QUASIADDITIVE PROPERTIES AND BILIPSCHITZ CONDITIONS

V. V. Aseev[1], M. K. Vamanamurthy, and M. Vuorinen

30 December 1996 (asvv35.tex)

Abstract. Some properties of quasiadditive functions are studied. Various Hölderian growth estimates are obtained, connections to the classes of Lipschitz and quasisymmetric functions are established, and functional inequalities proved for these functions. Important examples of quasiadditive functions include distortion functions of quasiconformal maps.

1991 Mathematics Subject Classification. Primary: 26 D 07 Secondary: 30 C 62

1 Introduction

Functional equations and inequalities have been studied extensively. See for instance the monographs [A], [K1] - [K3], [KCG], which also discuss applications to nonlinear analysis, economics and logistic systems. The important Cauchy's functional equation

$$f(x+y) = f(x) + f(y),$$

is studied in [K3, p. 120]. The solutions of this equation are called *additive functions*.

From the point of view of functional equations, one may consider this paper as an attempt to generalize the above additive property. We are thus lead to *quasiadditive functions* by which we mean increasing functions $g : (0,\infty) \to (0,\infty)$ satisfying for all $x, y \in (0,\infty)$ the condition

$$a \le \frac{g(x) + g(y)}{g(x+y)} \le b$$

where a and b are positive constants. Because g is increasing, we can always take $b = 2$.

Further motivation for the study of this condition is provided by nontrivial examples of quasiadditive functions. It was observed in [QVu, Lemma 3.48] (see also Theorem 7.6 below) that certain transcendental functions defined in terms of complete elliptic integrals have this quasiadditive property. Because these transcendental functions are nonlinear, the quasiadditive property is rather surprising, and it yields a useful technique for proving inequalities and growth estimates.

The following lemma provides a simple sufficient condition for a function to have this quasiadditive property.

1.1. Lemma. *Let $g : (0,\infty) \to (0,\infty)$ be a function and let $K > 0$.*

[1] **This research was partially supported by RFFI (Grant 94-01-01277).**

(1) If $g(t)/(t^K)$ is increasing, then

$$\frac{g(s)+g(t)}{g(s+t)} \le \max\{1, 2^{1-K}\}.$$

(2) If $g(t)/(t^K)$ is decreasing, then

$$\frac{g(s)+g(t)}{g(s+t)} \ge \min\{1, 2^{1-K}\},$$

for all $s, t \in (0, \infty)$.

(3) For $K \ge 1$, if $g(t)/(t^{1/K})$ is increasing and $g(t)/(t^K)$ is decreasing, then

$$2^{1-K} \le \frac{g(s)+g(t)}{g(s+t)} \le 2^{1-1/K},$$

for all $s, t \in (0, \infty)$.

This lemma is a consequence of some elementary properties of the weighted mean values [BB, pp. 16-18]. Part (3) of Lemma 1.1 was recently used in [QVu, Lemma 3.48] (see also Theorem 7.6 below), in connection with the study of distortion functions for plane quasiconformal maps. These distortion functions are homeomorphisms $h : \Delta_1 \to \Delta_2$ where $\Delta_j \subset R, j = 1, 2$ are intervals. Usually $\Delta_1 = \Delta_2 = (0, 1)$ or $(0, \infty)$. Distortion functions provide the extreme limits for the variation of distances under quasiconformal maps of subdomains of the complex plane. These functions have expressions in terms of elliptic integrals and related functions (see Section 7 below).

Section 2 gives the basic properties of quasiadditive (QAD) functions on $(0, \infty)$. The stability of QAD functions under admissible perturbations (such as composition of functions) is studied in Section 3. Section 4 deals with the class of strongly QAD functions and refines some of the results of Section 3. Section 5 studies QAD functions defined on R and characterizes such functions as odd bilipschitz functions. Connection between the QAD functions and quasisymmetric functions is studied in Section 6. Section 7 relates quasiconformal distortion functions with QAD functions.

2 Quasiadditive functions on $(0, \infty)$

Let $\mathcal{H}$ be the class of all positive nondecreasing functions f on $(0, +\infty)$, and let

(2.1) $$S_f(x, y) = (f(x) + f(y))/f(x + y).$$

2.2. Definition. A function $f \in \mathcal{H}$ is *quasiadditive* ($f \in$QAD) iff

$$\lambda_f \equiv \inf_{x,y>0} S_f(x, y) > 0.$$

For $f \in \mathcal{H}$, every nonnegative number $\lambda \le \lambda_f$ is called a *lower QAD bound* while λ_f is the *sharp lower bound*. Similarly, every number $\mu \ge \mu_f \equiv \sup_{x,y>0} S_f(x, y)$ is called an *upper QAD bound* while μ_f is the *sharp upper bound*.

2.3. Lemma. *For $f \in \mathcal{H}$, let $a = f(0+)$ and $b = \lim_{x\to\infty} f(x)$, so that $0 \le a < \infty$, $0 < b \le \infty$, and $a \le b$. Let $\lambda = \lambda_f$ and $\mu = \mu_f$. Then*
(i) $0 \le \lambda \le 1 + (a/b) \le \mu \le 2$.
(ii) If $a = 0$ or $b = \infty$, then $\lambda \le 1$.
(iii) If $\lambda > 1$, then $0 < a \le b \le a/(\lambda - 1)$.
(iv) If $a > 0$ or $b < \infty$, then $\mu = 2$.

Proof. (i) By (2.1), $0 < S_f(x, y) \le 2$, so that $0 \le \lambda \le \mu \le 2$.
Clearly, $\lambda \le 1 + (f(y)/f(x))$, for all $x, y \in (0, \infty)$. Now letting $y \to 0$ and then $x \to \infty$, we get $\lambda \le 1 + (a/b)$. Next, let x be a point of continuity of f. Then,

$$\mu \ge \lim_{y\to 0} S_f(x, y) = 1 + (a/f(x)) \ge 1 + (a/b).$$

(ii) This follows from (i).
(iii) This follows from (i) and (ii).
(iv) If $a > 0$, then

$$2 \ge \mu \ge S_f(x, x) = 2f(x)/f(2x) \to 2,$$

as $x \to 0$. If $b < \infty$, then letting $x \to \infty$ in the above inequality, the result follows. □

The converses of Lemma 2.3(ii),(iii), (iv) are false as seen from Examples 2.7(7),(8).

2.4. Lemma. *For $f, g \in \mathcal{H}$, and $c > 0$, a constant, let $F = \max\{f, g\}, G = \min\{f, g\}$ and let cf and f_c denote the functions given by $(cf)(x) = cf(x)$ and $f_c(x) = f(cx)$, for all $x \in (0, \infty)$. Then*
(i) $S_{cf}(x, y) = S_f(x, y)$ and $S_{f_c}(x, y) = S_f(cx, cy)$. Consequently, $\lambda_{cf} = \lambda_{f_c} = \lambda_f$, and $\mu_{cf} = \mu_{f_c} = \mu_f$.
(ii) $(1/2)S_f.S_g \le S_{f.g} \le \min\{S_f, S_g\} \le S_{f+g} \le \max\{S_f, S_g\}$.
Consequently, $(1/2)\lambda_f.\lambda_g \le \lambda_{f.g} \le \min\{\lambda_f, \lambda_g\} \le \lambda_{f+g}, \mu_{f.g} \le \min\{\mu_f, \mu_g\}$ and $\mu_{f+g} \le \max\{\mu_f, \mu_g\}$.
(iii) $\min\{S_f, S_g\} \le S_F$, and $S_G \le \max\{S_f, S_g\}$. Consequently, $\min\{\lambda_f, \lambda_g\} \le \lambda_F$, and $\mu_G \le \max\{\mu_f, \mu_g\}$.

Proof. (i) This is clear.
(ii) First,

$$[2S_{f.g}(x, y) - S_f(x, y)S_g(x, y)]f(x + y)g(x + y) = [f(x) - f(y)][g(x) - g(y)] \ge 0,$$

and the first inequality follows. Next,

$$[S_{f.g}(x, y)]f(x + y) \le f(x) + f(y), \text{ and } [S_{f.g}(x, y)]g(x + y) \le g(x) + g(y)$$

so that the second inequality follows. Next,

$$[S_{f+g}(x, y)][f(x + y) + g(x + y)] = S_f(x, y)f(x + y) + S_g(x, y)g(x + y)$$

whence, the third and fourth inequalities follow.
(iii) By symmetry, we may assume that $f(x + y) \le g(x + y)$. Then
$S_G(x, y)G(x + y)) = S_G(x, y)f(x + y) = G(x) + G(y) \le f(x) + f(y)$, so that $S_G(x, y) \le S_f(x, y) \le \max\{S_f(x, y), S_g(x, y)\}$. Similarly, $S_F(x, y) \ge \min\{S_f(x, y), S_g(x, y)\}$. □

2.5. Lemma. *Let α, β, A, B, x_0 be positive numbers such that $C = Bx_0^\beta / Ax_0^\alpha \geq 1$. Let the function f be defined by*

$$f(x) = \begin{cases} Ax^\alpha, & \text{for } 0 < x \leq x_0; \\ Bx^\beta, & \text{for } x > x_0. \end{cases}$$

Then $\lambda_f = C^{-1} \min\{1, 2^{1-\alpha}, 2^{1-\beta}\}$ and $\mu_f = \max\{1, 2^{1-\alpha}, 2^{1-\beta}\}$, respectively.

Proof. In view of Lemma 2.4 (i), we may assume $x_0 = A = 1$ and $B = C \geq 1$. The well-known estimates (cf. [BB])

$$\min\{1, 2^{1-p}\} \leq (x^p + y^p)/(x+y)^p \leq \max\{1, 2^{1-p}\}, \tag{2.6}$$

for all $p > 0$, will be used in the proof (see also [Vu, (3.9), p.623]). Denote $\delta = \min\{\alpha, \beta\}$, $\gamma = \max\{\alpha, \beta\}$, $m = \min\{1, 2^{1-\gamma}\}$, and $M = \max\{1, 2^{1-\delta}\}$.

Case 1. If $x, y, x+y \leq 1$ then (2.6) with $p = \alpha$ yields

$$S_f(x,y) = (x^\alpha + y^\alpha)/(x+y)^\alpha \in [\min\{1, 2^{1-\alpha}\}, \max\{1, 2^{1-\alpha}\}] \subset [m, M].$$

Case 2. If $x, y \leq 1$ while $x + y > 1$ then (2.6) applied to

$$(x^\gamma + y^\gamma)/(C(x+y)^\gamma) \leq (x^\alpha + y^\alpha)/(C(x+y)^\beta) = S_f(x,y) \leq C^{-1}(x^\delta + y^\delta)/(x+y)^\delta$$

gives $S_f(x,y) \in [C^{-1}\min\{1, 2^{1-\gamma}\}, C^{-1}\max\{1, 2^{1-\delta}\}] \subset [C^{-1}m, C^{-1}M]$.
Case 3. If $x, y > 1$ then (2.6) with $p = \beta$ implies

$$S_f(x,y) = (Cx^\beta + Cy^\beta)/(C(x+y)^\beta) \in [\min\{1, 2^{1-\beta}\}, \max\{1, 2^{1-\beta}\}] \subset [m, M].$$

Case 4. Let $x \leq 1 < y$. Then we employ estimates similar to those in [Vu, Th.3.10(3), p.624]. Since

$$S_f(x,y) = (x^\alpha + Cy^\beta)/(C(x+y)^\beta) = (1 + x^\alpha/(Cy^\beta))/(1 + x/y)^\beta$$

we obtain

$$C^{-1}\frac{y^\gamma + x^\gamma}{(x+y)^\gamma} \leq \frac{1 + C^{-1}x^\gamma/y^\gamma}{(1+x/y)^\gamma} \leq S_f(x,y) \leq \frac{1 + C^{-1}x^\delta/y^\delta}{(1+x/y)^\delta} \leq \frac{y^\delta + x^\delta}{(x+y)^\delta}.$$

Now (2.6) yields the desired estimates:

$$S_f(x,y) \in [C^{-1}\min\{1, 2^{1-\gamma}\}, \max\{1, 2^{1-\delta}\}] \subset [C^{-1}m, M].$$

Case 5. The case $y \leq 1 < x$ is the same as Case 4, since $S_f(x,y) \equiv S_f(y,x)$.
Hence the stated QAD bounds have been obtained and it only remains to prove their sharpness.
If $\alpha, \beta \geq 1$ then $M = 1$ and for $0 < x < 1,\ y > 1$,

$$S_f(x,y) = (x^\alpha + Cy^\beta)/C(y+x)^\beta \to 1 = M,$$

as $x \to 0$. If $\alpha \le \beta$ and $\alpha < 1$ then $M = 2^{1-\alpha}$ and $S_f(x,x) = 2^{1-\alpha} = M$, for $0 < x \le \frac{1}{2}$. If $\beta \le \alpha$ and $\beta < 1$ then $M = 2^{1-\beta}$ and $S_f(x,x) = 2^{1-\beta} = M$, for $x > 1$. Hence M is the sharp upper QAD bound in all cases.

