d'(y,z) = d(y,z) + |F(y) - F(z)|$.

Suppose ν is a complement of μ in $[\sigma,\tau]$. In particular, $U \in \mu \vee \nu$, so there exist $V \in \mu$ and $W \in \nu$ with $x \in V \cap W \subseteq U$. Shrinking V if necessary we can assume that $V = B(x, d', \varepsilon)$ for some $\varepsilon > 0$. Choose n large enough so that $2^{-n} < \varepsilon$. Let $O = W \cup B(x, d', 2^{-n})$. Then $O \in \mu\wedge\nu$. However, $O\cap\{\, x_m \mid m \in A_n \,\} = \varnothing$, and $x \in \overline{\{\, x_m \mid m \in A - n \,\}}^{\sigma}$, so $\mu \wedge \nu \neq \sigma$, a contradiction. □

3. Basic intervals in $\Sigma_m(X)$ with X countable

In the previous section we saw that every interval in Σ_m contains $\mathbb{P}(\omega)/\mathrm{fin}$, but is not isomorphic to it. In this section we will give two different characterisations of the simplest possible interval—that between the usual topology and the discrete topology on $\omega+1$. The first is as a quotient of ω^ω, and the second is as an extension of $\mathbb{P}(\omega)/\mathrm{fin}$ obtained by adding limits of increasing sequences. Finally we will see that, under CH, every basic interval in $\Sigma_m(X)$ with X countable is such an extension of either $\mathbb{P}(\omega)/\mathrm{fin}$ or a certain kind of power of $\mathbb{P}(\omega)/\mathrm{fin}$.

Define a relation $\preccurlyeq$ on ω^ω by declaring that $f \preccurlyeq g$ if and only if there is some (weakly) order-preserving function $\pi : \omega \to \omega$ such that for every $n \in \omega$, $g(n) \leqslant \pi(f(n))$. It is easy to see that $\preccurlyeq$ is a preorder (in other words, it is reflexive and transitive). Thus the relation $\approx$, defined by $f \approx g$ if and only if $f \preccurlyeq g$ and $g \preccurlyeq f$, is an equivalence relation, and $\preccurlyeq$ induces a partial order on $\omega^\omega/{\approx}$.

Theorem 3. *Let σ denote the usual topology on $X = \omega+1$, and let τ denote the discrete topology. Then $(\omega^\omega/{\approx}, \preccurlyeq)$ is isomorphic to the interval $[\sigma,\tau]_m$.*

Proof. If d is a metric on X with $\sigma \leqslant \rho_d \leqslant \tau$, define a function $f_d : \omega \to \omega$ by

$$f_d(n) = \min\{\, m \in \omega \mid 2^{-m} \leqslant d(n,\omega) \,\}.$$

Now suppose that e is another such metric. Suppose first that $\rho_d \leqslant \rho_e$. For each $m \in \omega$, $B(\omega, d, 2^{-m})$ is ρ_e-open, so there is some $k \in \omega$ such that $B(\omega, e, 2^{-k}) \subseteq B(\omega, d, 2^{-m})$. Define $\pi : \omega \to \omega$ by

$$\pi(m) = \min\{\, k \in \omega \mid B(\omega, e, 2^{-k}) \subseteq B(\omega, d, 2^{-m}) \,\}.$$

Since the sets $B(\omega, d, 2^{-m})$ and $B(\omega, e, 2^{-k})$ form decreasing sequences, π is an order-preserving function. Now let $n \in \omega$. Put $m = f_d(n)$. Then $d(n, \omega) \geqslant 2^{-m}$, so $n \notin B(\omega, d, 2^{-m})$, and therefore $n \notin B(\omega, e, 2^{-\pi(m)})$. So $2^{-\pi(m)} \leqslant e(n, \omega)$, and therefore $f_e(n) \leqslant \pi(m) = \pi(f_d(n))$. Since this holds for every $n \in \omega$, $f_d \preccurlyeq f_e$.

Conversely, suppose that d and e are metrics on X with $\rho_d, \rho_e \in [\sigma, \tau]_m$ and $f_d \preccurlyeq f_e$. Let $\pi : \omega \to \omega$ be an order preserving function such that for every $n \in \omega$, $f_e(n) \leqslant \pi(f_d(n))$. Fix $m \in \omega$. For each $n \in \omega$, if $d(n, \omega) \geqslant 2^{-m}$ then $f_d(n) \leqslant m$, so $f_e(n) \leqslant \pi(f_d(n)) \leqslant \pi(m)$, and thus $e(n, \omega) \geqslant 2{-}\pi(m)$. Hence $B(\omega, e, 2^{-\pi(m)}) \subseteq B(\omega, d, 2^{-m})$, so $B(\omega, d, 2^{-m})$ is an e-neighbourhood of ω. Since these sets form a neighbourhood basis at ω, and all the other points are isolated in both ρ_d and ρ_e, this implies that $\rho_d \leqslant \rho_e$.

