ANGLE BETWEEN SUBSPACES OF ANALYTIC AND ANTIANALYTIC FUNCTIONS IN WEIGHTED L_2 SPACE ON A BOUNDARY OF A MULTIPLY CONNECTED DOMAIN

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1. Introduction

The main idea of the solution of the problem on the basis of exponentials in $L_2([0,a])$ (see [1]) was that property of the system of exponentials to form the Riesz basis in $L_2([0,a])$ is equivalent to invertibility of orthogonal projection from the linear hull of these exponentials in $L_2([0,\infty))$ onto $L_2([0,a])$.

In a little bit more general setting (see for example [2,3]) the corresponding results from the harmonic analysis in the unit disk are as follows:

Let B and θ be the inner functions in the unit disk D and $K_B = H^2_+ \ominus BH^2_+$ and $K_\theta = H^2_+ \ominus \theta H^2_+$ be the corresponding coinvariant subspaces. Then the orthogonal projection $P_{K_\theta} K_B$ from K_B onto K_θ is an isomorphism if and only if the ratio B/θ can be represented in a following form

$$\frac{B}{\theta} = \frac{\bar{h}}{h},$$

where $h \in H^2_+$ is the so-called Helson-Segö's outer function, that is for which the angle in $L_2(|h|^2 d\varphi)$ on the unit circle between $\bigvee_{n<0}\{z^n\}$ and $\bigvee_{n\geq 0}\{z^n\}$ is nonzero.

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The equivalent condition is that h satisfies $A_2(\mathbb{T})$ Muckenhoupt condition on the unit circle $\mathbb{T}$

$$\sup_{\lambda\in D}\int_0^{2\pi}\frac{1-|\lambda|^2}{|1-\bar\lambda e^{i\varphi}|^2}|h|^2d\varphi\cdot\int_0^{2\pi}\frac{1-|\lambda|^2}{|1-\bar\lambda e^{i\varphi}|^2}\frac{1}{|h|^2}d\varphi<\infty. \tag{$A_2(\mathbb{T})$}$$

If, in particular, B is a Blaschke product with simple zeroes, the above condition on h together with the fact that zeroes of B satisfy Carleson condition gives the answer to the question when the projections on K_θ of reproducing kernels in the zeroes of B form the Riesz basis in K_θ.

It should be mentioned that the question on the completeness in $L_2([0,2a])$ of the system of resonance states of acoustic equation $u_{tt}=u_{xx}-qu$ on $(0,\infty)$ with $q(x)=0$ for $x>a$ is reduced to the following Sturm-Liouville problem: $-u_{xx}+qu=k^2u,\ u|_0=0,\ u'-iku|_{2a}=0$, with $q(x)=0$, for $x>a$, and can be studied using the same ideas by estimating the angle between two invariant subspaces of the shift operator.

It is the common place now (see [4, 5, 6]that the spectral analysis of periodic operators is based on harmonic analysis on the Riemann surface and the corresponding problem on the completeness of acoustic resonance states for periodic potential leads to the problems of harmonic analysis on the Riemann surface, which are similar to mentioned above problems in the unit disk.

In the present paper we are dealing with the similar problems in the Hardy spaces on finitely ($g+1$) connected domain Ω. The simplest problem similar to those in the unit disk is the following:

When do the projections of the normalized reproducing kernels in $H^2_+(\Omega)$ at the points $z_j\in\Omega,\ j=1,2\dots,$ form the Riesz basis in the linear hull of reproducing kernels corresponding to another sequence of points $t_1,t_2,\dots\in\Omega$? The first and essential difference of the case under consideration from that one in the unit disk is that as opposed to the case of the unit disk, where $L_2(\mathbb{T})=H^2_+\oplus H^2_-$ in our case we have the whole g-dimensional torus of similar decompositions: $L_2(\partial\Omega)=H^2_{+,\kappa}\oplus H^2_{-,\kappa}\oplus\mathfrak{M}_\kappa$, where $H^2_{+,\kappa},H^2_{-,\kappa}$ are so-called character-automorphic Hardy spaces on Ω, corresponding to character $\kappa\in\mathbb{R}^g/\mathbb{Z}^g$ and $\mathfrak{M}_\kappa$ is g-dimensional κ-automorphic defect space. All this spaces for different characters are of the equal importance and for each certain character the above problem on the projection from one system of reproducing kernels onto another can be posed.

It was natural to suspect that as above in this case the certain results on the angles between subspaces in weighted L_2 space on $\Gamma=\partial\Omega$, would play an essential role.

The first step towards the solution of this problem was done by the author in [7], where all the positive weights w on Γ, for which the angle in $L_2(\Gamma,w)$ between $L_2(\Gamma,w)\cap\{H^2_{-,\kappa}+\mathfrak{M}_\kappa\}$ and $L_2(\Gamma,w)\cap H^2_{+,\kappa}$ was nonzero, were described. These weights should satisfy the Muckenhoupt condition, which in this case has the same form as the classical one and the corresponding angles are nonzero for all characters simultaneously.

But it occurs that using this description one can get only sufficient but not necessary answer on the question about the invertibility of the corresponding projection, namely the problem can be reduced (this reduction and the solution of the above posed problem even in a more general setting is a matter of forthcoming paper) to the problem of description (in terms of necessary and sufficient conditions) of those weights w on Γ for which the angle in $L_2(\Gamma, w)$ between $L_2(\Gamma, w) \cap H^2_{-,\kappa}$ and $L_2(\Gamma, w) \cap H^2_{+,\kappa}$ is nonzero. The present paper contains the last mentioned description and the answer has a form similar to Muckenhoupt condition, but essentially depends on a character. Namely it may happen that for one and the same weight w on Γ for one character the corresponding angle is zero and for another character it is nonzero. The solution is essentially based on the results of papers [7,8].

2. Preliminaries

In what follows we consider the planar domain as a half of its Schottky double - the compact Riemann surface, and essentially use the function theory on it. Though the description of the basic objects and constructions on the doubles is well known (see for example [9,10]) and is already present in the previous papers [7, 8, 11] by the author, it seems to be important to repeat this stuff here once more. We start this section with the brief discussion of Schottky doubles, meromorphic differentials, divisors, Riemann-Roch theorem, Abel-Jacobi's thansform, Jacobian variety and multiplicative meromorphic functions. Then we make mention of the definition and basic properties of character-automorphic Hardy spaces and finish the section with recalling the notion of an angle between subspaces in a Hilbert space.

2.1. We consider our $g+1$ connected domain $\Omega = \Omega_+$ as a bordered Riemann surface and denote by $\hat{\Omega}$ its Schottky double - the compact Riemann surface of genus g, which is obtained topologically from two copies Ω_+ and Ω_- of the domain Ω by the identification of the points of $\partial\Omega_\pm \stackrel{def}{=} \Gamma$ (gluing a second copy Ω_- of Ω to Ω_+ along $\partial\Omega_+$) and the complex structure is obtained by "reflecting" the complex structure from $\Omega_+ \cup \partial\Omega_+$ to Ω_-. The involution $J : \hat{\Omega} \longrightarrow \hat{\Omega}$ which fixes the points of Γ and interchanges the same points on Ω_+ and Ω_- is antiholomorphic.

Fix a point $P_0 \in \Gamma_0$ and let $\gamma_1, \dots, \gamma_g$ be oriented crosscuts from P_0 to the boundary components $\Gamma_1, \dots, \Gamma_g$ respectively, which except of their end points lie in Ω_+ and intersect each other only in the point P_0. We define the homology basis on the compact Riemann surface $\hat{\Omega}$ of genus g in the following way: the b-cycles are the boundary components, $b_j = \Gamma_j,\ j = 1, \dots, g$ and the a-cycles are constructed from the crosscuts γ_j and their antiholomorphic reflections $J(\gamma_j)$ on Ω_-, $a_j = \gamma_j - J(\gamma_j),\ j = 1, \dots, g$.

We consider the g-dimensional vector space $\mathfrak{N}$ of holomorphic differentials on $\hat{\Omega}$ and the normalized basis $\omega_1, \dots, \omega_g$ in this space which is dual to the homology basis

$$\int_{a_j} \omega_k = \delta_{j,k}$$

where $\delta_{j,k}$ is the Kronecker symbol. The differentials ω_j are purely imaginary on Γ and are antisymmetric with respect to the antiholomorphic involution J, $J^*\omega \stackrel{def}{=} \overline{\omega_j(J(z))} = -\omega_j(z)$. Consequently the matrix of b-periods $\tau = \left\{ \int_{b_j} \omega_k \right\}_{j,k=1}^{g}$ has the form $\tau = iP$ where P is a real positive definite matrix.

The whole $g \times 2g$ Riemann period matrix of the holomorphic differentials $\omega_1, \ldots, \omega_g$ on $\hat{\Omega}$ can be represented in the following way

$$\left[\int_{a_1} \vec{\omega}, \ldots, \int_{a_g} \vec{\omega} : \int_{b_1} \vec{\omega}, \ldots, \int_{b_g} \vec{\omega}\right] = [I_g : \tau]$$

where $\vec{\omega} = (\omega_1, \ldots, \omega_g)^t$ and I^g is the $g \times g$ identity matrix.