If $\alpha, \beta \le 1$, then $m = 1$ and for $0 < x \le \frac{1}{2}$

$$S_f(2x, 1-x) = [(2x)^\alpha + (1-x)^\alpha]/C(1+x)^\beta \to C^{-1} = C^{-1}m$$

as $x \to 0$. If $\beta \le \alpha$ and $\alpha > 1$ then $m = 2^{1-\alpha}$ and for $\frac{1}{2} < x \le 1$,

$$S_f(x,x) = 2x^\alpha/C(2x)^\beta \to 2^{1-\alpha}/C = C^{-1}m$$

as $x \to \frac{1}{2}$. If $\alpha \le \beta$ and $\beta > 1$ then $m = 2^{1-\beta}$ and

$$S_f(1,1) = 2/C(2)^\beta = C^{-1}m.$$

Hence $C^{-1}m$ is the sharp lower QAD bound in each possible case. $\square$

2.7. Examples.

(1) For $c \in (0,1]$, let $f(x) = x^{1-\log_2 c}$, for all $x \in (0,\infty)$. Then $\lambda_f = c$, and $\mu_f = 1$.

(2) For $c \in (0,1]$, let $f(x) = 1$, if $x \in (0,1]$ and $= x^{1-\log_2 c}$, if $x \in (1,\infty)$. Then, $\lambda_f = c$ and $\mu_f = 2$.

(3) For $c \in [1,2)$, let $f(x) = x^{1-\log_2 c}$, for all $x \in (0,\infty)$. Then $\lambda_f = 1$ and $\mu_f = c$.

(4) For $c \in (0,2]$, $p > 0$ let $f(x) = 1$, if $x \in (0,1]$, and $= (2/c)^p$, if $x \in (1,\infty)$. Then, $\lambda_f = 2^{1-p}c^p$, and $\mu_f = 2$.

(5) Let $f(x) = \log(1+x)$, for all $x \in (0,\infty)$. Then, $\lambda_f = 1$, and $\mu_f = 2$.

(6) Let $f(x) = \tanh x$, for all $x \in (0,\infty)$. Then, $\lambda_f = 1$, and $\mu_f = 2$.

(7) Let $f(x) = \exp(x)$, for all $x \in (0,\infty)$. Then, $\lambda_f = 0$, and $\mu_f = 2$.

(8) For $a > 0$, let $f(x) = ax$, for all $x \in (0,\infty)$. Then, $\lambda_f = \mu_f = 1$.

(9) Let $f(x) = 4x^2 + 1$, if $x \in (0,1]$, and $= 5$, for all $x \in (1,\infty)$. Then, $\lambda_f = 4/5$, and $\mu_f = 2$.

(10) For $a, b > 0$, let $f(x) = ax + b$, for all $x \in (0,\infty)$. Then, $\lambda_f = 1$, and $\mu_f = 2$.

(11) Let $f(x) = x\exp(x)$, for all $x \in (0,\infty)$. Then, $\lambda_f = 0$, and $\mu_f = 1$.

(12) For $a \in (0,1]$, and $b \in [1,2)$, let $f(x) = x^{1-\log_2 a}$, if $x \in (0,1]$, and $= x^{1-\log_2 b}$, if $x \in [1,\infty)$. Then $\lambda_f = a$, and $\mu_f = b$.

(13) For $c \in (0,1]$, let $f(x) = 1$, if $x \in (0,1]$, and $= x/c$, if $x \in (1,\infty)$. Then, $\lambda_f = c$, and $\mu_f = 2$.

(14) For $c \in (0,1)$, let $f(x) = 1$, if $x \in (0,1]$, and $= x - 2 + (2/c)$, if $x \in (1,\infty)$. Then, $\lambda_f = c$, and $\mu_f = 2$.

For $\lambda \in (0,2]$, let $\mathcal{H}_\lambda$ be the family of all $f \in \mathcal{H}$ with a lower QAD bound λ, and let

$$\mathcal{H}^1 = \{f \in \mathcal{H} : \quad f(1) = 1\}.$$

By Examples 2.7, we see that the class $\mathcal{H}_\lambda \cap \mathcal{H}^1$ is nonempty, for each $\lambda \in (0,2]$.

2.8. Definition. For each $\lambda \in (0,2]$, we define the function k_λ on $(0,\infty)$, by

$$k_\lambda(x) = \sup_{f \in \mathcal{H}_\lambda} \sup_{y>0} \frac{f(xy)}{f(y)}.$$

2.9. Lemma. *For each $\lambda \in (0{,}2]$, the function k_λ is finite, nondecreasing and satisfies the following properties.*

(1) $k_\lambda(x) = \sup_{f\in\mathcal{H}_\lambda\cap\mathcal{H}^1} f(x)$.

(2) $k_\lambda(x) = 1$, *for all* x *in (0, 1].*

(3) $k_\lambda(2^n) \le (2/\lambda)^n$, *for all* $n = 1, 2, \ldots$, *with equality if* $\lambda \in (0, 1]$.

(4) If $x \in (1, 2]$, *then* $k_\lambda(x) = 2/\lambda$.

(5) If $\lambda \in (1, 2]$, *then* $k_\lambda(x) \le 1/(\lambda - 1)$, *for all* $x \in (0, \infty)$.

Proof. Monotoneity of k_λ follows from that of each f.

Assertion (1) follows from Lemma 2.4 (i), since for each $y > 0$, the function $f_y(x) \equiv f(xy)/f(y)$, belongs to $\mathcal{H}_\lambda \cap \mathcal{H}^1$, for all $f \in \mathcal{H}_\lambda$.

Assertion (2) follows from (1) and Example 2.7 (2).

If $f \in \mathcal{H}_\lambda \cap \mathcal{H}^\infty$, then $f(2^n) \le (2/\lambda)f(2^{n-1})$. Hence, by induction, $f(2^n) \le (2/\lambda)^n$, so that $k_\lambda(2^n) \le (2/\lambda)^n$. On the other hand, if $\lambda \in (0, 1]$, then by Example 2.7(1), it follows that $k_\lambda(2^n) \ge (2/\lambda)^n$, so that (3) follows.

Next, let $x \in (1, 2]$. From Example 2.7 (4), it follows that $k_\lambda(x) \ge 2/\lambda$. On the other hand, (3) implies that $k_\lambda(x) \le k_\lambda(2) \le 2/\lambda$, so that (4) follows. Finally, if $f \in \mathcal{H}_\lambda \cap \mathcal{H}^1$, then, by Lemma 2.3(iii), $f(x) \le f(1)/(\lambda - 1) = 1/(\lambda - 1)$, which yields (5). □

2.10. Lemma. *For each* $\mathcal{F} \subset \mathcal{H}_\lambda$ *the upper bound* $g(x) \equiv \sup_{f\in\mathcal{F}} f(x)$ *is either identically* $+\infty$ *or* $\in \mathcal{H}_\lambda$.

Proof. Let $g(x_0) = M < +\infty$. Since g is nondecreasing, $g(x) \le M$ for all $x \le x_0$. If $x > x_0$ then Lemma 2.9 gives the estimate $f(hx_0) \le f(x_0) \cdot k_\lambda(h)$ for every $f \in \mathcal{H}_\lambda$, $h \ge 1$. Letting $h = x/x_0 > 1$ gives $f(x) \le M \cdot k_\lambda(x/x_0) < +\infty$ for all $f \in \mathcal{F}$. Thus $g(x) < +\infty$ for all $x > 0$.

If $f \in \mathcal{F}$ and $x, y, \in (0, \infty)$, then $\lambda f(x + y) \le f(x) + f(y) \le g(x) + g(y)$. Hence, taking supremum over $f \in \mathcal{F}$, we get $\lambda g(x + y) \le g(x) + g(y)$, so that $g \in \mathcal{H}_\lambda$. □

2.11. Theorem. *For each* $\lambda \in (0, 2]$, *the function* k_λ *has sharp lower QAD bound* λ, *sharp upper QAD bound* 2 *and satisfies the submultiplicative property*

$$k_\lambda(xy) \le k_\lambda(x) \cdot k_\lambda(y), \tag{2.12}$$

for all $x, y > 0$.

Proof. Since $k_\lambda(1) = 1$, Lemma 2.10 implies that $k_\lambda \in \mathcal{H}_\lambda \cap \mathcal{H}^1$. Now,

$$S_{k_\lambda}(1, 1) = \frac{k_\lambda(1) + k_\lambda(1)}{k_\lambda(2)} = \frac{1 + 1}{(2/\lambda)} = \lambda,$$

by Lemma 2.9. Hence λ is the sharp lower *QAD* bound for k_λ.

On the other hand, $S_f(1/2, 1/2) = 2$, so that $\mu_f = 2$. Next, for $f \in \mathcal{H}_\lambda$, and $x, y, z \in (0, \infty)$, we have $f(xyz)/f(z) = [f(xyz)/f(yz)][f(yz)/f(z)] \le k_\lambda(x).k_\lambda(y)$. Hence taking supremum over all $z \in (0, \infty)$ and $f \in \mathcal{H}_\lambda$, (2.12) follows.

2.13. Theorem. *Let* $\lambda \in (0, 1]$. *Then*

$$x^{1-\log_2(\lambda)} \le k_\lambda(x) \le \max\{1, (2/\lambda)x^{1-\log_2 \lambda}\}, \tag{2.14}$$

for all $x \in (0,\infty)$. Moreover, equality holds in the first, iff $x = 2^n$, for some nonnegative integer n, and in the second iff, $x \in (0, 1/2]$.

Proof. The first inequality follows from Definition 2.8 and Example 2.7 (1). For $x \in (0, 1]$, the second inequality follows from Lemma 2.9(2). If $x > 1$, choose a natural number n, such that $(n-1) < \log_2 x \leq n$. Then

$$k_\lambda(x) \leq k_\lambda(2^n) \leq (2/\lambda)^n < (2/\lambda)x^{1-\log_2 \lambda},$$

so that the second inequality is strict, in this case.

For $x \in (0, 1]$, Lemma 2.9(2) implies that equality in the first, holds iff $x = 1$, and in the second, iff $x \in (0, 1/2]$. Finally, Examples 2.7(8), (13) and (14), show that the first inequality cannot reduce to identity on $[1,\infty)$. □

2.15. Theorem (Hölder estimate for QAD). *For each $f \in \mathcal{H}_\lambda$, $0 < \lambda \leq 2$,*

$$1/(k_\lambda(y/x)) \leq f(x)/f(y) \leq k_\lambda(x/y), \tag{2.16}$$

for all $x, y > 0$.

In particular, if $\lambda \in (0, 1]$, then

$$f(1)\min\{1, (\lambda/2)x^{1-\log_2 \lambda}\} \leq f(x) \leq f(1)\max\{1, (2/\lambda)x^{1-\log_2 \lambda}\} \tag{2.17}$$

for all $x > 0$.

Proof. These follow immediately from Definition 2.8 and Theorem 2.13. □

2.18. Definition. A function $f \in \mathcal{H}$ is called *quasimultiplicative* (QM), iff there exist positive constants a and b such that

$$a \leq M_f(x, y) \equiv f(x)f(y)/f(xy) \leq b, \tag{2.19}$$

for all $x, y \in (0,\infty)$.

2.20. Theorem. *(1) If $f \in \mathcal{H}$ is bounded away from 0 and ∞, then $f \in QM$, but not conversely.*

(2) If $f \in QM$, then $f \in QAD$, but not conversely. Moreover, if $\lambda \in (1, 2]$, then the function $k_\lambda \in QM$, but if $\lambda \in (0, 1]$, then $k_\lambda \notin QM$.