Hence if d and e are metrics on X with $\rho_d, \rho_e \in [\sigma, \tau]_m$ then $f_d \preccurlyeq f_e$ if and only if $\rho_d \leqslant \rho_e$. Thus the function $\Phi : [\sigma, \tau]_m \to \omega^\omega/{\approx}$ given by $\Phi(\rho_d) = [f_d]$ is well-defined and order-preserving. To show that it is an isomorphism, we will show that for every $f \in \omega^\omega$ there is a metric d with $\rho_d \in [\sigma, \tau]_m$ and $f_d = f$. Indeed, let d be the metric on X given by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 2^{-f(x)} & \text{if } x \in \omega,\ y = \omega \\ 2^{-f(y)} & \text{if } x = \omega,\ y \in \omega \\ 2^{-f(x)} + 2^{-f(y)} & \text{otherwise.} \end{cases}$$

Then it is easy to see that d is such a metric and $f_d = f$. □

Given the way that between two metrizable topologies we can find $\mathbb{P}(\omega)/\mathrm{fin}$ many metrizable topologies, a tempting conjecture is that all basic intervals between metrizable topologies, or at least all such intervals on a countable set, are isomorphic. However, this is not the case: there are at least two isomorphism classes, and under CH there are exactly two isomorphism classes. To show this, we will first introduce some new definitions.

Definition 1. Let $\sigma, \tau \in \Sigma_m$ with $\sigma \leqslant \tau$. We say that τ is a *successor* with respect to σ if $[\sigma, \tau]$ is a basic interval and there is some A such that $\tau = \langle \sigma, A \rangle$. We say that τ is a *limit* with respect to σ if there is a strictly increasing countable sequence $(\mu_n)_{n \in \omega}$ in $[\sigma, \tau]_m$ such that $\tau = \sup\{\, \mu_n \mid n \in \omega \,\}$.

We denote the set of successors with respect to σ in $[\sigma, \tau]_m$ by $[\sigma, \tau]_{ms}$. In other words,

$$[\sigma, \tau]_{ms} = \{\, \mu \in [\sigma, \tau]_m \mid \exists A\, (\mu = \langle \sigma, A \rangle) \,\}.$$

Notice that, despite our rather suggestive terminology, these notions are not *a priori* complementary. Notice also that not every topology of the form $\langle\sigma, A\rangle$ will be metrizable. Finally, notice that we regard σ as being a successor with respect to itself. In what follows, whenever we say something like "choose A such that $\mu = \langle\sigma, A\rangle$" we will assume that if $\mu = \sigma$ then the A we choose will be X.

Lemma 3. *Let σ be a regular topology on a set X, and let $x \in A \subseteq X$ with $A \setminus \operatorname{int}_\sigma(A) = \{x\}$. Then $\langle\sigma, A\rangle$ is regular if and only if there is some $U \in \sigma$ such that $x \in U$ and $U \cap A = U \cap \overline{A}^\sigma$.*

Proof. Suppose first that $\langle\sigma, A\rangle$ is regular. Then there is some $U \in \sigma$ such that $x \in U \cap A \subseteq \overline{U \cap A}^{\langle\sigma,A\rangle} \subseteq A$. As U is open, $U \cap \overline{A}^\sigma = \overline{U \cap A}^\sigma$, whence $U \cap \overline{A}^\sigma = U \cap A$.

Conversely, suppose that there is some $U \in \sigma$ such that $x \in U$ and $U \cap \overline{A}^\sigma = U \cap A$. Let $y \in V \in \langle\sigma, A\rangle$.

Case 1: $y \in \operatorname{int}_\sigma(V)$. Then, by regularity of σ, there is some $W \in \sigma \subseteq \langle\sigma, A\rangle$ with
$$y \in W \subseteq \overline{W}^{\langle\sigma,A\rangle} \subseteq \overline{W}^\sigma \subseteq \operatorname{int}_\sigma(V) \subseteq V.$$

Case 2: $y \notin \operatorname{int}_\sigma(V)$. Then we must have $y = x$ and there is some $V' \in \sigma$ with $x \in V' \cap A \subseteq V$. By regularity of σ, we can choose $W \in \sigma$ with $x \in W \subseteq \overline{W}^\sigma \subseteq V' \cap U$. Then $x \in W \cap A \in \langle\sigma, A\rangle$, and so
$$\begin{aligned}\overline{W \cap A}^{\langle\sigma,A\rangle} &= \overline{W \cap A}^\sigma \\ &\subseteq \overline{W}^\sigma \cap \overline{A}^\sigma \\ &\subseteq V' \cap U \cap \overline{A}^\sigma \\ &= V' \cap U \cap A \\ &\subseteq V\end{aligned}$$

Thus $\langle\sigma, A\rangle$ is regular, as required. □

Corollary 2. *Let X be a countable set and let $\sigma \in \Sigma_m(X)$. Then τ is a (metrizable) successor with respect to σ if and only if $[\sigma, \tau]$ is basic and $\tau = \langle\sigma, A\rangle$ for some σ-closed set A.*

Proof. Suppose that $\tau = \langle\sigma, A\rangle$ is a successor and $\sigma \neq \tau$. Let x be the base of $[\sigma, \tau]$. By Lemma 3 there is some $U \in \sigma$ with $x \in U \cap A = U \cap \overline{A}^\sigma$. Equivalently, we have $U \cap (\overline{A}^\sigma \setminus A) = \varnothing$, so we may shrink U to a σ-clopen set Q containing x (since σ is a metrizable topology on a countable set, and hence zero-dimensional). Since $x \in Q \in \sigma$, $\langle\sigma, A\rangle = \langle\sigma, Q \cap A\rangle$, and $Q \cap A$ is closed.