Let us denote by $d\eta_{P,Q}$, $P, Q \in \hat{\Omega}$, $P \neq Q$, the normalized differential of the third kind with only simple poles in the points P, Q with the residues $\frac{1}{2\pi i}$ and $-\frac{1}{2\pi i}$ respectively and $\int_{a_j} d\eta_{P,Q} = 0$, $j = 1, \ldots, g$. The bilinear relations between the differentials of the first and the third kind give us the following expressions for the b-periods of $d\eta_{P,Q}$

$$\int_{b_j} d\eta_{P,Q} = \int_Q^P \omega_j, \; j = 1, \ldots, g \tag{1}$$

where the path of integration does not intersect the a and b-cycles. Note that for the point $P \in \Omega_+$ the restriction of the differential $d\eta_{P,J(P)}$ on Γ determines exactly the harmonic measure $d\eta_P$. The function $m_j = 2\Re \int\limits_{P_0}^{z} \omega_j, z \in \Omega_+, \; j = 1, \ldots, g$ is a harmonic function in Ω_+, $m_j|_{\Gamma_j} = 1$, $m_j|_{\Gamma \setminus \Gamma_j} = 0$, which due to (1) is equal to the harmonic measure of Γ_j with respect to the point z

$$\int_{\Gamma_j} d\eta_{z,J(z)} = 2\Re \int_{P_0}^{z} \omega_j.$$

There is a close relation between the harmonic measure of Ω_+ with respect to the point P and the Green function $G(z, P)$ of the domain Ω_+ with singularity at the point P,

$$2\pi \Im \int\limits_{P_0}^{z} d\eta_{P,J(P)} = G(z, P) \quad \text{or} \quad 2\pi d\eta_{P,J(P)} = 2i \frac{\partial G(z, P)}{\partial z} dz.$$

2.2. The divisor on $\hat{\Omega}$ is the finite formal sum

$$\mathfrak{a} = \sum_{j=1}^{k} n_j P_j, \; n_j \in \mathbb{Z}, \; P_j \in \hat{\Omega}.$$

The divisor $\mathfrak{a}$ is called positive ($\mathfrak{a} \geq 0$) if $n_j \geq 0$ for all j.The order of the divisor is defined as

$$ord(\mathfrak{a}) = \sum_{j=1}^{k} n_j$$

The divisor of a meromorphic on $\hat{\Omega}$ function f is called the principal divisor and is denoted by (f). Obviously $ord(f) = 0$. Let us denote by $L(\hat{\Omega})$ the field of meromorphic functions on $\hat{\Omega}$ and by $L(\mathfrak{a})$ the vector space $\left\{ f \in L(\hat{\Omega}) : (f) \geq \mathfrak{a} \right\}, r(\mathfrak{a}) \stackrel{def}{=} \dim L(\mathfrak{a})$.

The divisors $\mathfrak{a}, \mathfrak{b}$ are equivalent ($\mathfrak{a} \sim \mathfrak{b}$) if their difference is the principal divisor.

The divisor of a meromorphic differential ω on $\hat{\Omega}$ is also denoted by (ω), the corresponding vector space of meromorphic differentials on $\hat{\Omega}$ is denoted by $\mathfrak{N}(\hat{\Omega})$ and $\mathfrak{N}(\mathfrak{a}) \stackrel{def}{=} \left\{\omega \in \mathfrak{N}(\hat{\Omega}) : (\omega) \geq \mathfrak{a}\right\}, i(\mathfrak{a}) \stackrel{def}{=} \dim \mathfrak{N}(\mathfrak{a})$. Any two canonical divisors are equivalent and the order of any canonical divisor is equal to $2g-2$. The action of the antiholomorphic involution J is obviously extended on the divisors.

The Riemann-Roch theorem asserts that

$$r(-\mathfrak{a}) = ord(\mathfrak{a}) - g + 1 + i(\mathfrak{a}).$$

The period lattice of the holomorphic differentials is the lattice $\mathbb{Z}^g + \tau\mathbb{Z}^g$ in $\mathbb{C}^g$. The complex torus $Jac(\hat{\Omega}) = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ is called the Jacobian variety of the Riemann surface $\hat{\Omega}$. The Abel-Jacoby mapping based at the point P_0 (recall that we chose P_0 on $\Gamma_0 = b_0$) is defined by

$$\varphi(P) = \int_{P0}^{P} \vec{\omega}, \ P \in \hat{\Omega}.$$

The mapping $\varphi : \hat{\Omega} \longrightarrow \mathbb{C}^g$ is multi-valued and, hence, is not correctly defined, but it is a correctly defined mapping from $\hat{\Omega}$ to $J(\hat{\Omega})$. φ is a one to one conformal map of $\hat{\Omega}$ onto its image in $Jac(\hat{\Omega})$. The mapping φ can be extended on $D_0(\hat{\Omega})$ - the divisors of order 0 on $\hat{\Omega}$. If

$$\mathfrak{a} = \sum_{j=1}^{k} n_j P_j, \ \sum_{j=1}^{k} n_j = 0$$

we define

$$\varphi(\mathfrak{a}) = \sum_{j=1}^{k} n_j \varphi(P_j).$$

The Abel theorem asserts that for $\mathfrak{a} \in D_0(\hat{\Omega})$ $\varphi(\mathfrak{a}) = 0$ if and only if $\mathfrak{a}$ is a principal divisor. Another property of φ is that if we define the action of antiholomorphic involution $\mathbf{J}$ on $Jac(\hat{\Omega})$ as $\mathbf{J}[z] = -[z]$ where $[z]$ denotes the equivalence class of z in $Jac(\hat{\Omega})$ then

$$\varphi \circ J = \mathbf{J} \circ \varphi.$$

The divisor of zeros of $\eta_{P,J(P)}$ has an order 2g and is symmetric with respect to J and has the form $Z^* + J(Z^*)$ where $Z^* \subset \Omega_+, \ Z^* = z_1^* + \dots + z_g^*, z_1^*, \dots, z_g^*$ are the critical points of the Green function $G(z, P)$. That's why the divisor $Z^* = Z^*(P)$ is often referred to as the critical Green's divisor. The φ-image of the divisor $Z^* - gP_0$ in $Jac(\hat{\Omega})$ is well determined in terms of Riemann theta functions on $Jac(\hat{\Omega})$ (see [9,12]).

2.3. For any divisor $\mathfrak{a}$ of order 0, $\mathfrak{a} = \sum_{j=1}^{k} Q_j - \sum_{j=1}^{k} P_j$ and any vector $\vec{c} = (c_1, \ldots, c_g) \in \mathbb{C}^g$ the function

$$f_{\mathfrak{a},\vec{c}} = e^{2\pi i \int\limits_{P0}^{P} \omega(\mathfrak{a},\vec{c})}, \quad \omega(\mathfrak{a},\vec{c}) = \sum_{j=1}^{k} \eta_{Q_j,P_j} + \sum_{l=1}^{g} c_l \omega_l \tag{2}$$

is the so-called multiplicative meromorphic function on $\hat{\Omega}$ corresponding to the divisor $\mathfrak{a}$ and vector $\vec{c} \in \mathbb{C}^g$. That is, $f_{\mathfrak{a},\vec{c}}(P)$ is a locally meromorphic multivalued function on $\hat{\Omega}$, whose multivalued behavior is determined by the $2g$-dimensional vector

$$\chi = (\int_{a_1} \omega(\mathfrak{a},\vec{c}), \ldots, \int_{a_g} \omega(\mathfrak{a},\vec{c}) : \int_{b_1} \omega(\mathfrak{a},\vec{c}), \ldots, \int_{b_g} \omega(\mathfrak{a},\vec{c}))$$

of a and b periods of the differential $\omega(\mathfrak{a},\vec{c})$ which due to (1) is equal to $(\vec{c} : \tau\vec{c} + \varphi(\mathfrak{a}))$ modulo $\mathbb{Z}^{2g}$. The values f_1, f_2 of $f_{\mathfrak{a},\vec{c}}$ in the point P corresponding to two different paths of integration l_1, l_2 from P_0 to P, for which the cycle $l = l_2 - l_1$ is homologous to $\sum_{j=1}^{g} n_j a_j + \sum_{j=1}^{g} m_j b_j$, are connected by the equality $f_2 = e^{2\pi i <\chi, \vec{\mathbf{n}}_{\mathbf{l}}>} f_1$ where $<\chi, \vec{\mathbf{n}}_{\mathbf{l}}> = \sum_{j=1}^{g} n_j \chi_j + \sum_{j=1}^{g} m_j \chi_{g+j}$. The inverse statement is also true (see for example [13]), any meromorphic multiplicative function on $\hat{\Omega}$ has the form (2).

2.4. We will consider the vector space $\mathcal{H}_\kappa$, $\kappa \in \mathbb{R}^g/\mathbb{Z}^g$ of functions f, locally analytic in Ω_+, with single-valued modulus and such that the analytic continuation of any functional element of f along a closed curve homologous to $\sum_{j=1}^{g} m_j b_j$ leads to multiplication of the initial value by the unimodular factor (character)

$$e^{2\pi i \sum_{j=1}^{g} m_j \kappa_j}.$$

From now on we fix the term character for the elements $\kappa \in \mathbb{R}^g/\mathbb{Z}^g$, this means that in place of usual multiplicative representation of the fundamental group of $\hat{\Omega}$ in $\mathbb{T}^g$ we use the (equivalent) additive one.

The functions from the space $\mathcal{H}_\kappa$ are called modulus automorphic or character automorphic corresponding to the character κ, or simply κ-automorphic. These functions can be considered as single-valued analytic functions f in the simply connected domain $\Omega'_+ = \Omega_+ \setminus \bigcup_{j=1}^{g} \gamma_j$ with $|f|$ continuous in Ω_+ and such that the limits $f(P\pm) = \lim_{z \to P\pm} f(z)$ exist on $\bigcup_{j=1}^{g} \gamma_j \cap \Omega_+$and satisfy $f(P+) = e^{2\pi i \lambda_j} f(P-)$, $P \in \gamma_j$, $j = 1, \ldots, g$. Here the limits as $z \to P+$, $z \to P-$ on γ_j are respectively from the left and from the right side of γ_j.

It should be noted that the unit disk D is the universal covering $\pi : D \longrightarrow \Omega_+$ for the domain Ω_+. The group of covering transformations Σ (the Fuchsian group of the second kind) is isomorphic to the fundamental group $\pi_1(\Omega_+, P_0)$ of the domain and any element $f \in \mathcal{H}_\kappa$ is lifted to a single valued holomorphic function $\tilde{f}$ on D such that $|\tilde{f} \circ \sigma| = |\tilde{f}|$, $\sigma \in \Sigma$ and $\tilde{f} \circ \sigma_j = e^{2\pi i \lambda_j} \tilde{f}$, where $\sigma_j \in \Sigma$ corresponds to the element ${\gamma_j}^{-1} b_j \gamma_j$ in $\pi_1(\Omega_+, P_0)$.The corresponding lifted space $\mathcal{H}_\kappa = \mathcal{H}_\kappa(D)$ will be also called the space of character automorphic functions.