(3) However, QM has no relation to the behaviour of upper QAD bound .

Proof. (1) Let $0 < m \leq f(x) \leq M < \infty$, for all $x \in (0,\infty)$. Then, $m \leq f(xy) \leq M$, and $m^2 \leq f(x)f(y) \leq M^2$, for all $x, y \in (0,\infty)$. Hence $m^2/M \leq f(x)f(y)/f(xy) \leq M^2/m$, so that $f \in QM$. On the other hand the function $f(x) \equiv x, \in QM$, but unbounded on $(0,\infty)$.

(2) If $f \in$ QM, then by (2.19), we get $a/f(2) \leq f(y)/f(2y) \leq b/f(2)$, for all $y \in (0,\infty)$. Hence, for $0 < x \leq y$, we have $S_f(x, y) \geq f(y)/f(2y) \geq a/f(2)$, so that $f \in$ QAD.

Next, the function f in Example 2.7(5), is in QAD, but not in QM, since $M_f(x, x) = [\log(1+x)]^2/[\log(1+x^2)]$, which tends to ∞, while $\log(1+x)\log(1+(1/x)) \to 0$, as $x \to \infty$.

Let $g = k_\lambda$. Then, by Theorem 2.11, $M_g(x, y) \geq 1$, for all $\lambda \in (0, 2]$, and $x, y \in (0,\infty)$.

Now, if $\lambda \in (1, 2]$, then, by Lemma 2.9(5), we get

$$M_g(x, y) \leq g(x)g(y) \leq 1/(\lambda-1)^2,$$

so that g is in QM. But, if $\lambda \in (0,1]$, then, by (2.14), $g(x)g(1/x) \geq g(x)$, which tends to ∞, as x tends to ∞.

(3) The constant function $f(x) \equiv 1, \in QM$, but $\mu_f = 2$. On the other hand, for $0 < \lambda \leq 1 \leq \mu < 2$, the function in Example 2.7(12) is not in QM although it has both lower QAD bound positive and upper QAD bound less than 2. □

We next show the rather surprising fact that for $f \in \mathcal{H}$, the lower and upper bounds of $S_f(x,y)$ on $(0,\infty) \times (0,\infty)$, are close to those of $S_f(x,x)$ on $(0,\infty)$.

2.21. Definition. For $f \in \mathcal{H}$, let

$$\lambda'_f = \inf_{x \in (0,\infty)} S_f(x,x), \tag{2.22}$$

and

$$\mu'_f = \sup_{x \in (0,\infty)} S_f(x,x). \tag{2.23}$$

2.24. Theorem. *For $f \in \mathcal{H}$,*

$$\lambda_f \leq \lambda'_f \leq 2\lambda_f, \tag{2.25}$$

and

$$\mu'_f \leq \mu_f \leq 1 + (\mu'_f/2). \tag{2.26}$$

Consequently, $\lambda_f > 0$, iff $\lambda'_f > 0$, and $\mu_f < 2$, iff $\mu'_f < 2$.

Proof. We only need to prove the second inequalities in (2.25) and (2.26). By symmetry, we may assume that $0 < x \leq y < \infty$. Then

$$(1/2)S_f(y,y) \leq S_f(x,y) \leq 1 + (1/2)S_f(x,x),$$

from which the results follow. □

2.27. Lemma. *For $f, g \in \mathcal{H}$, let $F = \max\{f,g\}$ and $G = \min\{f,g\}$. Then*

$$S_F(x,x) \leq \max\{S_f(x,x), S_g(x,x)\}, \tag{2.28}$$

and

$$S_G(x,x) \geq \min\{S_f(x,x), S_g(x,x)\}, \tag{2.29}$$

consequently,

$$\mu'_F \leq \max\{\mu'_f, \mu'_g\}, \lambda'_G \geq \min\{\lambda'_f, \lambda'_g\}. \tag{2.30}$$

Proof. May assume $f(x) \leq g(x)$. Then

$$S_F(x,x) = \frac{2F(x)}{F(2x)} = \frac{2g(x)}{F(2x)} \leq \frac{2g(x)}{g(2x)} = S_g(x,x).$$

and

$$S_G(x,x) = \frac{2G(x)}{G(2x)} = \frac{2f(x)}{G(2x)} \geq \frac{2f(x)}{f(2x)} = S_f(x,x).$$

□

The important result on algebraic closure of quasiadditivity follows immediately from Lemmas 2.4. and 2.27.

2.31. Theorem. *Let $f, g \in \mathcal{H}$.*
(i) If f and g have positive lower QAD bounds, then so do the functions $f + g$, $f.g, \max\{f, g\}$ and $\min\{f.g\}$.
(ii) If f and g have upper QAD bounds less than 2, then so do the functions $f + g, f.g, \max\{f, g\}$ and $\min\{f, g\}$.

3 QAD criteria and composition theorems

Results in this section will be useful in estimating the lower QAD bounds for various functions when direct calculation of $\inf S_f(x, y)$ is tedious.

3.1. Lemma. *Let functions $f, g \in \mathcal{H}$ be such that $0 < a \le f(x)/g(x) \le b < +\infty$, for all $x > 0$. Then*

(1) $\quad a/b \le S_f(x, y)/S_g(x, y) \le b/a,$

and

(2) $\quad a^2/b \le M_f(x, y)/M_g(x, y) \le b^2/a,$

for all $x, y \in (0, \infty)$.

Proof. The assertions are obvious. □

3.2. Theorem. *Let $f \in \mathcal{H}$, $h > 1$ and let*

$$\sup_{x>0} f(hx)/f(x) \le k < +\infty. \tag{3.3}$$

Then $f \in \mathcal{H}_\lambda$, with $\lambda \ge (1/k)^n \ge (1/k)^{1+\log_h 2}$, where the integer n is defined by $n - 1 < \log_h 2 \le n$. When $h = 2$, and $k > 1$, the estimate $\lambda \ge 1/k$, cannot be improved.

Proof. If $h \ge 2$, the inequality $f(2x)/f(x) \le f(hx)/f(x) \le k$ holds for all $x > 0$. Given $x, y > 0$, $z = \max\{x, y\}$, we have $S_f(x, y) \ge f(z)/f(2z) \ge 1/k$, so that $f \in \mathcal{H}_\lambda$ with $\lambda \ge 1/k$. Next, if $h < 2$, let $n - 1 < \log_h 2 \le n$; $n = 1, 2, \ldots$. Then $h^{n-1} < 2 \le h^n$. Hence by (3.3) and induction,

$$\frac{f(2x)}{f(x)} \le \frac{f(h^n x)}{f(x)} \le k^n,$$

for all $x > 0$. With $h = 2$ we get the desired result: $f \in \mathcal{H}_\lambda$ where $\lambda \ge (1/k)^n \ge (1/k)^{1+\log_h 2}$. The function $f(x) = k^n$ for $x \in (2^n, 2^{n+1}]; n = 0, \pm 1, \ldots$ gives the example where $f(2x)/f(x) \equiv k$ while

$$S_f(2, 2^{n+1} - 1) = (1 + k^n)/k^{n+1} \to 1/k \text{ as } n \to +\infty.$$

This relation shows that $1/k$ is the sharp lower QAD bound for this function. Thus in the case $h = 2, k > 1$, the estimate $\lambda \ge 1/k$ is best possible. □

3.4. Theorem. *Let $f \in \mathcal{H}$. Then the following three conditions are equivalent.*
(i) $f \in QAD$.
(ii) For each positive constant h, there exist positive constants a and b, such that

$$a \leq f(hx)/f(x) \leq b,$$

for all $x > 0$.
(iii) There exist positive constants, a, b, h, with $h \neq 1$, such that

$$a \leq f(hx)/f(x) \leq b,$$

for all $x > 0$.

Proof. (i) implies (ii). Let λ be a lower QAD bound for f. If $h = 1$, then $a = b = 1$, will do. Next, let h be $\neq 1$. First, if $h \geq 2$, then there exists a unique positive integer n, such that $2^n \leq h < 2^{n+1}$. Hence

$$a = 1 \leq f(hx)/f(x) \leq f(2^{n+1}x)/f(x) \leq (2/\lambda)^{n+1} \leq (2/\lambda)^{1+\log_2(h)} = b,$$

for all $x > 0$.
Next, if $1 < h < 2$, then

$$a = 1 \leq f(hx)/f(x) \leq f(2x)/f(x) \leq 2/\lambda = b,$$

for all $x > 0$.
Finally, if $0 < h < 1$, then $1/h > 1$, so that by the previous case, with $hx = y$,

$$1 \geq f(hx)/f(x) = f(y)/f(y/h) \geq 1/b,$$

for all $x > 0$.
(ii) implies (iii), is trivial.
(iii) implies (i). If $h > 1$, then the result follows from Theorem 3.2. If $0 < h < 1$, then $1/h > 1$, so that, with $y = x/h$, we have

$$1 \leq f(x/h)/f(x) = f(y)/f(hy) \leq 1/a,$$

so that the result again follows from Theorem 3.2. $\square$

3.5. Example. The function

$$f(x) = \begin{cases} 1/\log_2(2/x), & \text{for } 0 < x < 1; \\ \log_2(2x), & \text{for } x \geq 1. \end{cases}$$

has a lower QAD bound 4/9.

Proof. If $x < 1/2$ then

$$f(2x)/f(x) = (\log_2(2/x))/\log_2(2/(2x)) = 1 + 1/\log_2(1/x) < 2.$$

If $1/2 \leq x \leq 1$ then

$$f(2x)/f(x) = (\log_2(4x)) \cdot (\log_2(2/x)) \leq \max_{1/2 \leq x \leq 1} (2 + \log_2 x)(1 - \log_2 x) = 9/4.$$

If $x > 1$, then $f(2x)/f(x) = \log_2(4x)/\log_2(2x) = 1 + 1/\log_2(2x) \leq 2$. Thus (3.3) holds with $h = 2$ and $k = 9/4$. Then the result follows from Theorem 3.2. □

3.6. Remark. By a more careful calculation, the minimum of $S_f(x, y)$ in Example 3.5 can be shown to be $S_f(1/\sqrt{2}, 1/\sqrt{2}) = 8/9$, which is the sharp lower QAD bound for f.

Theorem 3.4 complements our earlier sufficient condition for QAD, given in Lemma 1.1.

The next result shows that the class of QAD functions on $(0, \infty)$ is closed under composition.

3.7. Theorem. *Let $f, g \in \mathcal{H}$ have positive lower QAD bounds λ_f, λ_g and upper QAD bounds μ_f, μ_g, respectively. Then the composition $f \circ g$ has a lower QAD bound $\lambda > 0$ and an upper QAD bound μ where*

$$\lambda \geq \lambda_f / k_{\lambda_f}(1/\lambda_g); \qquad \mu \leq \mu_f k_{\lambda_f}(\mu_g). \tag{3.8}$$

Proof. Let $x, y > 0$. From the inequalities

$$f(g(x)) + f(g(y)) \leq \mu_f f(g(x) + g(y)) \leq \mu_f f(\mu_g g(x + y));$$

$$f(g(x)) + f(g(y)) \geq \lambda_f f(g(x) + g(y)) \geq \lambda_f f(\lambda_g g(x + y))$$

we obtain the estimates

$$S_{f\circ g}(x, y) \leq \mu_f \sup_{t>0} \frac{f(\mu_g t)}{f(t)} \leq \mu_f k_{\lambda_f}(\mu_g);$$

$$S_{f\circ g}(x, y) \geq \lambda_f / \sup_{t>0} \frac{f(\lambda_g^{-1} t)}{f(t)} \geq \lambda_f / k_{\lambda_f}(1/\lambda_g),$$

which yield (3.8). □

The next theorem gives a necessary and sufficient condition for the QAD property in terms of the hyperbolic metric σ of R_+, defined by $\sigma(x, y) = |\log(x/y)|$, for $x, y > 0$.

3.9. Theorem. *Let $f \in \mathcal{H}$. Then $f \in QAD$ iff there exists a constant $M > 1$, such that*

$$\sigma(f(x), f(y)) \leq [1 + (c\, \sigma(x, y)/(\log 2))] \log M, \tag{3.10}$$

where $c = 0$, if $\sigma(x, y) \leq \log 2$, and $= 1$, otherwise.