Conversely, if A is closed and $[\sigma, \tau]$ is basic, then by Lemma 3, $\langle \sigma, A\rangle$ is regular. Since σ has a countable basis, $\langle \sigma, A\rangle$ has a countable basis. Thus $\langle \sigma, A\rangle$ is metrizable. $\square$

Proposition 3. *Let X be a countable set, and let $\sigma, \tau \in \Sigma_m(X)$ with $\sigma < \tau$ and $[\sigma, \tau]$ basic. Then there is some $\mu \in \Sigma_m$ with $\sigma < \mu \leqslant \tau$ and μ a successor with respect to σ.*

Proof. Let x be the base of $[\sigma, \tau]$. Choose some $A \in \tau \smallsetminus \sigma$. Notice that $x \in A$. Since τ is a metrizable topology on a countable set, it is zero-dimensional, so we can find some τ-clopen set B with $x \in B \subseteq A$. Now B is closed in τ and contains x, so it is closed in σ. Thus, by Lemma 3, $\mu = \langle \sigma, B\rangle \in \Sigma_m(X)$. Since $A \in \mu$ and $B \in \tau$ we have $\sigma < \mu \leqslant \tau$. $\square$

The hypothesis that X is countable is necessary in the previous result, as is shown by the following example.

Example 4. *Let σ be the usual topology on $\mathbb{R}$, and let $\tau = \sigma \vee \psi_f$, where $f : \mathbb{R} \to \mathbb{R}$ is given by*

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then no topology in $(\sigma, \tau]_m$ is a successor with respect to σ.

Proof. Notice that $[\sigma, \tau]$ is basic with base 0.

Suppose $\mu \in [\sigma, \tau]_m$ with $\mu = \langle \sigma, A\rangle$. In particular, μ is regular, so by Lemma 3 there is some neighbourhood U of x such that $U \cap \overline{A}^{\sigma} = U \cap A$. Shrinking U if necessary, we can assume that U is an open interval $(-x, x)$ with $x > 0$. Now $A \smallsetminus \{0\}$ is σ-open, so each component of $U \cap (A \smallsetminus \{0\})$ is an open interval. Since U contains no points of $\overline{A}^{\sigma} \smallsetminus A$, these components must be of the form $(-x, 0)$ or $(0, x)$. Since $0 \in \overline{(-x,0)}^{\tau} \cap \overline{(0,x)}^{\tau}$, both of these must be components of $U \cap (A \smallsetminus \{0\}$. In other words, $U \cap A = U$. But then $A \in \sigma$, so $\mu = \sigma$. $\square$

Proposition 4. *Let X be a countable set, and let $\sigma, \tau \in \Sigma_m(X)$ with $\sigma < \tau$ and $[\sigma, \tau]$ a basic interval. Then τ is a limit with respect to σ if and only if τ is not a successor with respect to σ. Moreover, if τ is a limit, then it is a limit of successors with respect to σ.*

Proof. Suppose first that τ is a limit. Let $(\mu_n)_{n \in \omega}$ be a strictly increasing sequence in $[\sigma, \tau]_m$ converging to τ, and suppose there is some $A \in \tau$ with $\tau = \langle \sigma, A\rangle$. Then, for some $n \in \omega$, $A \in \mu_n$. But then, for every $m \geqslant n$, $\tau = \langle \sigma, A\rangle \leqslant \mu_m < \mu_{m+1} \leqslant \tau$, a contradiction. So if τ is a limit, it cannot be a successor.

Conversely, suppose that τ is not a successor. Let x be the base of $[\sigma, \tau]$, and let $\{ B_n \mid n \in \omega \}$ be a local basis for τ at x consisting of clopen sets. Then, for each $n \in \omega$, $\mu_n = \langle \sigma, \bigcap_{m<n} B_m \rangle$ is in $[\sigma, \tau]_{ms}$. Furthermore, $(\mu_n)_{n\in\omega}$ is an increasing sequence, and $\tau = \sup\{ \mu_n \mid n \in \omega \}$. However, the sequence might not be strictly increasing. So we define a strictly increasing subsequence $(\mu_{n_k})_{k\in\omega}$ by

$$\begin{aligned} n_0 &= 0 \\ n_{k+1} &= \text{the least } m > k \text{ such that } \mu_{n_k} < \mu_m \end{aligned}$$

To see that this is a well-defined subsequence, it is enough to show that for every $n \in \omega$ there is some $m \in \omega$ with $\mu_n < \mu_m$. Suppose this were not so. Then there would be some n such that $B_m \in \mu_n$ for every $m \geqslant n$. But then we would have $\mu_n = \tau$, and so $\tau = \langle \sigma, \bigcap_{m<n} B_m \rangle$, contradicting the assumption that τ is not a successor. $\square$

Proposition 5. *Let X be a countable set. Then there exist at least two isomorphism classes of basic intervals in $\Sigma_m(X)$.*

Proof. We exhibit topologies $\sigma < \mu < \tau$ in $\Sigma_m(\omega+1)$ with $[\sigma, \tau]$ basic and $[\sigma, \mu]_m \ncong [\sigma, \tau]_m$.