From now on we fix the universal covering mapping $\pi : D \longrightarrow \Omega_+$ by the condition $\pi(0) = a$ for some fixed point $a \in \Omega_+$ and consider the harmonic measure

$d\eta_a$ as a basic measure on Γ. By $L_p(\Gamma), p \geq 1$, we denote the the usual L_p space on Γ with respect to measure $d\eta_a$.

The spaces $H^p_{+,\kappa}$, $1 \leq p \leq \infty$, mentioned in the previous section are the spaces of functions f from $\mathcal{H}_\kappa$ which have non-tangential limits on Γ a.e. $d\eta_a$ belonging to $L_p(\Gamma)$. The function f from $H^p_{+,\kappa}$ is lifted on the universal covering as a function $\tilde{f}$ from $\mathcal{H}_\kappa(D) \cap H^p_+(D)$, where $H^p_+(D)$ is the usual Hardy space in the unit disk. For the lifted space we will use the same abbreviation as for the space in the domain.

The multiplicative meromorphic functions on $\hat{\Omega}$ of the form (2) with $P_j = J(Q_j)$, $Q_j \in \Omega_+, \Re c_j = 0$, $j = 1, \dots, g$ for the point P and the path of integration from P_0 to P contained in $\overline{\Omega_+}$ give important examples of the character-automorphic functions. Thus, the function

$$b_Q \stackrel{def}{=} f_{Q-J(Q),\vec{\mathbf{0}}}, \ Q \in \Omega_+$$

is a character-automorphic function with $\kappa = \kappa(Q) = \Re\varphi(Q - J(Q))$, contractive in Ω_+, unimodular on Γ, with only one simple zero in the point Q (and only one simple pole in $J(Q)$). We will call this function an elementary Blaschke factor. By character automorphic Blaschke product we will mean the product (finite or infinite) of elementary Blaschke factors (for the conditions on the zero set of Blaschke factors which are necessary and sufficient for the uniform convergence of Blaschke product see for example [11]). The lifting of b_Q on the universal covering is the ordinary Blaschke product in the unit disk with the zero set $\pi^{-1}(Q)$.

In what follows we will denote by B_Z the Blaschke product with the divisor of zeros Z and the corresponding character by $\kappa(Z)$. Thus B_{Z^*} is the finite Blaschke product corresponding to the critical Green's divisor Z^* (with the zeros at the critical points of the Green function $G(z, a)$) with character $\kappa^* = \kappa_{Z^*} = \Re\varphi(Z^* - J(Z^*)) = 2\Re\varphi(Z^*)$.

By analogy with the theory in the unit disk we also consider the space

$$H^2_{-,\kappa} = \left\{ \bar{f} : f \in H^2_{+,-\kappa}, f(a) = 0 \right\}.$$

The spaces $H^2_{+,\kappa}$ and $H^2_{-,\kappa}$ are orthogonal since $< f, \bar{g} >= \int_\Gamma f g d\eta_a = f(a)g(a)$ for $f \in H^2_{+,\kappa}$, $g \in H^2_{+,-\kappa}$. The space $H^2_{-,\kappa}$ also can be treated as a Hardy space on the second "sheet" Ω_- of the double $\hat{\Omega}$.

The space $H^2_{+,\kappa}$ is a functional Hilbert space and for any $t \in \Omega_+$ there exists the reproducing kernel $k(\cdot, t, \kappa) = k^\kappa_t$, $< f, k^\kappa_t >= f(t)$, for any $f \in H^2_{+,\kappa}$ (for the expression of k^κ_t in terms of Riemann theta-functions on $Jac(\hat{\Omega})$ see [10]).

For any $\kappa \in \mathbb{R}^g/\mathbb{Z}^g$ and any inner function B in Ω_+, with character κ_B let us consider the co-invariant subspaces

$$K^\kappa_B = H^2_{+,\kappa} \ominus B H^2_{+,\kappa-\kappa_B}, \ \bar{K}^\kappa_B = H^2_{-,\kappa} \ominus \frac{1}{B} H^2_{,\kappa+\kappa_B}$$

For the critical Green's divisor $Z^* = z^*_1 + \dots + z^*_g$ we consider the g-dimensional space $K^\kappa_{Z^*} = H^2_{+,\kappa} \ominus B_{Z^*} H^2_{+,\kappa-\kappa^*}$, $\kappa^* = \kappa(Z^*) = \Re\varphi(Z^* - J(Z^*))$, which is spanned by the reproducing kernels at the points z^*_j, $j = 1, \dots, g$. Then

$$L^2(\Gamma) = H^2_{-,\kappa} \oplus \mathfrak{M}_\kappa \oplus H^2_{+,\kappa}, \ \mathfrak{M}_\kappa = \frac{1}{B_{Z^*}} K^{\kappa+\kappa^*}_{Z^*} = B_{Z^*} \bar{K}^{\kappa-\kappa^*}_{Z^*},$$

where $\mathfrak{M}_\kappa$ is the so-called κ-automorphic defect space, $dim \mathfrak{M}_\kappa = g$.

2.5. More generally by character automorphic meromorphic function on $\hat{\Omega}$ we understand the function of the form (2) with $\vec{c} \in \mathbb{Z}^g$ and $\varphi(\mathfrak{a}) = \kappa \in \mathbb{R}^g/\mathbb{Z}^g$ modulo $\mathbb{Z}^g + \tau\mathbb{Z}^g$. Clearly such a function corresponds exactly to the character κ. The analog of Abel's theorem (see for example [13]) asserts that a divisor $\mathfrak{a}$ of degree zero, is a divisor of a character-automorphic function if and only if $\varphi(\mathfrak{a}) = \kappa \in \mathbb{R}^g/\mathbb{Z}^g$, and κ is exactly the character of this function. The divisor of a character-automorphic function f will be also denoted by (f) (clearly $deg(f) = 0$). We denote by $L_\kappa(\hat{\Omega})$ the vector space of κ-automorphic meromorphic functions on $\hat{\Omega}$ and by $L_\kappa(\mathfrak{a})$ the vector space $\left\{f \in L_\kappa(\hat{\Omega}) : (f) \geq \mathfrak{a}\right\}$, $r^\kappa[\mathfrak{a}] = dim L_\kappa(\mathfrak{a})$. We can also consider the character-automorphic meromorphic differentials on $\hat{\Omega}$. Actually all such differentials are obtained by multiplication of meromorphic differentials by character-automorphic functions on $\hat{\Omega}$. The vector space of κ-automorphic meromorphic differentials will be denoted by $\mathfrak{N}_\kappa$ and $\mathfrak{N}_\kappa(\mathfrak{a}) = \{\omega \in \mathfrak{N}_\kappa : (\omega) \geq \mathfrak{a}\}, i^\kappa[\mathfrak{a}] = dim\mathfrak{N}_\kappa(\mathfrak{a})$, where by (ω) we again denote the divisor of ω. The version of Riemann-Roch theorem for character-automorphic functions and differentials asserts that for any divisor $\mathfrak{a}$ on $\hat{\Omega}$

$$r^\kappa[\mathfrak{a}] = -deg(\mathfrak{a}) - g + 1 + i^{-\kappa}[-\mathfrak{a}].$$

In these notations we clearly have

$$\mathfrak{M}_\kappa = L_\kappa(J(a) - Z^* - J(Z^*)),$$

and consequently $r^\kappa[J(a) - J(Z^*)] = 0$, since otherwise the nonzero function from $L_\kappa(J(a) - J(Z^*))$ would have been from $\mathfrak{M}_\kappa$ and $H^2_{+,\kappa}$ simultaneously.

2.6. Let L, M be the subspaces of the Hilbert space H and $L \cap M = \{0\}$. Then on the sum $L + M$ we can define the operator

$$\mathcal{P}_{L\|M} : l + m \longrightarrow l, \ l \in L, \ m \in M,$$

Which is the so-called skew projection onto L parallel to M. By the closed graph theorem the boundedness of this operator is equivalent to the fact that $L + M = clos(L + M)$. The angle between L and M is defined as follows:

$$< (L, M) \in [0, \pi/2], \ \cos(L, M) = \sup_{x \in L, y \in M} \frac{< x, y >}{\|x\|\|y\|}.$$

It is easy to see that

$$\cos(L, M) = \sup_{x \in L} \frac{\|P_M x\|}{\|x\|} = |P_M P_L|, \ \sin(L, M) = \inf_{x \in L} \frac{\|(I - P_M)x\|}{\|x\|} = |\mathcal{P}_{L\|M}|^{-1},$$

where P_L, P_M are orthogonal projections on L and M respectively.

3. Two technical Lemmas on divisors of character-automorphic functions on $\hat{\Omega}$.

Let $Z = z_1 + z_2 + \cdots + z_g$ be the positive divisor of degree g on $\hat{\Omega}$. By Abel's theorem for any $\kappa \neq 0 \in \mathbb{R}^g/\mathbb{Z}^g$ we can find another positive divisor $T = t_1 + \cdots + t_g$ of degree g, such that $T - Z$ is a divisor of some κ-automorphic function on $\hat{\Omega}$. Unfortunately in general we can say nothing about the positions of points $t_1, \dots, t_g$ on $\hat{\Omega}$. But if the character κ is not fixed and we are looking for a divisor T, such that $T - Z$ is a divisor of character-automorphic function for some character, then it is possible to choose T in two (positionally) different ways: T is totally contained inside Ω_+ and (or) T is totally contained in Γ, and in the last case T may be chosen in such a way that $t_1 \in \Gamma_1, \dots, t_g \in \Gamma_g$.

Probably the above statements are already known, but they are of an importance in what follows and we proof these facts in two following lemmas. The proofs are essentially based on the results from [12, chapter VI].