Proof. By symmetry, we may assume that $0 < y \leq x$.

(1) Let $f \in QAD$, with a QAD lower bound λ. First, if $\sigma(x, y) \leq \log 2$, then $y \leq x \leq 2y$, so that

$$\sigma(f(x), f(y)) \leq \log(f(2y)/f(y)) \leq \log(2/\lambda) = \log M,$$

say. Next, if $\sigma(x, y) > \log 2$, then let n be the natural number such that, $n \log 2 < \sigma(x, y) \leq (n + 1) \log 2$. Then

$$\sigma(f(x), f(y)) \leq \log[f(2^{n+1} y)/f(y)] \leq (n + 1) \log(2/\lambda) < [1 + (c\, \sigma(x, y)/(\log 2))] \log M,$$

where $M = 2/\lambda$.

(2) Conversely, suppose that (3.10) holds. Then $\sigma(x, x+y) \le \log 2$, so that $c = 0$ and

$$\sigma(f(x), f(x+y)) = \log(f(x+y)/f(x)) \le \log M,$$

which implies that $f \in QAD$, with a lower QAD bound $1/M$. □

3.11. Theorem. *Let $p > 0$. If a function $f \in \mathcal{H}$ has a lower QAD bound $\lambda > 0$ and an upper QAD bound μ then the function $\varphi(x) = [f(x)]^p$ has lower and upper QAD bounds $\tilde{\lambda}$ and $\tilde{\mu}$ such that*

$$\tilde{\lambda} \ge a \cdot \lambda^p; \quad \tilde{\mu} \le b \cdot \mu^p \tag{3.12}$$

where $a = \min\{1, 2^{1-p}\}$, and $b = \max\{1, 2^{1-p}\}$. If $\lambda \le 1$ or $p \ge 1$, the estimate for $\tilde{\lambda}$ in (3.12) is best possible.

Proof. From (2.26)

$$a\lambda^p \le a[S_f(x,y)]^p \le S_\varphi(x,y) \le b[S_f(x,y)]^p \le b\mu^p.$$

If $p \ge 1$, then $a = 2^{1-p}$. The function $f(x) = 1$ for $0 < x \le 1$, $f(x) = 2/\lambda$ for $x > 1$, has sharp lower QAD bound λ. Hence the function $[f(x)]^p = 1$ for $0 < x \le 1$, $[f(x)]^p = (2/\lambda)^p$ for $x > 1$, has sharp lower QAD bound $2(\lambda/2)^p = 2^{1-p}\lambda^p = a\lambda^p$ (by Example 2.7 (4)). Thus the estimate for $\tilde{\lambda}$ in (3.12) in this case cannot be improved.

If $p < 1, 0 < \lambda \le 1$, then $a = 1$. The function

$$f(x) = \begin{cases} x, & \text{for } 0 < x \le 1; \\ (1/\lambda)x, & \text{for } x > 1 \end{cases}$$

has (by Lemma 2.27) sharp lower QAD bound $\lambda \cdot \min\{1, 2^0\} = \lambda$ while the function

$$[f(x)]^p = \begin{cases} x^p, & \text{for } 0 < x \le 1; \\ (1/\lambda)^p x^p, & \text{for } x > 1 \end{cases}$$

has (by Lemma 2.27) sharp lower QAD bound $\lambda^p \cdot \min\{1, 2^{1-p}\} = \lambda^p = a\lambda^p$. Thus the estimate for $\tilde{\lambda}$ in this case is best possible. □

3.13. Remark. The question of sharpness of the estimate (3.12) for $\tilde{\lambda}$ in the case $p < 1, 1 < \lambda \le 2$, is open. We observe that the estimates in Theorem 3.11 are better than those given by Theorem 3.7 in the general situation.

3.14. Theorem. *Let $p > 0$. If $f \in \mathcal{H}$ has a lower QAD bound $\lambda > 0$ then the function $\varphi(x) = f(x^p)$ has lower QAD bound $\tilde{\lambda}$ such that*

$$\tilde{\lambda} \ge \lambda, \qquad \text{if } p \le 1; \tag{3.15}$$

$$\tilde{\lambda} \ge \lambda(\lambda/2)^n \quad \text{if } 1 \le n < p \le n+1, n = 1, 2, \ldots.$$

If $p \le 1$ or $0 < \lambda \le 1$, and $p > 1$ is an integer, the estimates for $\tilde{\lambda}$ in (3.15) are best possible.

Proof. We derive (3.15) from the general formula in Theorem 3.7. Let $p \le 1$. Then $g(x) = x^p$ has sharp lower bound $\lambda_g = \min\{1, 2^{1-p}\} = 1$. Since $k_\lambda(1) = 1$ for each λ, (3.8) gives the estimate $\tilde{\lambda} \ge \lambda$. Let us show that this estimate cannot be improved. For each $p \le 1$ and $\lambda \in (0, 2]$ the function defined by $f(x) = 1$ for $0 < x \le 1$ and $f(x) = (2/\lambda)$ for $x > 1$, has a lower QAD bound λ. Since $f(x^p) \equiv f(x)$ then $\tilde{\lambda} = \lambda$, which proves the sharpness of the estimate $\tilde{\lambda} \geq \lambda$ in this statement.

Let $1 \leq n < p \leq n+1$, n be an integer. Then $\lambda_g = \min\{1, 2^{1-p}\} = 2^{1-p}$ and the general estimate (3.8) gives in this case the inequality $\tilde{\lambda} \geq \lambda/k_\lambda(2^{p-1}) \geq \lambda/k_\lambda(2^n)$. Hence the estimate $\tilde{\lambda} \geq \lambda(\lambda/2)^n$ follows from Lemma 2.9 (3).

Let $1 \leq n < p \leq n+1$, and let n be an integer, and $\lambda \leq 1$. Let us prove that if $p = n+1$ then the estimate $\tilde{\lambda} \geq \lambda(\lambda/2)^{p-1}$ just obtained cannot be improved. Indeed, the function $f(x) = x^{1-\log_2 \lambda}$ has sharp lower QAD bound λ while $f(x^p) = x^{p-p\log_2 \lambda}$ has sharp lower bound (see Lemma 2.27)

$$\min\{1, 2^{1-p+p\log_2 \lambda}\} = \min\{1, 2(\lambda/2)^p\} = \lambda(\lambda/2)^{p-1}.$$

Thus the sharpness of the estimate for $\tilde{\lambda}$ in this case is proved. □

The next theorem shows the invariance of λ' and μ' (cf. Definition 2.21) under the operation which maps f to f^*, where $f^*(x) = 1/f(1/x)$.

3.16. Theorem. *If $f \in \mathcal{H}$, then $\lambda'_f = \lambda'_{f^*}$ and $\mu'_f = \mu'_{f^*}$. Consequently, $\lambda_{f^*} \leq 2\lambda_f \leq 4\lambda_{f^*}$ and $\mu_{f^*} \leq 1 + (\mu_f)/2 \leq 1.5 + (\mu_{f^*})/4$.*

Proof. Since $S_{f^*}(x,x) = S_f(\frac{1}{2x}, \frac{1}{2x})$, the assertions follow from Theorem 2.24. □

3.17. Remark. The question of sharpness of the estimates for QAD bounds in Theorems 3.16 and 3.11 is open (in the case $p > 1, 2 \geq \lambda > 1$, and in the case $\lambda \leq 1, p > 1$ being noninteger).

3.18. Example. Many examples of QAD functions may be obtained from the solution of the functional equation

$$f(hx) = k \cdot f(x) \qquad \text{with } \ h > 1, k \geq 1, \tag{3.19}$$

which is a simple special case of a more general Schröder equation (see [K1, Chapter 7]). Given an interval $I = [x_0, hx_0) \subset R_+$ and a bounded nondecreasing function $\varphi : I \to R_+$, there exists a unique function f which is an extension of φ over all R_+ and satisfies the equation (3.19) in R_+. This statement is a special case of the general theorem on solutions of the Schröder equation (see [K2, Theorem 1.1, p.4]). This solution is given by [K2, (1.3), p.4]:

$$f(x) = k^n \varphi(h^{-n}x) \quad \text{for } x \in [h^n x_0, h^{n+1}x_0),\ n = 0, \pm 1, \dots, \tag{3.20}$$

where

$$k \geq \sup_{x\in I} \varphi(x) / \inf_{x \in I} \varphi(x) = \lim_{x \to hx_0} \varphi(x)/\varphi(x_0). \tag{3.21}$$

It is obvious that f will be a nondecreasing function iff φ is.

(The general theorem on monotoneity for solutions of Schröder equation is given in [K2, Theorem 10.1, p.35].)

It follows from Theorem 3.2 that (3.20) also implies that $f \in$QAD. Thus we have

3.22. Corollary. *Every solution $f(x)$ of the Schröder equation* (3.19) *in the class $\mathcal{H}$ of all positive nondecreasing functions on R_+ is quasiadditive and $f \in \mathcal{H}_\lambda$ where*

$$\lambda \geq (1/k)^n \geq (1/k)^{1+\log_h 2}, \tag{3.23}$$

and the integer n is defined by $n-1 < \log_h 2 \leq n$.

3.24. Example. Let $h = 2$, $I = [x_0, 2x_0)$, $f(x) \equiv 1$ on I, and $k > 1$. Then the corresponding solution of the Schröder equation (3.19) (that is the function $f(x) = k^n$ for $x \in [2^n x_0, 2^{n+1} x_0)$; $n = 0, \pm 1, ...$) has the sharp lower QAD bound $1/k$ and the sharp upper QAD bound $1 + (1/k)$. Indeed, we already know that $1/k$ is a lower QAD bound for f. Its sharpness follows from the calculations:

$$S_f(x_0, 2^n x_0 - x_0/2) = (1 + k^{n-1})/k^n \to 1/k \quad \text{ as } n \to +\infty.$$

Next, if $x \leq y$, then, $S_f(x, y) \leq 1 + f(x)/f(2x) = 1 + (1/k)$, and $S_f(x_0, 2x_0) = 1 + (1/k)$, so that $\mu_f = 1 + (1/k)$.

4 Strongly quasiadditive functions on R_+

4.1. Definition. A function $f \in \mathcal{H}$ will be called *strongly quasiadditive* (SQAD) iff it is QAD and has an upper QAD bound < 2.

By Lemma 2.3 (iv), if $f \in$ SQAD, then

$$f(x) \to +\infty \quad \text{as } x \to +\infty; \qquad f(x) \to 0 \quad \text{as } x \to 0, \tag{4.2}$$

and $\lambda_f \leq 1$.

4.3. Remarks. (i) The functions in Example 3.5 and Examples 2.7(2), (4)- (7), (9), (10), (13), (14) are QAD, but not SQAD. Thus the condition (4.2) is not sufficient for a QAD function f to be SQAD.

(ii) Lemma 1.1 (3) yields a sufficient condition for a function f to be in SQAD.

4.4. Remark. For $1 \leq \mu < 2$, let $\mathcal{U}_\mu$, be the class of all functions $f \in \mathcal{H}$ for which μ is an upper QAD bound. As an immediate consequence of Theorem 2.31 we may note that $\mathcal{U}_\mu$ is closed under certain basic algebraic operations. Note that by Examples 2.7 (3), (12) the class $\mathcal{U}_\mu \cap \mathcal{H}^1$ is nonempty for each $\mu \in [1, 2)$.

4.5. Definition. For each $\mu \in [1, 2)$, the function u_μ is defined on $[1, \infty)$ by

$$u_\mu(x) = \inf_{f \in \mathcal{U}_\mu} \inf_{y > 0} \frac{f(xy)}{f(y)}. \tag{4.6}$$

4.7. Lemma. *The function u_μ is nondecreasing on $[1, \infty)$ and satisfies*

(1) $u_\mu(x) = \inf_{f \in \mathcal{U}_\mu \cap \mathcal{H}^1} f(x)$,

(2) $u_\mu(x) = 1$, *for all* $x \in [1, 2)$,

(3) $u_\mu(2^n) = \left(\dfrac{2}{\mu}\right)^n, n = 0, 1, 2, \ldots$

(4) $u_\mu(x) \geq \max\{1, (\mu/2)x^{1-\log_2 \mu}\}$.