Let σ and τ denote, respectively, the usual topology and the discrete topology on $\omega + 1$. Let $\{ A_n \mid n \in \omega \}$ be a partition of ω into infinite subsets, and let μ be the topology on $\omega + 1$ in which elements of ω are isolated and a basic neighbourhood of ω is of the form $\{\omega\} \cup \bigcup_{m\geqslant n} A_m$ for some $n \in \omega$. Notice that μ is a limit with respect to σ, since it is the limit of the strictly increasing sequence $(\mu_n)_{n\in\omega}$, where $\mu_n = \langle \sigma, \bigcup_{m\geqslant n} A_m \rangle$. On the other hand, $\tau = \langle \sigma, \{\omega\} \rangle$ is a successor with respect to σ, and is therefore not a limit. Since the property of being a limit is purely a lattice property, it is preserved by isomorphism. So $[\sigma, \mu]_m \ncong [\sigma, \tau]_m$. $\square$

Lemma 4. *Let σ be a topology on X, and let $A, B \subseteq X$ with $A \smallsetminus \operatorname{int}_\sigma(A) = B \smallsetminus \operatorname{int}_\sigma(B) = \{x\}$. Then*

(1) $\langle \sigma, A \rangle \wedge \langle \sigma, B \rangle = \langle \sigma, A \cup B \rangle$.
(2) $\langle \sigma, A \rangle \vee \langle \sigma, B \rangle = \langle \sigma, A \cap B \rangle$.
(3) *the following are equivalent:*
 (a) $\langle \sigma, A \rangle \leqslant \langle \sigma, B \rangle$,
 (b) $x \notin \overline{B \smallsetminus A}^{\sigma}$,
 (c) *for some $U \in \sigma$, $x \in U \cap B \subseteq A$.*

Proof. Straightforward. $\square$

Lemma 5. *Let X be a countable set and let $\sigma, \tau \in \Sigma_m(X)$ with $\sigma < \tau$ and τ a successor with respect to σ. Then $[\sigma, \tau]_{ms}$ is a Boolean algebra.*

Proof. By Lemma 4 we know that $[\sigma, \tau]_{ms}$ is a distributive lattice, and it is clearly bounded. So we only need to show that it is complemented. Let $\mu \in [\sigma, \tau]_{ms}$. By Corollary 2 we can choose A and B closed in σ with $\mu = \langle \sigma, A\rangle$ and $\tau = \langle \sigma, B\rangle$. Put $C = (X \setminus A) \cup B$ and $\nu = \langle \sigma, C\rangle$. Then C is τ-clopen (since both A and B are), and $x \in C$, so C is σ-closed. Hence $\nu \in \Sigma_m(X)$. Moreover, since $B \subseteq C$, $\nu \in [\sigma, \tau]$, so $\nu \in [\sigma, \tau]_{ms}$.

Now $\mu \wedge \nu = \langle \sigma, A \cup C\rangle = \langle \sigma, X\rangle = \sigma$, and $\mu \vee \nu = \langle \sigma, A \cap C\rangle = \langle \sigma, A \cap B\rangle = \mu \vee \tau = \tau$. Thus ν is a complement of μ in $[\sigma, \tau]_{ms}$, as required. $\square$

Recall that a Boolean algebra B has *property* H_ω (also known as the *strong countable separation property*) if for every pair $\{ a_n \mid n \in \omega \}$ and $\{ b_n \mid n \in \omega \}$ of countable subsets of B such that for every $n \in \omega$, $\bigvee_{m\leqslant n} a_m < \bigwedge_{m\leqslant n} b_m$ there is some $c \in B$ such that for every $n \in \omega$, $\bigvee_{m\leqslant n} a_m < c < \bigwedge_{m\leqslant n} b_m$.

Lemma 6. *Let X be a countable set, and let $\sigma, \tau \in \Sigma_m(X)$ with $\sigma < \tau$ and τ a successor with respect to σ. Then $[\sigma, \tau]_{ms}$ has property H_ω.*

Proof. Let d be a metric with $\rho_d = \sigma$. Let x be the base of $[\sigma, \tau]$. Let $\{ \alpha_n \mid n \in \omega \}$ and $\{ \beta_n \mid n \in \omega \}$ be subsets of $[\sigma, \tau]_{ms}$ such that for every $n \in \omega$, $\bigvee_{m\leqslant n} \alpha_m < \bigwedge_{m\leqslant n} \beta_m$. For each $n \in \omega$ choose σ-closed sets A_n and B_n with $\alpha = \langle \sigma, A_n\rangle$ and $\beta_n = \langle \sigma, B_n\rangle$. Now $\langle \sigma, \bigcap_{m\leqslant n} A_m\rangle \leqslant \langle \sigma, \bigcup_{m\leqslant n} B_m\rangle$, so there is some σ-open set U_n with $x \in U_n \cap \bigcup_{m\leqslant n} B_m \subseteq \bigcap_{m\leqslant n} A_m$. Shrinking U_n if necessary we can assume that U_n is clopen and that $U_n \subseteq B(x, d, 2^{-n})$. We also know that $\langle \sigma, \bigcup_{m\leqslant n} B_m\rangle \not\leqslant \langle \sigma, \bigcap_{m\leqslant n} A_m\rangle$, so $x \in \overline{\bigcap_{m\leqslant n} A_m \setminus \bigcup_{m\leqslant n} B_m}^{\sigma}$. Thus we can inductively choose distinct points x_n, y_n such that

$$x_n, y_n \in U_n \cap (\bigcap_{m\leqslant n} A_m \setminus \bigcup_{m\leqslant n} B_m) \setminus (\{ x_m \mid m < n \} \cup \{ y_m \mid m < n \}).$$