Lemma 3.1. *For any positive divisor Z of degree g on $\hat{\Omega}$, $\varphi(Z) = \mu + \tau\nu$, $\mu, \nu \in \mathbb{R}^g/\mathbb{Z}^g$, there exists the positive divisor T of degree g, $T = t_1 + \cdots + t_g$, $t_1 \in \Gamma_1, \dots, t_g \in \Gamma_g$, such that $\varphi(T) = (1, 1, \dots, 1)^t + \tau\nu$. That is $\Im\varphi(T - Z) = 0$ and, by Abel's theorem for character-automorphic function, there exists the $((1, \dots, 1) - \mu)$-automorphic function with the divisor $T - Z$.*

Proof. Due to [12, Propositions 6.2 and 6.3] the subvariety S of $Jac(\hat{\Omega})$, $S = \{s \in Jac(\hat{\Omega}) : [s] = -[s]\}$ is a disjoint union of 2^g real g-dimensional torii $S_\mu = \frac{1}{2}\mu + \tau\nu$, $\mu \in (\mathbb{Z}/2\mathbb{Z})^g$, $\nu \in \mathbb{R}^g/\mathbb{Z}^g$, and each torus $S_\mu, \mu = (\mu_1, \dots, \mu_g)^t$ consists of the φ-images of all divisor of the form $T - gb_0$, where $b_0 \in \Gamma_0$ is a base point and T is a positive symmetric (J(T)=T) divisor of degree g, containing, modulo 2, μ_k points on Γ_k, $k = 1, \dots, g$ [1)].

Now to prove the Lemma we should simply take that positive symmetric divisor T of degree g, for which $\varphi(T - gb_0) = (1, \dots, 1)^t + \tau\nu$, where $\nu = \Im\varphi(Z)$. Then, due to Fay's description, T contains exactly one point on each boundary component Γ_k, $k = 1, \dots, g$. ■

Lemma 3.2. *For any positive divisor Z of degree g on $\hat{\Omega}$, $\varphi(Z) = \mu_Z + \tau\nu_Z$, $\mu_Z, \nu_Z \in \mathbb{R}^g/\mathbb{Z}^g$, there exists the positive divisor T of degree g, which is contained totally inside Ω_+ and for which $\varphi(T) = \mu_T + \tau\nu$. That is $\Im\varphi(T - Z) = 0$ and, by Abel's theorem for character-automorphic function, there exists the $(\mu_T - \mu_Z)$-automorphic function with the divisor $T - Z$.*

Proof. First we note that by the previous Lemma it is enough to prove this fact only for positive divisors of degree g, which contain exactly one point from each of boundary components $\Gamma_1, \dots, \Gamma_g$.

Thus, let $Z = z_1 + \cdots + z_g$ and $z_k \in \Gamma_k$, $k = 1, \dots, g$. It is very easy to see (see for example [11]) that $i[Z] = 0$, i.e. there are no holomorphic differentials on $\hat{\Omega}$ which have zeros at all points $z_1, \dots, z_g$. This means that the matrix $\{\varphi_j(z_k)\}_{k,j=1}^g$

[1)]Since $Jac(\hat{\Omega})$ can be identified with D_0 - the group of divisors on $\hat{\Omega}$ of degree 0 modulo principal divisors, it is often convenient to say that divisor $\mathfrak{a} \in D_0$ belongs to S_μ, $\mu \in (\mathbb{Z}/2\mathbb{Z})^g$, if $\varphi(\mathfrak{a}) \in S_\mu$.

is nondegenerate. But the matrix $\{\varphi_j(u_k)\}_{k,j=1}^{g}$, $u_k \in \hat{\Omega}$, $k = 1, \ldots, g$, is exactly equal to the Jacobian of the analytic mapping $\Phi(u_1, \ldots, u_g) = \varphi(u_1 + \cdots + u_g - gb_0)$ from the (small enough) neighborhood U_Z of the point $(z_1, \ldots, z_g) \in \hat{\Omega} \times \cdots \times \hat{\Omega}$ into $\mathbb{C}^g$. So the Jacobian of this mapping is nondegenerate at $(z_1, \ldots, z_g)$ and there exists the inverse analytic mapping Φ^{-1} from a small neighborhood $V_{\varphi(Z)}$ of the point $\mu_Z + \tau\nu_Z$ onto the neighborhood of the point $(z_1, \ldots, z_g)$. We are looking at all the points $(u_1, \ldots, u_g)$ close to $(z_1, \ldots, z_g)$ such that $\Im\varphi(u_1 + \cdots + u_g - gb_0) = \Im\varphi(Z - gb_0)$. But these points form the real manifold of dimension g which is exactly the image of g-dimensional real subspace $i\Im\mathbb{C}^g$ of $\mathbb{C}^g$ under the mapping Φ^{-1} (here $i\Im(w_1, w_2, \ldots, w_g) = (i\Im w_1, i\Im w_2, \ldots, i\Im w_g)$). Now since we already know that the Jacobian of this mapping at the point $\mu_Z + \tau\nu_Z$ is nondegenerate and all its elements are pure imaginary (see Sec 2.1 for the properties of differentials $\varphi_1, \ldots, \varphi_g$), we conclude that the tangent space of this manifold $\Phi^{-1}(i\Im\mathbb{C}^g)$ at the point $(z_1, \ldots, z_g)$ is exactly $\Re\mathbb{C}^g$. All this analysis implies that for any fixed l, $1 \le l \le g$ in the neighborhood U_Z there is a point $(u_1, \ldots, u_g)$ with $\Im\varphi(u_1 + \cdots + u_g) = \tau\nu_Z$ and $u_l \ne z_l$. Indeed, otherwise, say for $l = 1$, $\Phi^{-1}(i\Im\mathbb{C}^g) \cap U_Z \subset \{(u_1, \ldots, u_g) \in U_Z : u_1 = z_1\}$. But the last inclusion means that the above mentioned tangent space should be $(g-1)$-dimensional, since all the vectors of the form $(x + iy, 0, \ldots, 0), \forall x, y \in \mathbb{R}$, are orthogonal to the manifold $\Phi^{-1}(i\Im\mathbb{C}^g) \cap U_Z$ itself and in particular to its tangent space at $(z_1, \ldots, z_g)$.

Thus, for each l, $1 \le l \le g$ there exists a character κ_l, such that the space $L_{\kappa_l}(-Z)$ contains the function f_l with nondegenerate pole at the point z_l. Then the functions $f_l(z)\overline{f_l(J(z))}$ are single-valid positive on Γ meromorphic functions on $\hat{\Omega}$ from $L(-2Z)$ and their sum is also positive on Γ and definitely has nondegenerate double poles at all points of Z. Hence, for positive constant C, the divisor of strictly positive on Γ function $f = \sum_{l=1}^{g} f_l(z)\overline{f_l(J(z))} + C$ is symmetric ($J((f)) = (f)$) and has the form $(f) = T + J(T) - 2Z$, where T is positive divisor of degree g totally contained inside Ω_+.

We claim that T is a desired divisor and $\sqrt{1/(fb_T)}$ is a desired function, where as above b_T is a finite character-automorphic Blaschke product, $(b_T) = T - J(T)$. To show this we must prove that $\Im(\varphi(Z - T)) = \tau(\nu_Z - \nu_T) = 0$. Actually, since f is single-valid meromorphic function on $\hat{\Omega}$, we already know that $2(\nu_Z - \nu_T) \in \mathbb{Z}^g$ and to see that, $(\nu_Z - \nu_T) \in \mathbb{Z}^g$ we should simply note that, by construction of function f, $\omega = (1/f)d\eta_a$ is a non-negative on Γ, symmetric meromorphic differential on $\hat{\Omega}$ with the divisor $(\omega) = 2Z - T - J(T) + Z^* + J(Z^*) - a - J(a)$. Hence, by [12, Proposition 6.8], $\varphi(Z - T + Z^* - a) + [\Delta_0] \in T_0 = \{\mu + \tau\nu : \mu \in \mathbb{R}^g/\mathbb{Z}^g, \nu \in \mathbb{Z}^g\}$, where Δ_0 is a vector of Riemann constants, which depends only on the choice of homology basis and basic point b_0. But $d\eta_a$ itself is a positive on Γ symmetric meromorphic differential on $\hat{\Omega}$. Hence, by the same Proposition from [12] $\varphi(Z^* - a) + [\Delta_0] \in T_0$ and, finally, $\Im\varphi(Z - T) \in \mathbb{Z}^g$. ■

4. The weight might be bad

4.1. Let $\mathcal{B}_+^\kappa$ be the space of all functions from $H_\kappa(\Omega_+)$ which are continuous up to the boundary Γ. It is known (see for example [7, Sec 3.] that $\mathcal{B}_+^\kappa$ is a dense set in $H^2_{+,\kappa}$. By $\mathcal{B}_-^\kappa$ we denote the corresponding dense subset of $H^2_{-,\kappa}$, $\mathcal{B}_-^\kappa = \{\overline{f \circ J} : f \in \mathcal{B}_+^{-\kappa}, f(a) = 0\}$.

The question we are interested in is the following: What are necessary and sufficient conditions for finite positive measure $d\mu$ on Γ that the angle between subspaces $Clos_{L_2(d\mu)}\mathcal{B}^\kappa_-$ and $Clos_{L_2(d\mu)}\mathcal{B}^\kappa_+$ is nonzero? The equivalent question is when (for what measures $d\mu$) the orthogonal projection P^κ_+ in $L_2(\Gamma, d\eta_a) = L_2(\Gamma)$ onto $H^2_{+,\kappa}$ can be extended from $\mathcal{B}^\kappa_- + \mathcal{B}^\kappa_+$ to $Clos_{L_2(d\mu)}\mathcal{B}^\kappa_- + Clos_{L_2(d\mu)}\mathcal{B}^\kappa_+$ as a bounded operator?