Proof. The same argument as in the proof of Lemma 2.9 proves (1) as well as the monotoneity of u_μ. It is also obvious that $u_\mu(1) = 1$ and $u_\mu(x) \geq 1$ for all $x \geq 1$. Since $2f(x) \geq \mu f(2x)$ for every $f \in \mathcal{U}_\mu$ and for all $x > 0$, the estimate

$$f(2^n x) \geq (2/\mu) f(2^{n-1}x) \geq ... \geq (2/\mu)^n f(x)$$

holds for each $n = 1, 2, ...$. It follows that $u_\mu(2^n) \geq (2/\mu)^n$ for all $n = 1, 2, ...$. Considering the function $f(x) = x^{1-\log_2 \mu}$ with sharp upper QAD bound (see Lemma 2.27) $\max\{1, 2^{\log_2 \mu}\} = \mu$, we obtain $f(2^n x)/f(x) \equiv 2^{n-n\log_2 \mu} = (2/\mu)^n$. Thus the inequality $u_\mu(2^n) \leq (2/\mu)^n$ holds, which together with the previous one, gives (3).

To show that $u_\mu(x) = 1$ for all $x \in [1, 2)$. Given a sufficiently small $\varepsilon > 0$ consider the function $f(x) = \varepsilon x$ for $0 < x < 1$, $f(x) = 1 - \varepsilon + \varepsilon x$ for $1 \leq x < 2$ $f(x) = x$ for $x \geq 2$. Let $0 < x \leq y$. If $x + y \geq 2$ then $f(x) \leq x, \quad f(y) \leq y$ and $S_f(x, y) \leq (x + y)/(x + y) = 1$. If $1 \leq x + y < 2$ then $x < 1$, $f(x) = \varepsilon x$, $f(y) \leq 1 - \varepsilon + \varepsilon y$ and

$$S_f(x, y) \leq (\varepsilon x + 1 - \varepsilon + \varepsilon y)/(1 - \varepsilon + \varepsilon(x + y)) = 1.$$

If $x + y < 1$ then $f(x) + f(y) = \varepsilon x + \varepsilon y = f(x + y)$ and $S_f(x, y) = 1$. Thus $f \in \mathcal{U}_1 \subset \mathcal{U}_\mu$ for each $\mu \geq 1$. Moreover, $f \in \mathcal{U}_\mu \cap \mathcal{H}^1$. Hence by (4.2) we obtain the estimate $u_\mu(x) \leq f(x) = 1 - \varepsilon + \varepsilon x$ for all $x \in [1, 2)$. Letting $\varepsilon \to 0$ gives $u_\mu(x) \leq 1$ for each $x \in [1, 2)$. Since $u_\mu(x) \geq 1$ for all $x \geq 1$, $u_\mu(x) = 1$ for all $x \in [1, 2)$.

Since $(\mu/2)x^{1-\log_2 \mu} \leq 1$ while $u_\mu(x) = 1$ for $1 \leq x < 2$, (4) holds for all $x \in [1, 2)$. In the case where $x \in [2^n, 2^{n+1})$; $\quad n = 1, 2, ...$ the monotoneity of u_μ implies $u_\mu(x) \geq u_\mu(2^n) = (2/\mu)^n$. Since $(\mu/2)x^{1-\log_2 \mu} \leq (\mu/2)(2^{n+1})^{1-\log_2 \mu} = (2/\mu)^n$, the desired estimate (4) follows.

□

4.8. Theorem(Hölder estimate for SQAD). *For* $\mu \in [1, 2)$, *let* $f \in \mathcal{U}_\mu$. *Then*

$$f(x) \geq f(1) \cdot \max\{1, (\mu/2)x^{1-\log_2 \mu}\}, \qquad for \quad x \geq 1, \tag{4.9}$$

and

$$f(x) \leq f(1) \cdot \min\{1, (2/\mu)x^{1-\log_2 \mu}\}, \qquad for \quad x \in (0, 1). \tag{4.10}$$

Proof. The estimate (4.9) follows immediately from Definition 4.5 and Lemma 4.7 (4). Since Lemma 4.7(4) also gives the estimate

$$1/u_\mu(1/x) \leq \min\{1, (2/\mu)x^{1-\log_2 \mu}\}, \quad \text{for } x \leq 1,$$

the inequality $f(1) \geq f(x) \cdot u_\mu(1/x)$ yields (4.10). □

4.11. Theorem. *Let $f \in \mathcal{H}$ and $h \in (1,2]$. If*

$$\inf_{x>0} f(hx)/f(x) = A > 1, \tag{4.12}$$

then f has an upper QAD bound $\mu \le 1 + A^{-n} < 2$, where n is the integer such that $n \le \log_h 2 < n+1$. For $h = 2$, this estimate for μ cannot be improved.

Proof. First, let $h = 2$, so that $n = 1$. If $0 < x \le y$, then $S_f(x,y) \le 1 + f(x)/f(2x) \le 1 + 1/A$, so that $1 + A^{-1}$ is an upper QAD bound for f. Next, let $n \le \log_h 2 < n+1$, $n = 2, 3, \ldots$. Then, by induction,

$$\frac{f(2x)}{f(x)} \ge \frac{f(h^n x)}{f(x)} \ge A^n.$$

Hence, $1 + A^{-n}$ is an upper QAD bound for f. The function

$$f(x) = A^k \quad \text{for} \quad x \in [2^k, 2^{k+1});\ k = 0, \pm 1, \ldots$$

satisfies (4.12) with $h = 2$. Since $f(2^{k+1} + 2^k) = A^{k+1}$,

$$S_f(2^k, 2^{k+1}) = (A^k + A^{k+1})/A^{k+1} = 1 + 1/A.$$

This relation shows that $1 + A^{-1}$ is the sharp upper QAD bound for f. Hence the estimate $\mu \le 1 + A^{-1}$ in the case $h = 2$ cannot be improved. $\square$

4.13. Remarks. (i) The condition (4.12) with the factor $h > 2$ gives no information about upper QAD bounds for f. Given $A > 1$, $h > 2$, the function $f(x) = A^n$ for $x \in [h^n, h^{n+1})$; $n = 0, \pm 1, \ldots$ satisfies (4.12) while $S_f(1,1) = 2$. Therefore 2 is the sharp upper QAD bound for f.

(ii) By Lemma 1.1 the function $f(x) = x^{1/K}, K \ge 1$, is SQAD on $(0, \infty)$ with $\mu_f = 2^{1-(1/K)}$. Then $1 - \log_2 \mu_f = 1/K$ and we see that the Hölder exponents in Theorem 4.8 are both sharp for this function f.

(iii) If $f \in \mathcal{H}, h \in [\frac{1}{2}, 1)$ and

$$\sup_{x>0} \frac{f(hx)}{f(x)} = \frac{1}{A} < 1,$$

then it is easy to see that the conclusion in Theorem 4.11 is also true.

4.14. Corollary. *Let $f \in \mathcal{H}$, $t \in (1, \sqrt{2}]$, and let*

$$f(t^{n+1})/f(t^n) \ge A > 1, \qquad \text{for all } n = 0, \pm 1, \ldots . \tag{4.15}$$

Then f has an upper QAD bound $1 + A^{-1} < 2$.

Proof. For $x > 0$, let n be the integer such that $t^n < x \le t^{n+1}$. Since $2x > t^{n+2}$, the inequalities $f(2x)/f(x) \ge f(t^{n+2})/f(t^{n+1}) \ge A$ hold for all $x > 0$. The desired result follows from Theorem 4.11. $\square$

4.16. Theorem. *For $1 \le \mu < 2$, $f \in \mathcal{U}_\mu$, and $p > 0$, the function $g(x) = [f(x)]^p$ has an upper QAD bound $1 + (\mu/2)^p < 2$.*

Proof. Since μ is an upper QAD bound for f, by Definition 4.5, $f(2x)/f(x) \geq 2/\mu > 1$ for all $x > 0$. It follows that $g(2x)/g(x) = [f(2x)/f(x)]^p \geq (2/\mu)^p > 1$ for all $x > 0$. Now the desired result follows from Theorem 4.11 (case $h = 2$). $\square$

4.17. Theorem. *For $f \in \mathcal{U}_\mu$, $1 \leq \mu < 2$, and for each $p \geq 1$, the function, $g(x) = f(x^p)$ has an upper QAD bound*

$$\tilde{\mu} \leq \min\{\mu, 1 + 1/u_\mu(2^p)\}. \tag{4.18}$$

Proof. Since $f \in \mathcal{U}_\mu$, with $x^p = y$, Lemma 4.7 gives the estimate

$$\frac{g(2x)}{g(x)} = \frac{f(2^p y)}{f(y)} \geq u_\mu(2^p) > 1 \qquad \text{for all } x > 0.$$

Then by Theorem 4.11 we obtain for g the second estimate of its upper QAD bound.

On the other hand, from (2.6) and the monotoneity of f,

$$S_g(x, y) \leq [f(x^p) + f(y^p)]/f(x^p + y^p) = S_f(x^p, y^p) \leq \mu$$

which implies $\tilde{\mu} \leq \mu$. $\square$

4.19. Remarks. (i) The estimate for upper QAD bounds of the function $(f(x))^p$ in Theorem 3.11, is essentially improved in Theorem 4.16. However, we do not know the sharp estimate for upper QAD bounds of all functions $[f(x)]^p$, $f \in \mathcal{U}_\mu$.

(ii) If $p < 1$ and $f \in \mathcal{U}_\mu$ one cannot expect the function $f(x^p)$ to have an upper QAD bound < 2. Indeed, let $\mu \in (1, 2)$ and $p \in (0, 1)$. Consider the function $f(x) = \gamma^n$ for $x \in [2^n, 2^{n+1})$; $n = 0, \pm 1, ...$, where $\gamma = (\mu - 1)^{-1}$. Then f has an upper QAD bound $1 + \gamma^{-1} = \mu < 2$, i.e. $f \in \mathcal{U}_\mu$. Since $g(x) = f(x^p) \equiv 1$ on $[1, 2^{1/p})$ and $2^{1/p} > 2$ then $S_g(1, 1) = 2$. Thus 2 is the sharp upper QAD bound for the function $f(x^p)$. Thus, under composition, $\mathcal{U}_\mu$ is not closed, where as, by Theorem 3.7, $\mathcal{H}_\lambda$ is closed, for $\lambda \in (0, 2)$, $\mu \in (1, 2)$.

4.20. Theorem. *If a function $f \in SQAD$, with $\lambda_f = \lambda$ and $\mu_f = \mu$, then*

$$\begin{cases} f(1)(\mu/2)x^b \leq f(x) \leq f(1)(2/\lambda)x^a & \text{for } x \geq 1; \\ f(1)(\lambda/2)x^a \leq f(x) \leq f(1)(2/\mu)x^b & \text{for } 0 < x \leq 1\ ; \end{cases} \tag{4.21}$$

where $a = 1 - \log_2 \lambda$, $b = 1 - \log_2 \mu$. These exponents are sharp.

Proof. The inequalities in (4.21) follow immediately from Theorems 2.15 and 4.8 . To show that the exponents a and b in (4.21) are sharp fix $K > 1$. With $\alpha = 1/K, \beta = K, C = 1$ the function f in Lemma 2.27 has $\lambda = 2^{1-K}, \mu = 2^{1-1/K}$ and thus the sharpness of the exponents a and b readily follows. $\square$

4.22. Example. The Hölder bounds similar to (4.21) are not sufficient for f to be quasiadditive. Consider the function

$$f(x) = \begin{cases} x, & \text{for } 0 < x < 2; \\ 2^{2^{n+1}}, & \text{for } x \in [2^{2^n}, 2^{2^{n+1}}); \quad n = 0, 1, ... \end{cases}$$

It is clear that $f \in \mathcal{H}$ and f has the Hölder bounds: $x \leq f(x) \leq x^2$, for $x \geq 1$; and $x^2 \leq f(x) \leq x$, for $x \in (0,1)$. Since $2^{n+1} - 1 > 2$ then

$$S_f(2^{2^{n+1}-1}, 2^{2^{n+1}-1}) = 2 \cdot 2^{2^{n+1}} / 2^{2^{n+2}} = 2 \cdot 2^{-2^{n+1}}$$

which tends to 0 as $n \to +\infty$. Therefore $\lambda_f = 0$, i.e. f is not QAD. Moreover, $\mu_f = 2$.