The only limit point of $\{ x_n \mid n \in \omega \} \cup \{ y_n \mid n \in \omega \}$ is x, so we can choose clopen sets D_n and E_n for $n \in \omega$ such that for each $n \in \omega$,

$$x_n \in D_n \subseteq U_n \cap (\bigcap_{m\leqslant n} A_m \setminus \bigcup_{m\leqslant n} B_m),$$

$$y_n \in E_n \subseteq U_n \cap (\bigcap_{m\leqslant n} A_m \setminus \bigcup_{m\leqslant n} B_m), \text{ and}$$

$$D_n \cap (\bigcup_{m\in\omega\setminus\{n\}} D_m \cup \bigcup_{m\in\omega} E_m) = \varnothing = E_n \cap (\bigcup_{m\in\omega} D_m \cup \bigcup_{m\in\omega\setminus\{n\}} E_m).$$

For each $n \in \omega$ let $C_n = ((U_n \setminus \bigcup_{m\leqslant n} D_m) \cap \bigcup_{m\leqslant n} B_m) \cup E_n$. Let $C = \bigcup_{n\in\omega} C_n$, and let $\gamma = \langle \sigma, C\rangle$.

Claim: C is closed.

For: Since $C_n \subseteq U_n \subseteq B(x, d, 2^{-n})$ for each n, every limit point of C must be either a limit point of some C_n or be x itself. Since each C_n is closed and $x \in C$, this means that every limit point of C is an element of C.

Claim: For each n, $\bigvee_{m \leqslant n} \alpha_m \leqslant \gamma$.

For: Let $V_n = U_n \smallsetminus \bigcup_{m \leqslant n} E_m$. Then $x \in V_n \in \sigma$, so $\langle \sigma, C \rangle = \langle \sigma, V_n \cap C \rangle$. Now $V_n \cap C = (V_n \cap \bigcup_{m \leqslant n} C_m) \cup (V_n \cap \bigcup_{m > n} C_m)$. Notice that for $m \leqslant n$ we have $C_m \subseteq \bigcup_{k \leqslant m} B_k \cup E_m$, so $\bigcup_{m \leqslant n} C_m \smallsetminus \bigcup_{m \leqslant n} E_m \subseteq \bigcup_{m \leqslant n} B_m$. Thus $V_n \cap \bigcup_{m \leqslant n} C_m \subseteq U_n \cap \bigcup_{m \leqslant n} B_m \subseteq \bigcap_{m \leqslant n} A_m$. Also for $m > n$ we have $C_m \subseteq \bigcap_{k \leqslant m} A_k \subseteq \bigcap_{k \leqslant n} A_k$, so $V_n \cap \bigcup_{m > n} C_m \subseteq \bigcap_{m \leqslant n} A_m$. Thus $V_n \cap C \subseteq \bigcap_{m \leqslant n} A_m$, as required.

Claim: For each n, $\gamma \nleqslant \bigvee_{m \leqslant n} \alpha_m$.

For: Notice that $\{ x_m \mid m \geqslant n \} \subseteq \bigcap_{m \leqslant n} A_m$ for each n. We will show that for every m, $x_m \notin C$, by showing that $x_m \notin C_n$ for every n. So let $m, n \in \omega$. If $m \leqslant n$ then $x_m \in \bigcup_{m \leqslant n} D_m$ and $x_m \notin E_n$, so $x_m \notin C_n$. On the other hand, if $m > n$ then $C_n \subseteq \bigcup_{k \leqslant n} B_k \cup E_n \subseteq \bigcup_{k \leqslant m} B_k \cup E_n$. Since x_m was chosen to be an element of $\bigcap_{k \leqslant m} A_k \smallsetminus \bigcup_{k \leqslant m} B_k$, and $x_m \notin E_n$, $x_m \notin C_m$.

Thus for every n we have $\{ x_m \mid m \geqslant n \} \subseteq \bigcap_{m \leqslant n} A_m \smallsetminus C$, so $x \in \overline{\bigcap_{m \leqslant n} A_m \smallsetminus C}^{\sigma}$, as required.

Claim: For each n, $\gamma \leqslant \bigwedge_{m \leqslant n} \beta_m$.

For: We have $U_n \smallsetminus \bigcup_{m \leqslant n} D_m \in \sigma$, and $(U_n \smallsetminus \bigcup_{m \leqslant n} D_m) \cap \bigcup_{m \leqslant n} B_m \subseteq C_n \subseteq C$.

Claim: For each n, $\bigwedge_{m \leqslant n} \beta_n \nleqslant \gamma$.

For: Notice that for each m, $y_m \in E_m \subseteq C_m$. However, for $m \geqslant n$, $y_m \notin \bigcup_{k \leqslant n} B_k$. Thus $\{ y_m \mid m \geqslant n \} \subseteq C \smallsetminus \bigcup_{m \leqslant n} B_m$, so $x \in \overline{C \smallsetminus \bigcup_{m \leqslant n} B_m}^{\sigma}$.

This completes the proof. □

Lemma 7. *Let X be a countable set, and let $\sigma, \tau \in \Sigma_m(X)$ with $\sigma < \tau$ and $[\sigma, \tau]$ basic. Let $(\mu_n)_{n \in \omega}$ be a (not necessarily strictly) increasing sequence in $[\sigma, \tau]_{ms}$ and let $\mu = \sup\{ \mu_n \mid n \in \omega \}$. Then μ is metrizable, and if $\nu \in [\sigma, \tau]_{ms}$ with $\nu \leqslant \mu$ then $\nu \leqslant \mu_n$ for some $n \in \omega$.*

Proof. For each n choose A_n σ-closed with $\mu_n = \langle \sigma, A_n \rangle$. Then $\mu = \langle \sigma \cup \{ A_n \mid n \in \omega \} \rangle$. Since each A_n is closed, a similar argument to the proof of Lemma 3 shows that μ is regular. Since σ has a countable basis, so does μ. Thus μ is metrizable.