Proposition 4.1. *If the angle in $L_2(d\mu)$ between $Clos_{L_2(d\mu)}\mathcal{B}^\kappa_-$ and $Clos_{L_2(d\mu)}\mathcal{B}^\kappa_+$ is nonzero, then $d\mu$ is absolutely continuous with respect to $d\eta_a$, $d\mu = wd\eta_a$ and $\log w \in L_1(\Gamma)$.*

Proof. Let $d\mu = wd\eta_a + d\mu_s$, where $w \in L_1(d\eta_a)$ and $d\mu_s$ is a singular component of $d\mu$. The statement about w follows immediately from character-automorphic analog of Segö's theorem ([7,Theorem 3.1, Corollary 3.1], which in particular asserts that if $\log w \notin L_1(d\eta_a)$ then $Clos_{L_2(d\mu)}\mathcal{B}^\kappa_+ = L_2(d\mu)$. But this is not true in our case since the closures of $\mathcal{B}^\kappa_+$ and $\mathcal{B}^\kappa_-$ in $L_2(d\mu)$ are at nonzero angle. As for the absence of the singular component $d\mu_s$, this assertion was actually proved in [7, proposition 4.1], which contained the same statements on the measures in a case when $Clos_{L_2(d\mu)}\{\mathcal{B}^\kappa_- + \mathfrak{M}_\kappa\}$ and $Clos_{L_2(d\mu)}\mathcal{B}^\kappa_+$ are at nonzero angle. ∎

From now on we denote by h the character-automorphic outer function in H^2_{+,κ_0}, with a certain character κ_0, such that $|h|^2\mid_\Gamma = w$, $d\mu = |h|^2\, d\eta_a$.

4.2. As in the case of the unit disk for $\lambda \in \Omega_+, f \in L_1(\Gamma)$ we denote by $f(\lambda)$ the value at the point λ of the harmonic extension of f in Ω_+, $f(\lambda) = \int_\Gamma f d\eta_\lambda$.

Remark 4.1. Since we know that the measure $d\mu = wd\eta_a = |h|^2 d\eta_a$ is absolutely continuous with respect to harmonic measure $d\eta_a$, it is easy to see that in place of the spaces $\mathcal{B}^\kappa_+, \mathcal{B}^\kappa_-$ we can use the spaces $H^\infty_{+,\kappa}, H^\infty_{-,\kappa} \subset L_2(d\mu)$. Indeed, we have $\mathcal{B}^\kappa_+ \subset H^\infty_{+,\kappa}$, $\mathcal{B}^\kappa_- \subset H^\infty_{-,\kappa}$ and

$$\mathcal{D}^\kappa_+ = Clos_{L_2(d\mu)}H^\infty_{+,\kappa} = Clos_{L_2(d\mu)}\mathcal{B}^\kappa_+ = \frac{1}{h}H^2_{+,\kappa+\kappa_0},$$

$$\mathcal{D}^\kappa_- = Clos_{L_2(d\mu)}H^\infty_{-,\kappa} = Clos_{L_2(d\mu)}\mathcal{B}^\kappa_- = \frac{1}{\bar{h}}H^2_{-,\kappa-\kappa_0} + .$$

To see that, for example, the first of these chains of equalities is valid one should simply note that due to the properties of outer functions, $h\mathcal{B}^\kappa$ (as well as $hH^\infty_{+,\kappa}$) is dense in $H^2_{+,\kappa+\kappa_0}$ and that the convergence of a sequence of functions $\{f_n\}_{n=1}^\infty$ to the function f in $L_2(d\mu)$ is equivalent to the convergence of a sequence $\{hf_n\}_{n=1}^\infty$ to the function hf in L_2.

Proposition 4.2. *If the "Riesz" projection P^κ_+ can be extended from $\mathcal{B}^\kappa_- + \mathcal{B}^\kappa_+$ to $Clos_{L_2(d\mu)}\mathcal{B}^\kappa_- + Clos_{L_2(d\mu)}\mathcal{B}^\kappa_+$ as a bounded operator then the following alternative takes place:*

(1) *$1/h$ is $(-\kappa_0)$-automorphic outer function and $|h|^2$ satisfies $A'_2(\Gamma)$ Muckenhoupt condition (see [7]):*

$$\sup_{\lambda\in\Omega_+} (|h|^2)(\lambda) \cdot (1/|h|^2)(\lambda) < \infty.$$

(2) *There exists a positive divisor $U_h = u_1 + \cdots + u_k$ of degree $k \le g$ contained totally in Γ, such that $1/h$ is locally square summable (from L_2^{loc}) at any point from $\Gamma \setminus U_h$, is not square summable at any of the points u_j, $1 \le j \le k$, but at each of these points the function $(z-u_j)^{\nu_j}/h$ is locally square summable, where $\nu_j = degU_h|_{u_j}$, the degree of divisor U_h at the point u_j (note that some of the points u_j may coincide). Moreover U_h is a minimal divisor possessing these properties in a sense that if U is any other divisor with the same properties, then $U_h \le U$.*

Proof. If the orthogonal projection $P_{\mathfrak{M}_\kappa}$ from $L_2(d\eta_a)$ onto defect space $\mathfrak{M}_\kappa$ can be extended from $\mathcal{B}_-^\kappa + \mathfrak{M}_\kappa + \mathcal{B}_+^\kappa$ onto $Clos_{L_2(d\mu)}\{\mathcal{B}_-^\kappa + \mathfrak{M}_\kappa + \mathcal{B}_+^\kappa\}$ as a bounded operator, then the same is valid for P_+^κ. Indeed, the "Riesz" projection can be expressed in the following way

$$P_+^\kappa = P_+^\kappa(I - P_{\mathfrak{M}_\kappa}).$$

where in the right-hand side of this identity it acts on the elements from $H^2_{-,\kappa} + H^2_{+,\kappa}$. Now if $P_{\mathfrak{M}_\kappa}$ is bounded in $L_2(d\mu)$ and P_+^κ is bounded on $Clos_{L_2(d\mu)}\{\mathcal{B}_-^\kappa + \mathcal{B}_+^\kappa\}$ then P_+^κ is bounded also in $L_2(d\mu)$ and, by [7, Theorem 4.1], the weight $|h|^2$ satisfies $A_2'(\Gamma)$ Muckenhoupt condition.

Now suppose that $P_{\mathfrak{M}_\kappa}$ is unbounded operator in $L_2(d\mu)$.

Let us put for any positive ε $w_\varepsilon = w + \varepsilon$ and $d\mu_\varepsilon = w_\varepsilon d\eta_a$. Then for any $f \in \mathcal{B}_-^\kappa + \mathfrak{M}_\kappa + \mathcal{B}_+^\kappa$

$$\lim_{\varepsilon \to 0} \int_\Gamma |f|^2 w_\varepsilon d\eta_a = \int_\Gamma |f|^2 w d\eta_a,$$

by Lebesgue's theorem on monotone convergence. Now

$$\begin{aligned}|P_{\mathfrak{M}_\kappa}|_{L_2(d\mu_\varepsilon)} = & \sup_{\substack{f_{\mathfrak{M}_\kappa} \in \mathfrak{M}_\kappa \\ \|f_{\mathfrak{M}_\kappa}\|_2 = 1}} \frac{\|f_{\mathfrak{M}_\kappa}\|_{L_2(d\mu_\varepsilon)}}{dist_{L_2(d\mu_\varepsilon)}(f_{\mathfrak{M}_\kappa}, \mathcal{B}_-^\kappa + \mathcal{B}_+^\kappa)} \xrightarrow[\varepsilon \to 0]{} \\ & \sup_{\substack{f_{\mathfrak{M}_\kappa} \in \mathfrak{M}_\kappa \\ \|f_{\mathfrak{M}_\kappa}\|_2 = 1}} \frac{\|f_{\mathfrak{M}_\kappa}\|_{L_2(d\mu)}}{dist_{L_2(d\mu)}(f_{\mathfrak{M}_\kappa}, \mathcal{B}_-^\kappa + \mathcal{B}_+^\kappa)} = \infty. \end{aligned} \tag{3}$$

Taking into account that $\left(\sqrt{w_\varepsilon}\{\mathcal{B}_-^\kappa + \mathcal{B}_+^\kappa\}\right)^\perp = \frac{1}{\sqrt{w_\varepsilon}}\mathfrak{M}_\kappa$, we can evaluate the first distance in (3) in a following manner,

$$dist^2_{L_2(d\mu_\varepsilon)}(f_{\mathfrak{M}_\kappa}, \mathcal{B}_-^\kappa + \mathcal{B}_+^\kappa) = dist^2_{L_2}(\sqrt{w_\varepsilon} f_{\mathfrak{M}_\kappa}, \sqrt{w_\varepsilon}\{\mathcal{B}_-^\kappa + \mathcal{B}_+^\kappa\}) =$$

$$\sup_{\varphi \in \mathfrak{M}_\kappa} \frac{|< f_{\mathfrak{M}_\kappa}, \varphi >|^2}{\int_\Gamma |\varphi|^2 \frac{1}{w_\varepsilon} d\eta_a} = \left\{ \inf_{\substack{\varphi_{\mathfrak{M}_\kappa} \in \mathfrak{M}_\kappa \\ \varphi_{\mathfrak{M}_\kappa} \perp f_{\mathfrak{M}_\kappa}}} \int_\Gamma |f_{\mathfrak{M}_\kappa} + \varphi_{\mathfrak{M}_\kappa}|^2 \frac{1}{w_\varepsilon} d\eta_a \right\}^{-1}.$$

Now since $\mathfrak{M}_\kappa$ is finite dimensional space, on which all the norms are equivalent and since

$$\lim_{\varepsilon \to 0} dist^2_{L_2(d\mu_\varepsilon)}(f_{\mathfrak{M}_\kappa}, \mathcal{B}_-^\kappa + \mathcal{B}_+^\kappa) = dist^2_{L_2(d\mu)}(f_{\mathfrak{M}_\kappa}, \mathcal{B}_-^\kappa + \mathcal{B}_+^\kappa),$$

using (3) we get

$$\sup_{\substack{f_{\mathfrak{M}_\kappa}\in\mathfrak{M}_\kappa\\ \|f_{\mathfrak{M}_\kappa}\|_2=1}} \inf_{\substack{\varphi_{\mathfrak{M}_\kappa}\in\mathfrak{M}_\kappa\\ \varphi_{\mathfrak{M}_\kappa}\perp f_{\mathfrak{M}_\kappa}}} \int_\Gamma |f_{\mathfrak{M}_\kappa}+\varphi_{\mathfrak{M}_\kappa}|^2\frac{1}{w}d\eta_a=\infty.$$