4.23. Theorem. *Suppose f is a nonnegative differentiable function on $[0,\infty)$ such that, $f(0) = 0$, and $0 < a \leq f'(x) \leq b < \infty$, for all x in $(0,\infty)$, then f is SQAD on $(0,\infty)$. In fact,*

$$a/b \leq S_f(x,y) \leq 2 - a/b,$$

for all $x, y \in (0,\infty)$.

Proof. By integrating, we get $ax \leq f(x) \leq bx$, for all $x \in (0,\infty)$. Replacing x by y and $x+y$, we get $ay \leq f(y) \leq by$, and $a(x+y) \leq f(x+y) \leq b(x+y)$. Adding the first two inequalities, we get $a(x+y) \leq f(x) + f(y) \leq b(x+y)$, so that $a/b \leq S_f(x,y) \leq b/a$. Next, integrating on $[x, x+y]$ and $[y, x+y]$, respectively, and adding, we get

$$(a/b)f(x+y) \leq a(x+y) \leq 2f(x+y) - f(x) - f(y),$$

so that $S_f(x,y) \leq 2 - (a/b)$. □

4.24. Theorem. *For $0 < a \leq b < \infty$, let $f : R \to R$ be differentiable such that $a \leq f'(x) \leq b$. Let g be defined on $(0,\infty)$ by $g(x) = \exp\,(f(\log x))$. Then g is SQAD on $(0,\infty)$, with*

$$\min\{1, 2^{1-b}\} \leq S_g(x,y) \leq \max\{1, 2^{1-a}\},$$

for all $x, y \in (0,\infty)$.

Proof. We have $\log g(t) = f(\log t)$. Hence $(d/dt)(\log g(t)) = [f'(\log t)]/t$, for all $t \in (0,\infty)$. Hence

$$a/t \leq (d/dt)(\log g(t)) \leq b/t.$$

Integrating over the interval $[x, x+y]$, we get

$$a \log[(x+y)/x] \leq \log[g(x+y)/g(x)] \leq b \log[(x+y)/x].$$

Interchanging x and y, we get

$$a \log[(x+y)/y] \leq \log[g(x+y)/g(y)] \leq b \log[(x+y)/y].$$

Hence

$$[x/(x+y)]^b \leq g(x)/[g(x+y)] \leq [x/(x+y)]^a,$$

and

$$[y/(x+y)]^b \leq g(y)/[g(x+y)] \leq [y/(x+y)]^a.$$

Adding these, we get

$$[x^b + y^b]/(x+y)^b \leq S_g(x,y) \leq [x^a + y^a]/(x+y)^a.$$

Hence the result follows from (2.6). □

5 Quasiadditive homeomorphisms on R

Throughout this section, let $\mathcal{F}$ be the class of all increasing homeomorphisms $f : R \to R$, $R = (-\infty, +\infty)$. For $f \in \mathcal{F}$ the function $S_f(x,y) = (f(x)+f(y))/f(x+y)$ is defined for all x, y such that $f(x+y) \neq 0$. Though one cannot expect S_f to be bounded for arbitrary $f \in \mathcal{F}$, the following result gives a simple necessary condition for the boundedness of S_f.

5.1. Proposition. *Let $f \in \mathcal{F}$. If $\sup |S_f(x,y)| < +\infty$, where the supremum is taken over all $x, y \in R$ such that $f(x+y) \neq 0$, then $f(0) = 0$ and $f(-x) \equiv -f(x)$, (that is, f is an odd function.)*

Proof. Let $a = f^{-1}(0)$ and $|S_f(x,y)| \leq M < +\infty$, for all x, y such that $x + y \neq a$. If $x \neq a$, then $2|f(x/2)| \leq M|f(x)|$, so that, letting $x \to a$, we get $f(a/2) = 0$. Hence $a = 0 = f(0)$. Next, if $y \neq -x$, then $|f(x)+f(y)| \leq M|f(x+y)|$, so that, letting $y \to -x$, we get $f(-x) = -f(x)$. □

The example $f(x) = x^3$, shows that the converse of Proposition 5.1 is false.

5.2. Definition. A mapping $f \in \mathcal{F}$ is said to be *quasiadditive* (QAD) if there exist $\lambda, \mu > 0$ such that

$$\lambda|f(x+y)| \leq |f(x)+f(y)| \leq \mu|f(x+y)|, \text{ for all } x, y \in R. \tag{5.3}$$

In this section we show that the class of all quasiadditive homeomorphisms of R coincides with the class of all increasing bilipschitz mappings f such that $f(-x) \equiv -f(x)$. We use the concept of a free quasisymmetric mapping which was introduced by D. Ivaşcu [Iv] in 1978 and developed in [Sh], and [ASh].

5.4. Definition. Let $A \subset R$. A homeomorphic embedding $f : A \to R$ is said to be *free $k-$quasisymmetric* ($k \geq 1$) iff

$$\frac{1}{k} \leq \frac{|f(x)-f(y)|}{|f(u)-f(v)|} \leq k, \tag{5.5}$$

for all $x, y, u, v \in A$ such that $|x-y| = |u-v| \neq 0$.

Let $A \subset R$. Recall that a homeomorphic embedding $f : A \to R$ is L-bilipschitz ($L \geq 1$) iff $L^{-1}|x-y| \leq |f(x)-f(y)| \leq L|x-y|$ holds for all $x, y \in A$.

5.6. Lemma. (see [Sh, Theorem 2.1, p.18]). *For a, b, c, d in $\mathbb{R}$, with $a < b,\ c < d$, let $f : [a,b] \to [c,d]$, be an increasing homeomorphism.*
(i) *If f is L-bilipschitz then f is free L^2-quasisymmetric mapping.*
(ii) *If f is free k-quasisymmetric then f is $\tilde{L}$-bilipschitz with*

$$\tilde{L} = k \cdot \max\{|d-c|/|b-a|, |b-a|/|d-c|\}.$$

Proof. (i) This statement is clear.
(ii) Let $p = b - a$, $q = d - c$, $\delta = \max\{p/q, q/p\}$. For $N = 2^n$ where $n = 1, 2, \ldots$, denote $A_n = \{x_i = a + ip/N : \quad i = 0, 1, \ldots, N\}$, $y_i = f(x_i)$, $m = \min_s(y_s - y_{s-1})$, $M = \max_s(y_s - y_{s-1})$ where $s = 1, \ldots, N$. Since f is free k-quasisymmetric we have $M \leq km$ and

$mN \le q \le MN$. If $0 \le i < j \le N$ then $x_j - x_i = (j-i)p/N$ and the following estimates are valid:

$$y_j - y_i = \sum_s (y_s - y_{s-1}) \le M(j-i) = MN(x_j - x_i)/p \le kq(x_j - x_i)/p \le k\delta(x_j - x_i);$$

$$y_j - y_i \ge m(j-i) = mN(x_j - x_i)/p \ge k^{-1}q(x_j - x_i)/p \ge (k\delta)^{-1}(x_j - x_i).$$

Thus $f|A_n$ is $k\delta$-bilipschitz for each $n = 1, 2, \ldots$. Since $A_1 \subset A_2 \subset \ldots$ then f is also $k\delta$-bilipschitz on the set $\cup_n A_n$ which is dense in $[a,b]$. Hence f is $k\delta$-bilipschitz on $[a,b]$. $\square$

5.7. Theorem. (see [Sh], [ASh]). *If $f \in \mathcal{F}$ is free k-quasisymmetric then f is L-bilipschitz, where*

$$L = k^2 \max\{f(1) - f(0), \quad (f(1) - f(0))^{-1}\}.$$

Proof. Denote $d = f(1) - f(0)$, $\delta = \max\{d, 1/d\}$. Since f is free k-quasisymmetric, $d/k \le f(p+1) - f(p) \le dk$, for all $p \in R$, so that

$$\max\{f(p+1) - f(p), (f(p+1) - f(p))^{-1}\} \le k\delta.$$

By Lemma 5.6, f is $k^2\delta$-bilipschitz on each segment $[p, p+1]$. It follows, in particular, that f is absolutely continuous, and hence differentiable a.e. on R and $(k^2\delta)^{-1} \le f'(x) \le k^2\delta$ at every point of differentiability. Hence f is $k^2\delta$- bilipschitz on R. $\square$

5.8. Remark. The situation will be even more interesting if we consider free quasisymmetric embeddings of a line segment into R^n. In [Sh] A. Shalaginov has proved that the canonical parametrization of Koch's curve is free quasisymmetric. We refer to [ASh] for further information on free quasisymmetric embeddings of metric spaces and their connections with self-similarity.

The next result shows the equivalence of symmetric Bilipschitz and QAD classes.

5.9. Theorem. *Let $f \in \mathcal{F}$.*

(i) *If $f \in$QAD with a lower QAD bound $\lambda > 0$ and an upper QAD bound μ then $f(-x) \equiv -f(x)$ and f is an L-bilipschitz mapping, where*

$$L = (\mu/\lambda)^2 \max\{f(1), f(1)^{-1}\}.$$

(ii) *If $f(-x) \equiv -f(x)$ and f is $\tilde{L}$- bilipschitz, then $f \in$QAD with a lower QAD bound $\tilde{L}^{-2}$ and an upper QAD bound $\tilde{L}^2$.*

Proof. (i) Proposition 5.1 gives the equality $f(-x) \equiv -f(x)$. For each quadruple $x, y, u, v \in R$ such that $|x - y| = |u - v| \ne 0$ we have the estimates

$$\lambda\mu^{-1} \le \frac{|S_f(x, -y)|}{|S_f(u, -v)|} = \frac{|f(x) - f(y)|}{|f(u) - f(v)|} \le \mu\lambda^{-1}$$

which means that f is free $\mu\lambda^{-1}$-quasisymmetric. By Theorem 5.7, the assertion (i) follows.

(ii) This statement is clear. $\square$

5.10. Example. The homeomorphism $f \in \mathcal{F}$ defined by

$$f(x) = \begin{cases} \sqrt{|x|}, & \text{for } x \geq 0; \\ -\sqrt{|x|}, & \text{for } x < 0 \end{cases}$$

is not bilipschitz. Hence, by Theorem 5.9 f is not quasiadditive on R. However, $|S_f(x,y)| \leq \sqrt{2}$. Moreover, $f \in$ SQAD, on $(0,\infty)$, with $\lambda_f = 1$ and $\mu_f = \sqrt{2}$.

5.11. Theorem. *For each homeomorphism $f : [0,+\infty) \to [0,+\infty)$ the following two conditions are equivalent:*

(i) *there exists a constant $K \geq 1$ such that*

$$\min\{(f(xy))^K, (f(xy))^{1/K}\} \leq f(x)f(y) \leq \max\{(f(xy))^K, (f(xy))^{1/K}\}, \tag{5.12}$$

for all $x, y > 0$.

(ii) *$f(1/x) \equiv 1/f(x)$ and there exists a constant $L \geq 1$ such that*

$$\min\{(x/y)^L, (x/y)^{1/L}\} \leq \frac{f(x)}{f(y)} \leq \max\{(x/y)^L, (x/y)^{1/L}\} \tag{5.13}$$

for all $x, y > 0$.

The constant in (ii) *may be taken to be*

$$L = K^4 \max\{\log f(e), 1/\log f(e)\},$$

where K is the constant in (i). *In particular,* (ii) *implies*

$$\min\{x^L, x^{1/L}\} \leq f(x) \leq \max\{x^L, x^{1/L}\}, \quad \textit{for all } x > 0. \tag{5.14}$$

Moreover, each of the conditions (i), (ii) *implies that f is absolutely continuous in* $(0,\infty)$ *and*

$$\frac{1}{L} \leq \frac{xf'(x)}{f(x)} \leq L, \quad \textit{for almost all } x > 0. \tag{5.15}$$

Proof. We associate with f the homeomorphism $g \in \mathcal{F}$ defined by $g(t) = \log f(e^t)$. Then it is easy to show that (i) is equivalent to the inequalities

$$|g(u+v)|/K \leq |g(u) + g(v)| \leq K|g(u+v)|,$$

so that $g \in$ QAD on R. By Theorem 5.9, this is equivalent to the condition that g is odd and $L-$ bilipschitz, with $L = K^4 \max\{\log f(e), 1/(\log f(e))\}$. This in turn, is equivalent to $f(1/x) = 1/f(x)$, and (5.13). The particular case (5.14) follows from (5.13) with $y = 1$.