Now let $\nu \in [\sigma, \tau]_{ms}$ with $\nu \leqslant \mu$. Choose B with $\nu = \langle \sigma, B \rangle$. Then $B \in \langle \sigma \cup \{ A_n \mid n \in \omega \} \rangle$, so there is some $n \in \omega$ and some $U \in \sigma$ such that $x \in U \cap \bigcap_{m \leqslant n} A_m \subseteq B$. But then $B \in \langle \sigma, A_0, \ldots, A_n \rangle = \mu_n$, so

$\nu \leqslant \mu_n$. □

If P is a partially ordered set in which no strictly increasing sequence has a supremum, let $S(P)$ denote the partially ordered set obtained by adding suprema of increasing sequences. More formally, let $S'(P)$ denote the set of sequences $(x_n)_{n\in\omega}$ such that for all $n \in \omega$, $x_n \leqslant x_{n+1}$. Extend $\leqslant$ to $S'(P)$ by declaring that $(x_n)_{n\in\omega} \leqslant (y_n)_{n\in\omega}$ if and only if for every $m \in \omega$ there is some $n \in \omega$ such that $x_m \leqslant y_n$. This is clearly a preorder on $S'(P)$: let $S(P)$ denote the quotient obtained by declaring $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ to be equivalent if and only if $(x_n)_{n\in\omega} \leqslant (y_n)_{n\in\omega}$ and $(y_n)_{n\in\omega} \leqslant (x_n)_{n\in\omega}$. We denote the equivalence class of $(x_n)_{n\in\omega}$ by $[x_n]_{n\in\omega}$. It is then straightforward to show that every increasing sequence in $S(P)$ has a supremum, and P can be embedded in $S(P)$ by identifying x with $[x]_{n\in\omega}$.

Lemma 8. *Let X be a countable set, and let $\sigma, \tau \in \Sigma_m(X)$ with $\sigma < \tau$ and $[\sigma, \tau]$ basic. Then $[\sigma, \tau]_m \cong S([\sigma, \tau]_{ms})$.*

Proof. By Proposition 4, every topology in $[\sigma, \tau]_m$ is the supremum of a (not necessarily strictly) increasing sequence in $[\sigma, \tau]_{ms}$. So we define $\Phi : [\sigma, \tau]_m \to S([\sigma, \tau]_{ms})$ by declaring that for every μ, $\Phi(\mu) = [\mu_n]_{n\in\omega}$, where $(\mu_n)_{n\in\omega}$ is an increasing sequence with supremem μ. To show that Φ is well-defined, we must verify that if $(\mu_n)_{n\in\omega}$ and $(\nu_n)_{n\in\omega}$ are increasing sequences with the same supremum then $[\mu_n]_{n\in\omega} = [\nu_n]_{n\in\omega}$, in other words that for each $m \in \omega$ there is some $n \in \omega$ with $\mu_m \leqslant \nu_n$, and some $k \in \omega$ with $\nu_m \leqslant \mu_k$. This follows from Lemma 7.

Now let $\mu, \nu \in [\sigma, \tau]_m$ and choose increasing sequences $(\mu_n)_{n\in\omega}$ and $(\nu_n)_{n\in\omega}$ with suprema μ and ν respectively. Suppose that $\mu \leqslant \nu$. For each $m \in \omega$ we have $\mu_m \leqslant \mu \leqslant \sup\{\, \nu_n \mid n \in \omega \,\}$, so $\mu_m \leqslant \nu_n$ for some $n \in \omega$. Since this holds for every $m \in \omega$, $(\mu_n)_{n\in\omega} \leqslant (\nu_n)_{n\in\omega}$, and thus $[\mu_n]_{n\in\omega} \leqslant [\nu_n]_{n\in\omega}$, i.e. $\Phi(\mu) \leqslant \Phi(\nu)$. Suppose instead that $\mu \nleqslant \nu$. Then there is some $A \in \mu \setminus \nu$. Shrinking A if necessary, we can assume that A is clopen in μ and hence closed in σ. Put $\lambda = \langle \sigma, A \rangle$. Then $\lambda \in [\sigma, \tau]_{ms}$ but $\lambda \nleqslant \nu$, so $\lambda \nleqslant \nu_n$ for every n. Thus $\Phi(\lambda) \nleqslant \Phi(\nu)$. On the other hand, $\lambda \leqslant \mu$, so $\lambda \leqslant \mu_n$ for some n. Thus $\Phi(\lambda) \leqslant \Phi(\mu)$. Therefore $\Phi(\mu) \nleqslant \Phi(\nu)$.

Thus for every $\mu, \nu \in [\sigma, \tau]_m$ we have $\mu \leqslant \nu$ if and only if $\Phi(\mu) \leqslant \Phi(\nu)$.