Now again using the compactness arguments for the finite dimensional space $\mathfrak{M}_\kappa$ we obtain that there exists $f_{\mathfrak{M}_\kappa}\in\mathfrak{M}_\kappa$ such that for any $\varphi_{\mathfrak{M}_\kappa}\in\mathfrak{M}_\kappa$, $\varphi_{\mathfrak{M}_\kappa}\perp f_{\mathfrak{M}_\kappa}$,

$$\int_\Gamma |f_{\mathfrak{M}_\kappa}+\varphi_{\mathfrak{M}_\kappa}|^2\frac{1}{w}d\eta_a=\infty. \tag{4}$$

On the other hand, since $\log w\in L_1$, we know (see [7, Section 3] that for any fixed $\lambda\in\Omega_+$ the distance in $L_2(wd\eta_a)$ between the reproducing kernel k^κ_λ in $H^2_{+,\kappa}$ at the point λ and $b_\lambda\mathcal{B}_+^{\kappa-\kappa(\lambda)}$ is nonzero and, hence the orthogonal projection $P^+_{k^\kappa_\lambda}$ in $H^2_{+,\kappa}$ onto one-dimensional space spanned by k^κ_λ can be extended from $\mathcal{B}^\kappa_+$ as a bounded operator to $Clos_{L_2(d\mu)}\mathcal{B}^\kappa_+$:

$$|P^+_{k^\kappa_\lambda}|_{L_2(d\mu)}=\frac{\|k^\kappa_\lambda\|_{L_2(d\mu)}}{dist_{L_2(d\mu)}(k^\kappa_\lambda,b_\lambda\mathcal{B}_+^{\kappa-\kappa(\lambda)})}<\infty.$$

The orthogonal projection $P_{k^\kappa_\lambda}$ in $L_2\Gamma$ onto the same one-dimensional space spanned by k^κ_λ can be extended from $\mathcal{B}^\kappa_-+\mathcal{B}^\kappa_+$ as a bounded operator as well to $Clos_{L_2(d\mu)}\mathcal{B}^\kappa_-+Clos_{L_2(d\mu)}\mathcal{B}^\kappa_+$. Indeed, this fact immediately follows from the expression $P_{k^\kappa_\lambda}=P^+_{k^\kappa_\lambda}P^\kappa_+$.Therefore

$$|P_{\{k^\kappa_\lambda\}}|_{L_2(d\mu)}=\frac{\|k^\kappa_\lambda\|_{L_2(d\mu)}}{dist_{L_2(d\mu)}(k^\kappa_\lambda,\mathcal{B}^\kappa_-+b_\lambda\mathcal{B}_+^{\kappa-\kappa(\lambda)})}<\infty. \tag{5}$$

Now using again the same trick with the measures $d\mu_\varepsilon$ we get

$$dist^2_{L_2(d\mu_\varepsilon)}(k^\kappa_\lambda,\mathcal{B}^\kappa_-+b_\lambda\mathcal{B}_+^{\kappa-\kappa(\lambda)})=dist^2_{L_2}(\sqrt{w_\varepsilon}k^\kappa_\lambda,\sqrt{w_\varepsilon}(\mathcal{B}^\kappa_-+b_\lambda\mathcal{B}_+^{\kappa-\kappa(\lambda)}))=$$
$$\sup_{\substack{\varphi\in\left\{\sqrt{w_\varepsilon}\left(\mathcal{B}^\kappa_-+b_\lambda\mathcal{B}_+^{\kappa-\kappa(\lambda)}\right)\right\}^\perp\\ \|\varphi\|_2=1}}|<\sqrt{w_\varepsilon}k^\kappa_\lambda,\varphi>|^2=\sup_{\varphi\in\{\mathbb{C}k^\kappa_\lambda\}+\mathfrak{M}_\kappa}\frac{|<k^\kappa_\lambda,\varphi>|^2}{\int_\Gamma|\varphi|^2\frac{1}{w_\varepsilon}d\eta_a}=$$
$$(k^\kappa_\lambda(\lambda))^2\left(\inf_{\varphi\in\mathfrak{M}_\kappa}\int_\Gamma|k^\kappa_\lambda+\varphi|^2\frac{1}{w_\varepsilon}d\eta_a\right)^{-1},$$

which, by the compactness arguments in finite dimensional space $\mathfrak{M}_\kappa$, means that there exists $\varphi_\varepsilon\in\mathfrak{M}_\kappa$, such that

$$dist^2_{L_2(d\mu_\varepsilon)}(k^\kappa_\lambda,\mathcal{B}^\kappa_-+b_\lambda\mathcal{B}_+^{\kappa-\kappa(\lambda)})=(k^\kappa_\lambda(\lambda))^2\left(\int_\Gamma|k^\kappa_\lambda+\varphi_\varepsilon|^2\frac{1}{w_\varepsilon}d\eta_a\right)^{-1},$$

Since $\lim_{\varepsilon\to0}dist_{L_2(d\mu_\varepsilon)}(k^\kappa_\lambda,\mathcal{B}^\kappa_-+b_\lambda\mathcal{B}_+^{\kappa-\kappa(\lambda)})=dist^2_{L_2(d\mu)}(k^\kappa_\lambda,\mathcal{B}^\kappa_-+b_\lambda\mathcal{B}_+^{\kappa-\kappa(\lambda)})$, taking into account (5) we see that $\lim_{\varepsilon\to0}\int_\Gamma|k^\kappa_\lambda+\varphi_\varepsilon|^2\frac{1}{w_\varepsilon}d\eta_a$ exists and is finite. Now if we assume that the norms $\|k_\lambda^\kappa + \varphi_\varepsilon\|_\infty$ are not bounded for all positive ε, then there exists a subsequence $\{\varepsilon_n\}$ of positive numbers with a zero limit, such that $\|k_\lambda^\kappa + \varphi_{\varepsilon_n}\|_\infty \underset{n\to\infty}{\longrightarrow} \infty$ and the functions $\varphi'_n = (k_\lambda^\kappa + \varphi_{\varepsilon_n})/\|k_\lambda^\kappa + \varphi_{\varepsilon_n}\|_\infty$ are from $(g+1)$-dimensional space $\mathbb{C}k_\lambda^\kappa + \mathfrak{M}_\kappa$ and are of unit L_∞-norm. Hence, without loss of generality, by compactness arguments, we can assume that this sequence itself is convergent and let $\varphi' = \lim_{n\to\infty} \varphi'_n$. Clearly $\varphi' \in \mathbb{C}k_\lambda^\kappa + \mathfrak{M}_\kappa$, $\|\varphi'\|_\infty = 1$. But, by the definition of the functions φ'_n, $|\varphi'_n|^2/w_{\varepsilon_n} \underset{n\to\infty}{\longrightarrow} |\varphi'|^2/w$ almost everywhere and by Fatou's Lemma

$$\lim_{n\to\infty} \int_\Gamma \frac{|\varphi'_n|^2}{w_{\varepsilon_n}} d\eta_a = \int_\Gamma \frac{|\varphi'|^2}{w} d\eta_a = 0.$$

But this is impossible, since φ' is nonzero locally analytic on Γ function and hence the equality $\int_\Gamma \frac{|\varphi'|^2}{w} d\eta_a = 0$ implies that $w = |h|^2$ is equal to infinity on a set of positive Lebesgue measure on Γ.

This contradiction shows that the norms $\|k_\lambda^\kappa + \varphi_\varepsilon\|_\infty$ are bounded for all positive ε, and again, by compactness arguments, we can find sequence of decreasing to zero positive numbers ε_n, such that $(k_\lambda^\kappa + \varphi_{\varepsilon_n})$ tends to $k_\lambda^\kappa + \varphi_\lambda$, $\varphi_\lambda \in \mathfrak{M}_\kappa$, uniformly on Γ as n tends to infinity. Finally,

$$\int_\Gamma |k_\lambda^\kappa + \varphi_\lambda|^2 \frac{1}{w} d\eta_a = \lim_{\varepsilon\to 0} \int_\Gamma |k_\lambda^\kappa + \varphi_\varepsilon|^2 \frac{1}{w_\varepsilon} d\eta_a < \infty. \tag{6}$$

Now let us compare the expressions (4) and (6). On one hand expression (4) tells us that $1/w \notin L_1(\Gamma)$ but on another hand (6) tells us that for any $\lambda \in \Omega_+$ it is possible to find $\varphi_\lambda \in \mathfrak{M}_\kappa$, such that $|k_\lambda^\kappa + \varphi_\lambda|^2/w \in L_1(\Gamma)$. Using Holder's inequality, for any measurable set $X \subset \Gamma$ we have

$$\left(\int_X \frac{1}{w} d\eta_a\right)^2 \le \int_X \frac{|k_\lambda^\kappa + \varphi_\lambda|^2}{w} d\eta_a \cdot \int_X \frac{1}{|k_\lambda^\kappa + \varphi_\lambda|^2} d\eta_a,$$

that, since $k_\lambda^\kappa + \varphi_\lambda$ is locally analytic on Γ, means that $1/w$ may be not locally summable only at zeros of $k_\lambda^\kappa + \varphi_\lambda$ on Γ. Since $k_\lambda^\kappa + \varphi_\lambda \in L_\kappa(J(a) - J(\lambda) - Z^* - J(Z^*))$ (see for example [8, Proposition 3.2] it may have at most $2g$ zeros on Γ. Thus the number of points on Γ at which $1/w$ is not locally summable is also finite. So $1/h$ is not locally square summable on Γ only at the finite number of points.

Let us denote by $U = U_h$ the positive divisor of these points on Γ. Note that due to (6) the degree of U_h at the zero of $k_\lambda^\kappa + \varphi_\lambda$ is less or equal to the order of this zero. Therefore if $\nu = degU_h|_u$, $u \in \Gamma$ then $(z-u)^\nu/h$ is locally square summable at u.