The differential inequalities (5.15) follow from the absolute continuity on R of the L-bilipschitz function g. □

5.16. Proposition. *If a nondecreasing function $f : (0,+\infty) \to (0,+\infty)$ satisfies the condition (ii) in Theorem 5.11, then f is SQAD, and*

$$2^{1-L} \leq S_f \leq 2^{1-1/L}.$$

Proof. Since $L \geq 1$ the condition (ii) implies

$$[x/(x+y)]^L \leq f(x)/f(x+y) \leq [x/(x+y)]^{1/L},$$

$$[y/(x+y)]^L \leq f(y)/f(x+y) \leq [y/(x+y)]^{1/L}.$$

Hence, the result follows from (2.6). □

Example 4.22 shows that (5.14) is not sufficient for f to be in QAD on $(0, \infty)$.

6 QAD and quasisymmetric properties

Quasisymmetric embeddings (QS) of metric spaces were introduced by P.Tukia and J.Väisälä and these maps have found many applications [TV]. For a survey of this topic, see [V2]. We give a slight modification of the definition in [TV, 2.1,p.98].

For simplicity we denote the metric $d(x,y)$ by $|x-y|$, though no linear structure is assumed.

6.1. Definition. Let η be a self–homeomorphism of $(0,\infty)$ such that $\eta(t) \to 0$ as $t \to 0$. An injective mapping $f : \mathcal{X} \to \mathcal{Y}$ of metric spaces is called *η-quasisymmetric at a point $a \in \mathcal{X}$* ($f \in \eta - QS(a)$) if

(6.2) $$\frac{|f(x)-f(a)|}{|f(y)-f(a)|} \leq \eta(\frac{|x-a|}{|y-a|}),$$

for all $x, y \in \mathcal{X}\backslash\{a\}$. A mapping $f : \mathcal{X} \to \mathcal{Y}$ is *η-quasisymmetric* ($f \in \eta - QS$) iff it is η-quasisymmetric at each point $a \in \mathcal{X}$.

6.3. Remarks. If a is a limit point in $\mathcal{X}$ then letting $x \to a$, keeping y fixed in $\mathcal{X}\backslash\{a\}$, we see that f is continuous at a. Thus if $\mathcal{X}$ has no isolated points, η-quasisymmetric mappings are always continuous.

Letting $x = y$ in (6.2), we see that $\eta(1) \geq 1$. For a limit point $a \in \mathcal{X}$ letting $y \to a$, keeping x fixed in $\mathcal{X}\backslash\{a\}$, we see that $\eta(t) \to \infty$ as $t \to +\infty$.

Interchanging x and y in (6.2), we also get the reverse inequality

(6.4) $$\frac{|f(x)-f(a)|}{|f(y)-f(a)|} \geq 1/[\eta(\frac{|y-a|}{|x-a|})],$$

so that

(6.5) $$\eta(t) \cdot \eta(1/t) \geq 1,$$

for all $t > 0$.

6.6. Lemma. *If $f \in \mathcal{H}$, $f(0) = 0$, is η-quasisymmetric at 0 then $f \in \mathcal{H}_\lambda$ with a lower QAD bound $\lambda \geq 1/\eta(2)$.*

Proof. Let $z = \max\{x, y\}$. Then (6.2) with $a = 0$, implies the estimate:

$$S_f(x,y) = (f(x)+f(y))/f(x+y) \geq f(z)/f(2z) \geq 1/\eta(2). \qquad \square$$

6.7. Remark. The function f from Example 3.5 is QAD, but not quasisymmetric at 0. Indeed, suppose that $f \in \eta - QS(0)$ for some distortion function η. Then for every fixed $t \in (0,1)$ we obtain the estimate

$$\eta(t) \geq \sup_{x>0} \frac{f(tx)}{f(x)} \geq \limsup_{x\to+\infty} \frac{\log_2(2tx)}{\log_2(2x)} = 1,$$

which contradicts the condition $\eta(t) \to 0$ as $t \to 0$ in Definition 6.1.

Thus the converse of Lemma 6.6 is false.

The next result gives a criterion for f to be quasisymmetric at 0, similar to that in [T1, Proposition 4,p.135], (see also [T2, C,p.153]).

6.8. Lemma. *Let $f \in \mathcal{H}$, $f(0) = 0$.*

(i) If $f \in \eta - QS(0)$, then there exists $h > 1$ such that

$$1 < A \leq f(hx)/f(x) \leq B, \tag{6.9}$$

for all $x > 0$, where $A = 1/\eta(1/h)$ and $B = \eta(h)$.

(ii) If (6.9) holds, then $f \in \eta - QS(0)$ with a distortion function

$$\eta(t) \leq B \cdot t^{(\log C)/(\log h)} \tag{6.10}$$

where $C = A$ for $t \in (0,1)$ and $C = B$ for $t \geq 1$.

Proof. (i) Since $\eta(t) \to 0$ as $t \to 0$, there exists $h > 1$ such that $\eta(1/h) < 1$. The condition $f \in \eta - QS(0)$ implies $f(hx)/f(x) \leq \eta(h)$, so that

$$f(hx)/f(x) = [f(x)/f(hx)]^{-1} \geq [\eta(1/h)]^{-1} > 1.$$

(ii) Let $\eta_0(t) = \sup_{x>0} f(tx)/f(x)$. Fix $t \geq 1$. Then there is an integer $n \geq 0$ such that $h^n \leq t < h^{n+1}$. By (6.9), the inequality

$$f(tx)/f(x) < f(h^{n+1}x)/f(x) \leq B^{n+1} \leq B \cdot B^{(\log t)/(\log h)} = B \cdot t^{(\log B)/(\log h)},$$

holds for all $x > 0$. Hence

$$\eta_0(t) \leq B \cdot t^{(\log B)/(\log h)}, \qquad \text{for all} \quad t \geq 1.$$

If $t < 1$, then there exists an integer $n \geq 0$ such that $h^n < 1/t \leq h^{n+1}$. Since $A > 1$, (6.9) gives for all $x > 0$ the inequality

$$f(tx)/f(x) \leq f(x/h^n)/f(x) \leq A^{-n} \leq A(1/t)^{(\log(1/A))/(\log h)} = A \cdot t^{(\log A)/(\log h)}.$$

Since $A \leq B$ it follows that

$$\eta_0(t) \leq B \cdot t^{(\log A)/(\log h)}, \qquad \text{for all} \quad t \in (0,1).$$

Thus $\eta_0(t)$ satisfies (6.10) and $f \in \eta_0 - QS(0)$ because $f(y)/f(x) \leq \eta_0(y/x)$ for all $x, y > 0$.
$\square$

6.11. Corollary. *If $f \in \mathcal{H}$, $f(0) = 0$, is η-quasisymmetric at 0 then $f \in \omega - QS(0)$ with a distortion function ω of the form $\omega(t) = C \cdot t^\gamma$ where $\gamma = \alpha \in (0, 1]$ for $t \in [0, 1]$ and $\gamma = \beta > 1$ for $t > 1$, and the positive constants C, α, β depend only on η.*

This statement follows immediately from Lemma 6.8. The general situation is described in [TV, Theorem 3.10, p.107]. Some estimates for the growth of QS maps of the real axis are given in [HH].

6.12. Theorem. *Let $f \in \mathcal{H}$, $f(0) = 0$. If $f \in SQAD$ with lower and upper QAD bounds $\lambda > 0$ and $\mu < 2$, respectively, then $f \in \eta - QS(0)$ with a distortion function η depending only on λ, μ and such that $\eta(1/2) \le \mu/2 < 1$.*

Proof. Lemma2.9 and (4.7) give, for all $x > 0$, the inequality

$$1 < 2/\mu = u_\mu(2) \le f(2x)/f(x) \le k_\lambda(2) = 2/\lambda.$$

Hence by Lemma 6.8(ii), $f \in \eta - QS(0)$ with a distortion function

$$\eta(t) = \sup_{x>0} f(tx)/f(x) \le (2/\lambda) \cdot t^{1-\log_2 \gamma}$$

where $\gamma = \mu$ for $t \le 1$ and $\gamma = \lambda$ for $t > 1$. The estimate

$$f(x/2)/f(x) = S_f(x/2, x/2)/2 \le \mu/2 < 1$$

holds for all $x > 0$, hence $\eta(1/2) \le \mu/2 < 1$. □

6.13. Theorem. *Let $f \in \mathcal{H}$, $f(0) = 0$. If $f \in \eta - QS(0)$ and $\eta(1/2) < 1$, then $f \in SQAD$ with a lower QAD bound $1/\eta(2)$ and an upper QAD bound $1 + \eta(1/2) < 2$.*

Proof. Since $1/[\eta(1/2)] \le f(2x)/f(x) \le \eta(2)$, by Theorem 4.11, f has an upper QAD bound $1 + \eta(1/2) < 2$. Theorem 3.2 gives the lower QAD bound $1/\eta(2)$ for f. □

6.14. Example. We shall show that $f \in \eta - QS(0)$ does not imply $f \in$SQAD. Therefore the condition $\eta(1/2) < 1$ in Theorem 6.13 is essential.

Let $p_k = 2^{2k}$ and $q_k = 2p_k$ for $k = 0, 1, \ldots$. The homeomorphism f of $[0, +\infty)$ onto itself is defined by the following conditions: $f(x) = x$ for $0 \le x \le 1$; $f(x) = (x + kp_k)/(k + 1)$ for $x \in [p_k, q_k], k = 0, 1, \ldots$ and $f(x) = [(3k + 2)x - 4kp_k]/2(k + 1)$ for $x \in [q_k, p_{k+1}], k = 0, 1, \ldots$. It is easy to see that $x/2 \le f(x) \le x$ for all $x \ge 0$, and it follows that $f \in \eta - QS(0)$ with $\eta(t) = 2t$. Since $S_f(p_k, p_k) = 2f(p_k)/f(q_k) = 2(k+1)/(k+2)$, which tends to 2 as $k \to +\infty$, it follows that f is not SQAD. □

7 Distortion functions

We now give an application to special functions of quasiconformal analysis. We first consider quasiconformal maps in $\overline{R}^n$, and then the special case $n = 2$, where more information is available.

7.1. Application. Recall the definition of the function $\eta_{K,n}$ which plays an important role in theory of quasiconformal mappings. For $1 \le K < \infty$, let QC_K be the class of all K-

quasiconformal self-mappings of $\overline{R}^n = R^n \cup \{\infty\}$ with fixed points $0, e, \infty$, where $|e| = 1$, [V1]. Following [Vu, (1.5), p.618] for each $K \geq 1$, $t > 0$, we define

$$\eta_{K,n}(t) = \sup_{f \in QC_K} (\sup_{|x|=t} |f(x)|). \tag{7.2}$$

Then $QC_K \subset \eta_{K,n} - QS$. It is clear that $\eta_{K,n} \in \mathcal{H}$.

The next result shows that $\eta_{K,n} \in QAD$, for each $n \geq 2$, while Theorem 7.6 gives a stronger result for $n = 2$.

7.3. Theorem. *For every $K \geq 1, n \geq 2$, the function $\eta_{K,n}$ has the following submultiplicative property*

$$\eta_{K,n}(st) \leq \eta_{K,n}(s) \cdot \eta_{K,n}(t); \quad s, t > 0. \tag{7.4}$$

In particular, $\eta_{K,n}$ is quasiadditive with a lower QAD bound $1/\eta_{K,n}(2)$.

Proof. Let $f \in QC_K, |x| = st$ and $|y| = t$. Let g, h be Möbius transformations of $\overline{R}^n$, fixing $0, \infty$, such that $g(y) = e = h(f(y))$, with $|e| = 1$. Then

$$F \equiv h \circ f \circ g^{-1} \in QC_K.$$

If $g(x) = u$, then $|u| = |x|/|y| = s$, and

$$|f(x)|/|f(y)| = |g(u)| \leq \eta_{K,n}(|x|/|y|) = \eta_{K,n}(s).$$

Hence, $|f(x)| \leq \eta_{K,n}(s)|f(y)| \leq \eta_{K,n}(s).\eta_{K,n}(t)$. Taking supremum over all $|x| = st$ and $f \in QC_K$, (7.4) follows. Letting $s = 2$ in (7.4) gives the estimate $\eta_{K,n}(2t)/\eta_{K,n}(t) \leq \eta_{K,n}(2) < +\infty$, and by Theorem 3.2 the result follows. $\square$

7.5. Conjecture. For each natural number $n \geq 3$, and all real $K > 1$, the function $\eta_{K,n} \in$ SQAD, on $(0, \infty)$.