To show that Φ is an isomorphism it remains only to show that it is surjective. So let $[\mu_n]_{n\in\omega} \in S([\sigma, \tau]_{ms})$. By Lemma 7, $\mu = \sup\{\, \mu_n \mid n \in \omega \,\}$ is metrizable, and $\Phi(\mu) = [\mu_n]_{n\in\omega}$. □

Corollary 3. *For $i = 1, 2$ let X_i be a countable set and let $\sigma_i, \tau_i \in \Sigma_m(X_i)$ with $\sigma_i < \tau_i$ and $[\sigma_i, \tau_i]$ basic. If $[\sigma_1, \tau_1]_{ms} \cong [\sigma_2, \tau_2]_{ms}$ then $[\sigma_1, \tau_1]_m \cong [\sigma_2, \tau_2]_m$.*

Lemma 9. *Let X be a countable set and let $\sigma, \tau \in \Sigma_m(X)$ with $\sigma < \tau$. Then $[\sigma, \tau]_{ms}$ has cardinality $\mathfrak{c}$.*

Proof. As previously remarked, we know that $\Sigma_m(X)$ has cardinality $\mathfrak{c}$, so it is enough to show that there are at least $\mathfrak{c}$ many topologies in $[\sigma, \tau]_{ms}$. We will show that $\mathbb{P}(\omega)/\text{fin}$ can be embedded in $[\sigma, \tau]_{ms}$.

As in the proof of Theorem 1, we choose some sequence $(x_n)_{n\in\omega}$ of distinct elements of X which converges in σ to x but which does not converge in τ, and τ-neighbourhoods U_n of x_n for $n \in \omega$ and U of x such that $\{U\} \cup \{\, U_n \mid n \in \omega \,\}$ is a discrete collection in (X, τ). Without loss of generality, U and all the U_n are clopen in τ, and U_n is contained in the d-ball about x_n whose radius is $\frac{1}{3}$ of the least distance from x_n to x or to any of the other points x_i (where d is a metric with $\sigma = \rho_d$).

For $A \in \mathbb{P}(\omega)$, let $S_A = U \cup \bigcup_{n\in\omega\smallsetminus A} U_n$ and let $\mu_A = \langle \sigma, S_A \rangle$. Since each S_A is closed in σ and open in τ, $\mu_A \in [\sigma, \tau]_{ms}$ for each A. Clearly $\mu_A \leqslant \mu_B$ if and only if $A \subseteq^* B$. Thus the function $A \mapsto \mu_A$ is an order-embedding, and $|[\sigma, \tau]_{ms}| \geqslant |\mathbb{P}(\omega)/\text{fin}| = \mathfrak{c}$. ☐

Lemma 10. *Assume CH. Let X be a countable set and, for $i = 1, 2$ let σ_i, τ_i be metrizable topologies on X with $[\sigma_i, \tau_i]$ basic and τ_i a successor with respect to σ_i. Then $[\sigma_1, \tau_1]_{ms} \cong [\sigma_2, \tau_2]_{ms}$.*

Proof. Assuming CH, any Boolean algebra of cardinality at most $\mathfrak{c}$ with property H_ω is isomorphic to $\mathbb{P}(\omega)/\text{fin}$ [3, Theorem 1.1.6]. Thus, by Lemmas 5, 6 and 9, both $[\sigma_1, \tau_1]_{ms}$ and $[\sigma_2, \tau_2]_{ms}$ are isomorphic to $\mathbb{P}(\omega)/\text{fin}$ and hence to one another. ☐

Definition 2. Let $\{\, L_i \mid i \in I \,\}$ be a family of bounded lattices. The *lower weak product* of the family is given by

$$\Pi^{lw}_{i\in I} L_i = \{\, f \in \prod_{i\in I} L_i \mid f(i) = 0 \text{ for all but finitely many } i \in I \,\},$$

ordered componentwise. This is a lattice with a least element but no greatest element (unless I is finite).

Lemma 11. *Let X be a countable set, and let $\sigma, \tau \in \Sigma_m(X)$ with $\sigma < \tau$ and τ a limit with respect to σ. Then there exist subsets X_n of X for $n \in \omega$ such that, for each n, $\tau \restriction X_n$ is a successor with respect to $\sigma \restriction X_n$, and*

$$[\sigma, \tau]_{ms} \cong \Pi^{lw}_{n\in\omega} [\sigma \restriction X_n, \tau \restriction X_n]_{ms}$$

Proof. Let x be the base of $[\sigma, \tau]$. Choose a strictly increasing sequence $(\mu_n)_{n\in\omega}$ such that $\sigma = \mu_0$, $\tau = \sup\{\, \mu_n \mid n \in \omega \,\}$. For each $n \in \omega$ choose a closed set A_n with $\mu_n = \langle \sigma, A_n\rangle$, in such a way that $A_{n+1} \subseteq A_n$ for each n.

For each $n \in \omega$ let $B_n = A_n \setminus A_{n+1}$ and let $X_n = B_n \cup \{x\}$. Notice that $\tau \restriction X_n = \langle \sigma \restriction X_n, \{x\}\rangle$, so $\tau \restriction X_n$ is a successor with respect to $\sigma \restriction X_n$.

We define $\theta : [\sigma, \tau]_{ms} \to \Pi^{lw}_{n\in\omega}[\sigma \restriction X_n, \tau \restriction X_n]_{ms}$ as follows: for $\mu \in [\sigma, \tau]_{ms}$ choose a closed set B with $\mu = \langle \sigma, B\rangle$. Then $\theta(\mu)(n) = \langle \sigma \restriction X_n, B \cap X_n\rangle$ for each n.