To finish the proof of proposition 4.2 we should show that $degU_h \le g$. Note that since (6) is valid for all $\lambda \in \Omega_+$, $k_\lambda^\kappa + \varphi_\lambda \in L^\kappa(U_h + J(a) - J(\lambda) - Z^* - J(Z^*))$ for all $\lambda \in \Omega_+$. Hence $r^\kappa[U_h + J(a) - J(\lambda) - Z^* - J(Z^*)] \ge 1$ for all $\lambda \in \Omega_+$. The application of Riemann-Roch theorem gives us

$$\begin{aligned} 1 \le r^\kappa[U_h + J(a) - J(\lambda) - Z^* - J(Z^*)] = \\ g + 1 - deg(U_h) + i^{-\kappa}[J(\lambda) + Z^* + J(Z^*) - U_h - J(a)], \end{aligned}$$

$$r^{\kappa}[U_h + J(a) - Z^* - J(Z^*)] = g - deg(U_h) + i^{-\kappa}[Z^* + J(Z^*) - U_h - J(a)]. \quad (7)$$

Clearly $r^{\kappa}[U_h + J(a) - J(\lambda) - Z^* - J(Z^*)] \geq r^{\kappa}[U_h + J(a) - Z^* - J(Z^*)]$ and $i^{-\kappa}[Z^* + J(Z^*) - U_h - J(a)] \geq i^{-\kappa}[J(\lambda) + Z^* + J(Z^*) - U_h - J(a)]$. We will show that

$$i^{-\kappa}[Z^* + J(Z^*) - U_h - J(a)] = 0. \quad (7')$$

This will imply that $i^{-\kappa}[J(\lambda) + Z^* + J(Z^*) - U_h - J(a)] = 0$ and,by (7), that $degU_h \leq g$. Suppose for a moment that $i^{-\kappa}[Z^* + J(Z^*) - U_h - J(a)] > 0$, then the equality $i^{-\kappa}[Z^* + J(Z^*) - U_h - J(a)] = i^{-\kappa}[J(\lambda) + Z^* + J(Z^*) - U_h - J(a)]$ can be valid only for finite number of points $\lambda \in \Omega_+$. Indeed, this equality means that all differentials from $\mathfrak{N}_{\kappa}(Z^* + J(Z^*) - U_h - J(a))$ have a zero at the point λ, which is not true if λ does not belong to the set of zeros of one certain differential from this space. Now for such λ $i^{-\kappa}[Z^* + J(Z^*) - U_h - J(a)] = i^{-\kappa}[J(\lambda) + Z^* + J(Z^*) - U_h - J(a)] + 1$ and, by (7)

$$\begin{aligned} r^{\kappa}[U_h + J(a) - Z^* - J(Z^*)] = \\ g + 1 - deg(U_h) + i^{-\kappa}[J(\lambda) + Z^* + J(Z^*) - U_h - J(a)] = \\ r^{\kappa}[U_h + J(a) - J(\lambda) - Z^* - J(Z^*)]. \end{aligned}$$

But due to (6) $k_{\lambda}^{\kappa} + \varphi_{\lambda} \in L_{\kappa}(U_h + J(a) - J(\lambda) - Z^* - J(Z^*))$ and $k_{\lambda}^{\kappa} + \varphi_{\lambda} \notin L_{\kappa}(U_h + J(a) - Z^* - J(Z^*))$. This contradiction proves (7′) and finishes the proof of the Proposition 4.2. ■

Corollary 4.1. *For any $\lambda \in \Omega_+$ and function $k_{\lambda}^{\kappa} + \varphi_{\lambda}$ from (6)*

$$\frac{k_{\lambda}^{\kappa} + \varphi_{\lambda}}{h} \in \mathfrak{M}_{\kappa - \kappa_0} + H^2_{+,\kappa-\kappa_0}.$$

This fact immediately follows from the observation that this function is square summable on Γ and meromorphic on Ω_+ with the divisor $(k_{\lambda}^{\kappa} + \varphi_{\lambda}) \geq -J(Z^*)$ on Ω_+.

Corollary 4.2.

(1)
$$r^{\kappa}[U_h + J(a) - Z^* - J(Z^*)] = g - degU_h,$$

and for any function $\varphi_{\mathfrak{M}_{\kappa}} \in L_{\kappa}(U_h + J(a) - Z^* - J(Z^*))$

$$\frac{\varphi_{\mathfrak{M}_{\kappa}}}{h} \in \mathfrak{M}_{\kappa-\kappa_0} + H^2_{+,\kappa-\kappa_0}, \ \frac{\varphi_{\mathfrak{M}_{\kappa}}}{\bar{h}} \in \mathfrak{M}_{\kappa+\kappa_0} + H^2_{+,\kappa+\kappa_0}.$$

(2) *Any function $\varphi_{\mathfrak{M}_{\kappa}} \in \mathfrak{M}_{\kappa}$ for which*

$$\frac{\varphi_{\mathfrak{M}_{\kappa}}}{h} \in \mathfrak{M}_{\kappa-\kappa_0} + H^2_{+,\kappa-\kappa_0}, \ \frac{\varphi_{\mathfrak{M}_{\kappa}}}{\bar{h}} \in \mathfrak{M}_{\kappa+\kappa_0} + H^2_{+,\kappa+\kappa_0},$$

belongs to $L_{\kappa}(U_h + J(a) - Z^* - J(Z^*))$

The proof immediately follows from (7),(7′) and the same observation as for Corollary 4.1.

Remark 4.2. It follows from the proof of Proposition 4.2, that for any character $\nu \in \mathbb{R}^g/\mathbb{Z}^g$ and any $\lambda \in \Omega_+$

$$r^\nu[U_h + J(a) - J(\lambda) - Z^* - J(Z^*)] \geq 1,$$

and for any $\psi \in L_\nu(U_h + J(a) - J(\lambda) - Z^* - J(Z^*))$ ψ/h is square summable on Γ.

The inverse statement is also true: If $\psi//h \in L^2(\Gamma)$, $\psi \in k^\nu_\lambda + \mathfrak{M}_\nu$, then $\psi \in L_\nu(U_h + J(a) - J(\lambda) - Z^* - J(Z^*))$.

5. The weight could be improved

5.1. Only case (2) of the alternative from the Proposition 4.2 is of interest for us, so in what follows we suppose that $0 < degU_h \leq g$. Our nearest aim is to study function h and divisor U_h of its "bad behavior" on Γ. To do this we will use the technique, similar to that in [14 , 2, 3], which involves Hankel and Teoplitz operators.

As above P^ν_+, $\nu \in \mathbb{R}^g/\mathbb{Z}^g$ is a "Riesz" orthogonal projection from $L_2(\Gamma)$ onto $H^2_{+,\nu}$ and let $P^\nu_- = I - P^\nu_+$, be the orthogonal projection from $L_2(\Gamma)$ onto $H^2_{-,\nu} + \mathfrak{M}_\nu$. Recall that (see for example [8]) for any $f \in L_\infty(\Gamma)$ and $\mu, \nu \in \mathbb{R}^g/\mathbb{Z}^g$ the Hankel operator $\mathbf{H}^{\nu,\mu}_f$ with a symbol is an operator acting from $H^2_{+,\nu}$ into $H^2_{-,\nu+\mu} + \mathfrak{M}_{\nu+\mu}$ according to the formula

$$\mathbf{H}^{\nu,\mu}_f g = P^{\nu+\mu}_- fg, \quad \forall g \in H^2_{+,\nu}.$$

The character-automorphic Teoplitz operator $\mathbf{T}^{\nu,\mu}_f$ with a symbol f is acting from $H^2_{+,\nu}$ into $H^2_{+,\nu+\mu}$,

$$\mathbf{T}^{\nu,\mu}_f g = fg - \mathbf{H}^{\nu,\mu}_f g = P^{\nu+\mu}_+ fg, \quad \forall g \in H^2_{+,\nu}.$$

Here character ν describes the Hardy space, on which the operators act, and character μ prescribes the character to the symbol $f \in L_\infty(\Gamma)$[2)]

If f is unimodular on Γ function then clearly for any $g \in H^2_{+,\nu}$

$$\|\mathbf{H}^{\nu,\mu}_f g\|_2^2 + \|\mathbf{T}^{\nu,\mu}_f g\|_2^2 = \|g\|_2^2, \tag{8}$$

and the adjoint operator $\left(\mathbf{T}^{\nu,\mu}_f\right)^*$ is a Teoplitz operator $\mathbf{T}^{\nu+\mu,-\mu}_{\bar f}$.

Just as in the case of the unit disk theory the estimation of the angles between subspaces in weighted L_2 on Γ is equivalent to the estimation of the norms of corresponding Hankel operators:

$$\cos(Clos_{L_2(d\mu)}\mathcal{B}^\kappa_-, Clos_{L_2(d\mu)}\mathcal{B}^\kappa_+) = \sup_{f\in\mathcal{B}^\kappa_+, g\in\mathcal{B}^\kappa_-} \frac{|< fh, gh >|}{\|fh\|_2 \cdot \|g\bar h\|_2} =$$

$$\sup_{\substack{f\in H^2_{+,\kappa+\kappa_0}\\ g\in H^2_{-,\kappa-\kappa_0}}} \frac{|< f\frac{\bar h}{h}, g >|}{\|f\|_2 \cdot \|g\|_2} = \sup_{\substack{f\in H^2_{+,\kappa+\kappa_0}\\ g\in \left(H^2_{+,\kappa-\kappa_0+\kappa^*}\right)^\perp}} \frac{|< B_{Z^*}\frac{\bar h}{h} f, g >|}{\|f\|_2 \cdot \|g\|_2} =$$

$$= \left|\mathbf{H}^{\kappa+\kappa_0,-2\kappa_0+\kappa^*}_{B_{Z^*}\frac{\bar h}{h}}\right|.$$

2) Note that (see [8]) $L^\infty(\Gamma)$ can be identified with $L^\infty_\mu(\Gamma)$ for all characters μ.