For $n = 2$, the answer is affirmative, by Theorem 7.6.

For the two-dimensional case we recall some notation from [AVV2]. In this case the function $\eta_{K,2} \equiv \eta_K$ has the following explicit representation in terms of elliptic integrals:

$$\eta_K(t) = \frac{u^2}{1 - u^2}, t > 0, \quad u = \varphi_K\left(\sqrt{\frac{t}{1+t}}\right),$$

$$\varphi_K(r) = \mu^{-1}(\frac{1}{K}\mu(r)), \quad \mu(r) = \frac{\pi}{2}\frac{\mathcal{K}'(r)}{\mathcal{K}(r)},$$

$$\mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}, \quad \mathcal{K}'(r) = \mathcal{K}(r'), \quad r' = \sqrt{1-r^2},$$

for $r \in [0, 1)$.

These functions have been studied extensively, for instance in [AVV2], [LV], [VV].

In [QVu, Theorem 3.49], it was shown that the function η_K satisfies the following quasiadditive property. The proof of Theorem 7.6 in [QVu] was based, in part, on an application of Lemma 1.1 (3).

7.6. Theorem. *The inequality*

$$2^{1-K} \le \frac{\eta_K(x)+\eta_K(y)}{\eta_K(x+y)} \le 2^{1-(1/K)}$$

holds for all $K \ge 1, x, y \in (0,\infty)$ *so that the function* $\eta_K \equiv \eta_{K,2} \in$ *SQAD, on* $(0,\infty)$.

As the above definitions indicate, the homeomorphisms $\eta_K : (0,\infty) \to (0,\infty)$ and $\varphi_K : (0,1) \to (0,1)$ are nonlinear. The derivative formulas

$$\frac{d\varphi_K(r)}{dr} = \frac{1}{K}\frac{ss'^2\,\mathcal{K}(s)^2}{rr'^2\,\mathcal{K}(r)^2} = K\frac{ss'^2\,\mathcal{K}'(s)^2}{rr'^2\,\mathcal{K}'(r)^2} = \frac{ss'^2\,\mathcal{K}(s)\,\mathcal{K}'(s)}{rr'^2\,\mathcal{K}(r)\,\mathcal{K}'(r)} \tag{7.7}$$

show that $\varphi_K'(r) \to \infty$, as $r \to 0$, and $\varphi_K'(r) \to 0$ as $r \to 1, K > 1$. In order to regularize the function φ_K we now consider a suitable auxiliary transformation.

7.8. Linearization of functions. We now apply a simple transformation which changes $\varphi_K(r)$ into a function whose derivative has range $(1/K, K)$. Due to this moderating effect on the functions, we call this transformation a *linearization*. The functions p, q in Theorem 5.9 were used in [SV] for similar purposes.

7.9. Theorem. *Let the functions* $q : (-\infty,\infty) \to (0,1)$ *and* $p \equiv q^{-1} : (0,1) \to (-\infty,\infty)$ *be defined by* $q(x) = e^x/(1+e^x)$ *and* $p(x) = \log(x/(1-x))$. *Then the composite function* $g(x) \equiv p(\varphi_K(q(x)))$ *is an increasing convex automorphism of* $(-\infty,\infty)$. *In fact,* g' *is strictly increasing from* $(-\infty,\infty)$ *onto* $(1/K, K)$.

Proof. Let $r = q(x)$ and $s = \varphi_K(r)$. Then $g(x) = p(s)$. Hence, by the Chain Rule and formula (7.7),

$$g'(x) = p'(s)\frac{ds}{dr}q'(x) = \frac{1}{s(1-s)}\frac{ss'^2\,\mathcal{K}(s)^2}{Krr'^2\,\mathcal{K}(r)^2}r(1-r) = \frac{1}{K}\frac{1+s}{1+r}\Big(\frac{\mathcal{K}(s)}{\mathcal{K}(r)}\Big)^2.$$

Next, by Landen's Transformation and the identity [AVV2, (3.8)]

$$\varphi_K(2\sqrt{r}/(1+r)) = 2\sqrt{\varphi_K(r)}/(1+\varphi_K(r)).$$

With $R = 2\sqrt{r}/(1+r)$ and $S = 2\sqrt{s}/(1+s) = \varphi_K(R)$, we obtain

$$g'(x) = \frac{1}{K}\frac{\mathcal{K}(S)\,\mathcal{K}(s)}{\mathcal{K}(R)\,\mathcal{K}(r)},$$

so that the result follows from [AVV2, Theorem 2.10]. □

The next two results show that slight modifications of the Grötzsch ring modulus $M_n(r)$ in n-space [AVV1, (1.9)], and that of the plane quasiconformal distortion function will yield functions having the QAD, but not SQAD property.

7.10. Theorem. *For* $n \ge 2$, *let* f *be the function defined on* $(0,\infty)$, *by* $f(x) = M_n(\exp(-x))$. *Then* $\lambda_f = 1$ *and* $\mu_f = 2$.

Proof. Since $M_n(r)/\log(1/r)$ is increasing from $(0,1)$ onto $(1,\infty)$, [AVV1, Lemma 2.6(2)], $f(x)/x$ is decreasing from $(0,\infty)$ onto $(1,\infty)$. Hence by Lemma 1.1 (2), $S_f(x,y) \geq 1$, so that $\lambda_f \geq 1$. Next,

$$\lim_{x\to\infty} S_f(x,x) = \lim_{x\to\infty} 2[f(x)/x][x]/[f(2x)/(2x)2x] = 1,$$

so that $\lambda_f = 1$.

On the other hand, $M_n(r)(\mathrm{arth}r)^{1/(n-1)}$ is increasing from $(0,1)$ onto $(0,C)$, where C is a positive constant, depending only on n. Hence,

$$\lim_{x\to 0} S_f(x,x) = \lim_{r\to 1} 2[(\mathrm{arth}r^2)/(\mathrm{arth}r)]^{1/(n-1)} = 2,$$

so that $\mu_f = 2$. $\square$

7.11. Theorem. *For $K \in [1,\infty)$, let f be defined on $(0,\infty)$ by $f(x) = \varphi_K(x/(x+1))$. Then $\lambda_f = 1$ and $\mu_f = 2$.*

Proof. For $x, y \in (0,\infty)$, by He's theorem [He], we have

$$f(x)/(x/(x+1))^{1/K} \geq f(x+y)/((x+y)/(x+y+1))^{1/K},$$

and

$$f(y)/(y/(y+1))^{1/K} \geq f(x+y)/((x+y)/(x+y+1))^{1/K}.$$

Hence, by (2.6),

$$S_f(x,y) \geq (x+y+1)^{1/K}[(x/(x+1))^{1/K}+(y/(y+1))^{1/K}]/(x+y)^{1/K} \geq [x^{1/K}+y^{1/K}]/(x+y)^{1/K} \geq 1,$$

so that $\lambda_f \geq 1$. Clearly, for fixed $x \in (0,\infty), S_f(x,y)$ tends to 1, as y tends to 0, so that $\lambda_f = 1$. On the other hand, clearly, $S_f(x,x)$ tends to 2 as x tends to ∞, so that $\mu_f = 2$. $\square$

Acknowledgement. The first author wishes to thank the Department of Mathematics, University of Helsinki, for assistance and hospitality during his visit in June 1995, where this work was initiated.

References

[A] J. Aczél: *On applications and theory of functional equations.* - Birkhäuser Verlag, Basel, 1969.

[AVV1] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen: *Special functions of quasiconformal theory.* - Expo. Math. 7 (1989), 97-136.

[AVV2] G.D. Anderson, M. K. Vamanamurthy, and M. Vuorinen: *Inequalities for plane quasiconformal mappings.* - In: The Mathematical Legacy of Wilhelm Magnus-Groups, Geometry and Special Functions, ed. by W. Abikoff, J.S. Berman and K. Kuiken, Contemporary Math., vol. 169, Amer. Math. Soc., Providence, RI, 1994, 1–27.

[ASh] V.V. Aseev and A.A. Shalaginov: *Mappings that boundedly distort distance ratios.* - Russian Acad. Sci. Dokl. Math., 49 (1994), No.2, 248-250.

[BB] E.F. Beckenbach and R. Bellman: *Inequalities.* - 2nd corr. printing, Ergebnisse der Math. Band 30, Springer-Verlag, 1965.

[HH] W. K. Hayman and A. Hinkkanen: *Distortion estimates for quasisymmetric functions.* - Ann. Univ. Mariae Curie Sklodowska Sec. A Math. 36/37 (1982/1983), 51-67 (1985).

[He] C.-Q. He: *Distortion estimates of quasiconformal mappings.* - Sci. Sinica Ser. A 27 (1984), 225-232.

[Iv] D. Ivaşcu: *Remarks on a class of quasisymmetric transformations.* - Rev. Roum. math. pure at appl., 23 (1978), No.5, 745-750.

[K1] M. Kuczma: *Functional equations in a single variable.* - PWN, Warszawa, 1968.

[K2] M. Kuczma: *On the Schröder equation.* - Rozprawy Mat., 34 (1963), 1-50.

[K3] M. Kuczma: *An introduction to the theory of functional equations and inequalities.* - Panstwowe Wydawnictwo Naukowe, Warzawa- Krakow, 1985.

[KCG] M. Kuczma, B. Choczewski and R. Ger: *Iterative Functional Equations.* - Encyclopedia of Mathematics and its Applications, Volume 32, Cambridge University Press, 1990.

[LV] O. Lehto and K.I. Virtanen: *Quasiconformal mappings in the plane, 2nd ed.* - Die Grundlehren der math. Wissenschaften, Band 126, Springer-Verlag, New York - Heidelberg - Berlin, 1973.

[QVu] S.-L. Qiu and M. Vuorinen: *Quasimultiplicative properties for the η-distortion function.* - Complex Variables Theory Appl. 30 (1996), 77-96.

[SV] K. Samuelsson and M. Vuorinen: *Computation of capacity in 3D by means of a posteriori estimates for adaptive FEM*, Preprint, Royal Institute of Technology, Stockholm, August 1995.

[Sh] A.A. Shalaginov: *On mappings of selfsimilar curves.* - Sibirsk. Mat. Zh., 34 (1993), No.6, 210-215.

[T1] P. Tukia: *The space of quasisymmetric mappings.* - Math. Scand. 40 (1977), 127-142.

[T2] P. Tukia: *Hausdorff dimension and quasisymmetric embeddings.* - Math. Scand. 65 (1989), 152-160.

[TV] P. Tukia and J.Väisälä: *Quasisymmetric embeddings of metric spaces.* - Ann. Acad. Sci. Fenn. Ser. A I Math. 5 (1980), 97-114.

[V1] J. Väisälä: *Lectures on n-Dimensional Quasiconformal Mappings.* - Lecture Notes in Math. Vol. 229, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[V2] J. Väisälä: *Domains and maps*, Lecture Notes in Math. 1508, Springer-Verlag, 1992, 119-131.

[VV] M. K. Vamanamurthy and M. Vuorinen: *Functional inequalities, Jacobi products, and quasiconformal maps.* - Illinois J. Math. 38 (1994), 394-419.

[Vu] M. Vuorinen: *Quadruples and spatial quasiconformal mappings.* - Math. Z. 205 (1990), 617-628.

V.V. Aseev
Institute of Mathematics
630090 Novosibirsk
Russia
FAX: 383-2-350652
email: ase@math.nsc.ru

M. K. Vamanamurthy
Mathematics Department
University of Auckland
P.B. 92019
Auckland
New Zealand
FAX: 64-9-3737457
email: m.vamanamurthy@auckland.ac.nz

M. Vuorinen
Mathematics Department
P.O. Box 4 (Hallitusk. 15)
FIN-00014 University of Helsinki
FINLAND
FAX: 358-9-19123213
email:vuorinen@csc.fi