Now if B is closed in X then $B \cap X_n$ is closed in X_n so $\langle \sigma \restriction X_n, B \cap X_n\rangle \in [\sigma \restriction X_n, \tau \restriction X_n]_{ms}$. Moreover, for any σ, B, C and Y, if $\langle \sigma, B\rangle = \langle \sigma, C\rangle$ then $\langle \sigma \restriction Y, B \cap Y\rangle = \langle \sigma \restriction Y, C \cap Y\rangle$. Thus, to show that θ is well-defined we only need to show that, for every $\mu \in [\sigma, \tau]_{ms}$, $\theta(\mu)(n) = \sigma \restriction X_n$ for all but finitely many $n \in \omega$. So suppose $\mu = \langle \sigma, B\rangle \in [\sigma, \tau]_{ms}$. Then $B \in \tau = \langle \sigma \cup \{\, A_n \mid n \in \omega \,\}\rangle$ so, for some $n \in \omega$, $B \in \langle \sigma, \bigcap_{m\leqslant n} A_m\rangle = \langle \sigma, A_n\rangle$. Thus there is some $U \in \sigma$ with $x \in U \cap A_n \subseteq B$. But then for every $m \geqslant n$,

$$x \in U \cap X_m \subseteq U \cap A_n \subseteq B,$$

so $B \cap X_m \in \sigma \restriction X_m$. Thus $\langle \sigma \restriction X_m, B \cap X_m\rangle = \sigma \restriction X_m$ for all $m \geqslant n$, as required.

For any σ, B, C and Y, if $\langle \sigma, B\rangle \leqslant \langle \sigma, C\rangle$ then $\langle \sigma \restriction Y, B \cap Y\rangle \leqslant \langle \sigma \restriction Y, C \cap Y\rangle$. Thus if $\mu, \nu \in [\sigma, \tau]_{ms}$ with $\mu \leqslant \nu$ then $\theta(\mu) \leqslant \theta(\nu)$.

Conversely, suppose $\mu, \nu \in [\sigma, \tau]_{ms}$ with $\theta(\mu) \leqslant \theta(\nu)$. Choose B, C closed with $\mu = \langle \sigma, B\rangle$, $\nu = \langle \sigma, C\rangle$. Then $B \in \tau$ so for some $n_1 \in \omega$ and $U_1 \in \sigma$, $x \in U_1 \cap A_{n_1} \subseteq B$. Similarly $x \in U_2 \cap A_{n_2} \subseteq C$ for some $n_2 \in \omega$, $U_2 \in \sigma$. Put $n = \max\{n_1, n_2\}$ and $U = U_1 \cap U_2$. Then $U \cap B \cap X_m = U \cap C \cap X_m = U \cap X_m$ for all $m \geqslant n$. Now $\theta(\mu) \leqslant \theta(\nu)$ so for each $m < n$ we have $\langle \sigma \restriction X_m, B \cap X_m\rangle \leqslant \langle \sigma \restriction X_m, C \cap X_m\rangle$. Thus for each $m < n$ there is some $V_m \in \sigma$ with $x \in V_m \cap C \cap X_m \subseteq B \cap X_m$. Put $V = \bigcap_{m<n} V_m$. Then $x \in U \cap V \cap C \subseteq B$, so $\langle \sigma, B\rangle \leqslant \langle \sigma, C\rangle$, i.e. $\mu \leqslant \nu$.

Therefore we have $\theta(\mu) \leqslant \theta(\nu)$ if and only if $\mu \leqslant \nu$, so θ is an order-embedding. It remains only to show that θ is onto. So let $f \in \Pi^{lw}_{n\in\omega}[\sigma \restriction X_n, \tau \restriction X_n]_{ms}$. Choose n so that for all $m > n$, $f(m) = \sigma \restriction X_m$. For each $m \leqslant n$ choose C_m closed in X_m with $f(m) = \langle \sigma \restriction X_m, C_m\rangle$. Notice that, because each A_m is τ-clopen, each X_m is closed in X, so each C_m is closed. Note also that each B_m is σ-open, so $\bigcup_{m>n} X_m = X \setminus \bigcup_{m\leqslant n} B_m$ is closed. Thus

$$C = \bigcup_{m\leqslant n} C_m \cup \bigcup_{m>n} X_m$$

is closed, contains x and is open in τ. Thus $\mu = \langle \sigma, C \rangle \in [\sigma, \tau]_{ms}$, and clearly $\theta(\mu) = f$, as required. $\square$

Combining these results, we obtain the following:

Theorem 4. *Assume CH. Let X be a countable set. Then every basic interval in $\Sigma_m(X)$ is isomorphic to either $S(\mathbb{P}(\omega)/\mathit{fin})$ or $S(\Pi^{lw}_{n\in\omega}\mathbb{P}(\omega)/\mathit{fin})$.*

It is well-known that the assertion that all Boolean algebras of cardinality at most $\mathfrak{c}$ with property H_ω are isomorphic to $\mathbb{P}(\omega)/\mathrm{fin}$ is equivalent to CH. This leads us to the following questions:

Question. Is it consistent that there exist three non-isomorphic basic intervals in $\Sigma_m(X)$ for X countable?

Question. Does the negation of the Continuum Hypothesis imply the existence of three non-isomorphic basic intervals in $\Sigma_m(X)$ for X countable?

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