Therefore the case (1) of alternative from proposition 4.2 corresponds to the case $\left|\mathbf{H}^{\nu,2\kappa_0}_{\frac{\bar h}{h}}\right| < 1$ for all $\nu \in \mathbb{R}^g/\mathbb{Z}^g$ (see [7]), which, in particular, implies that $\left|\mathbf{H}^{\kappa+\kappa_0,-2\kappa_0+\kappa^*}_{B_{Z^*}\frac{\bar h}{h}}\right| < 1$, and case (2) exactly corresponds to the case $\left|\mathbf{H}^{\nu,2\kappa_0}_{\frac{\bar h}{h}}\right| = 1$ for all $\nu \in \mathbb{R}^g/\mathbb{Z}^g$ (see again [7]) and $\left|\mathbf{H}^{\kappa+\kappa_0,-2\kappa_0+\kappa^*}_{B_{Z^*}\frac{\bar h}{h}}\right| < 1$. In what follows, to simplify the formulae, we will omit the upper indices for Hankel and Teoplitz operators and the reader should keep in mind that the characters of the symbols, which would be the products or quotients of character-automorphic functions, could be restored from these expressions, and the characters of the Hardy spaces could be restored from the context. For example, as above, the character of the symbol $\bar h/h$ is the difference of the characters of $\bar h$ and h, i.e. is equal to $-2\kappa_0$ and the character of the symbol $B_{Z^*}\bar h/h$ in the same manner is equal to $\kappa^* - 2\kappa_0$.

5.2.

Proposition 5.1. *Let the Teoplitz operator* $\mathbf{T}_{B_{Z^*}\frac{\bar h}{h}}$ *act on a space* $H^2_{+,\kappa+\kappa_0}$ *and* $\mathbf{T}_{\overline{B_{Z^*}}\frac{h}{\bar h}}$ *be its adjoint acting on a space* $H^2_{+,\kappa-\kappa_0+\kappa^*}$. *Then*

$$\mathcal{M} = \mathbf{T}_{B_{Z^*}\frac{\bar h}{h}}\left(H^2_{+,\kappa+\kappa_0}\right)\bigcap K^{\kappa-\kappa_0+\kappa^*}_{Z^*} \neq \{0\}, \quad dim\mathcal{M} = degU_h$$

$$dimKer\mathbf{T}_{\bar B_{Z^*}\frac{h}{\bar h}} = r^{\kappa}[U_h + J(a) - Z^* - J(Z^*)] = g - degU_h,$$

$$Ker\mathbf{T}_{\bar B_{Z^*}\frac{h}{\bar h}} = \frac{B_{Z^*}}{h}L_\kappa(U_h + J(a) - Z^* - J(Z^*)).$$

Proof. Using the definition of Teoplitz operators it is easy to see that $\mathbf{T}_{\frac{\bar h}{h}} = T_{\bar B_{Z^*}}T_{B_{Z^*}\frac{\bar h}{h}}$. Note also that for $g \in H^2_{+,\nu}$ represented in a form $g = k_{Z^*} + B_{Z^*}f$, where $k_{Z^*} \in K^{\nu}_{Z^*}$, $f \in H^2_{+,\nu-\kappa^*}$,

$$\|T_{\bar B_{Z^*}}g\|_2 = \|P^{\nu-\kappa^*}_{+}\frac{1}{B_{Z^*}}(k_{Z^*} + B_{Z^*}f)\|_2 = \|f\|_2 = dist(g, K^{\nu}_{Z^*})$$

Now, since $|\mathbf{H}_{\frac{\bar h}{h}}| = 1$, due to (8)

$$\inf_{\substack{x\in H^2_{+,\kappa+\kappa_0}\\ \|x\|_2=1}} \|\mathbf{T}_{\frac{\bar h}{h}}x\|_2 = \inf_{\substack{x\in H^2_{+,\kappa+\kappa_0}\\ \|x\|_2=1}} \|\mathbf{T}_{B_{Z^*}\frac{\bar h}{h}}x - P_{K^{\kappa-\kappa_0+\kappa^*}_{Z^*}}\mathbf{T}_{B_{Z^*}\frac{\bar h}{h}}x\|_2 = 0$$

But this means that

$$dist(\mathbf{T}_{B_{Z^*}\frac{\bar h}{h}}\left(H^2_{+,\kappa+\kappa_0}\right), K^{\kappa-\kappa_0+\kappa^*}_{Z^*}) = 0$$

and, since $dimK^{\kappa-\kappa_0+\kappa^*}_{Z^*} = g$,

$$\mathcal{M} = \mathbf{T}_{B_{Z^*}\frac{\bar h}{h}}\left(H^2_{+,\kappa+\kappa_0}\right)\bigcap K^{\kappa-\kappa_0+\kappa^*}_{Z^*} \neq \{0\}.$$

Let us find the dimension of subspace $\mathcal{M}$. To do this note that

$$\mathbf{T}_{B_{Z^*}\frac{\bar h}{h}}\left(H^2_{+,\kappa+\kappa_0}\right) \oplus Ker\mathbf{T}_{\bar B_{Z^*}\frac{h}{\bar h}} = H^2_{+,\kappa-\kappa_0+\kappa^*}$$

and let us first find the dimension of $Ker\mathbf{T}_{\bar{B}_{Z^*}\frac{h}{\bar{h}}}$ in $H^2_{+,\kappa-\kappa_0+\kappa^*}$. If $q \in Ker\mathbf{T}_{\bar{B}_{Z^*}\frac{h}{\bar{h}}}$ then, by inner-outer factorization for character-automorphic functions $q = kv$, where k is character-automorphic outer function and v is character-automorphic inner function. Now, since $\bar{B}_{Z^*}\frac{h}{\bar{h}}q \in \left(H^2_{+,\kappa-\kappa_0+\kappa^*}\right)^{\perp} = H^2_{+,\kappa-\kappa_0+\kappa^*} \oplus \mathfrak{M}_{\kappa-\kappa_0+\kappa^*} = B_{Z^*}H^2_{+,\kappa-\kappa_0}$ and the symbol $\bar{B}_{Z^*}\frac{h}{\bar{h}}$ is unimodular on Γ we see that there exists the character-automorphic inner function v' such that

$$\frac{B_{Z^*}}{b_a}\overline{kv'} = \bar{B}_{Z^*}\frac{h}{\bar{h}}kv,$$

or equivalently

$$\frac{B_{Z^*}}{b_a}\overline{hkv'} = \bar{B}_{Z^*}hkv. \tag{9}$$

Note that the both sides of this equality are summable on Γ and, moreover $\bar{B}_{Z^*}hkv \in H^1_{+,\kappa} + \mathfrak{M}_\kappa$, $\frac{B_{Z^*}}{b_a}\overline{hkv'} \in H^1_{-,\kappa} + \mathfrak{M}_\kappa$. Now, since this function is "orthogonal" to all functions from $H^\infty_{+,\kappa} + H^\infty_{-,\kappa}$, by analog of Riesz brothers theorem (see for example [8]), it belongs to $\mathfrak{M}_\kappa$. If we denote this function by $\varphi_{\mathfrak{M}_\kappa}$, then, by (9),

$$\frac{\varphi_{\mathfrak{M}_\kappa}}{h} \in \mathfrak{M}_{\kappa-\kappa_0} + H^2_{+,\kappa-\kappa_0}, \quad \frac{\varphi_{\mathfrak{M}_\kappa}}{\bar{h}} \in \mathfrak{M}_{\kappa+\kappa_0} + H^2_{-,\kappa+\kappa_0},$$

and, by part (2) of Corollary 4.2 $\varphi_{\mathfrak{M}_\kappa} \in L_\kappa(U_h + J(a) - Z^* - J(Z^*))$.

Conversely, if $\varphi_{\mathfrak{M}_\kappa} \in L_\kappa(U_h + J(a) - Z^* - J(Z^*))$ then, by part (1) of Corollary 4.2, $q = B_{Z^*}\frac{\varphi_{\mathfrak{M}_\kappa}}{h} \in H^2_{\kappa-\kappa_0+\kappa^*}$ and

$$\bar{B}_{Z^*}\frac{h}{\bar{h}}q = \frac{\varphi_{\mathfrak{M}_\kappa}}{\bar{h}} \in \mathfrak{M}_{\kappa+\kappa_0} + H^2_{-,\kappa+\kappa_0}$$

and $q \in Ker\mathbf{T}_{\bar{B}_{Z^*}\frac{h}{\bar{h}}}$. Thus we proved that

$$\begin{aligned} dimKer\mathbf{T}_{\bar{B}_{Z^*}\frac{h}{\bar{h}}} &= r^\kappa[U_h + J(a) - Z^* - J(Z^*)] = g - degU_h, \\ Ker\mathbf{T}_{\bar{B}_{Z^*}\frac{h}{\bar{h}}} &= \frac{B_{Z^*}}{h}L_\kappa(U_h + J(a) - Z^* - J(Z^*)). \end{aligned} \tag{10}$$

Now for any $q \in Ker\mathbf{T}_{\bar{B}_{Z^*}\frac{h}{\bar{h}}}$ let $q = k_q + B_{Z^*}f_q$, where $k_q = P_{K^{\kappa-\kappa_0+\kappa^*}_{Z^*}}q$, $f \in H^2_{+,\kappa-\kappa_0}$, then $\mathcal{M} = \{g \in K^{\kappa-\kappa_0+\kappa^*}_{Z^*} : 0 =< g, q >=< k_q, g >, \forall q \in Ker\mathbf{T}_{\bar{B}_{Z^*}\frac{h}{\bar{h}}}\}$. Note, that for any such q, $k_q \neq 0$. The last fact follows from (10), since otherwise the expression

$$q = B_{Z^*}\frac{\varphi_{\mathfrak{M}_\kappa}}{h} = B_{Z^*}f_q, \quad \varphi_{\mathfrak{M}_\kappa} \in L_\kappa(U_h + J(a) - Z^* - J(Z^*)) \subset \mathfrak{M}_\kappa$$

would imply that $\varphi_{\mathfrak{M}_\kappa} = hf_q \in H^1_{+,\kappa}$.

This means that

$$dimP_{K^{\kappa-\kappa_0+\kappa^*}_{Z^*}}Ker\mathbf{T}_{\bar{B}_{Z^*}\frac{h}{\bar{h}}} = dim \bigvee_{q \in Ker\mathbf{T}_{\bar{B}_{Z^*}\frac{h}{\bar{h}}}} \{k_q\} = g - degU_h.$$

Since $\mathcal{M} = K^{\kappa-\kappa_0+\kappa^*}_{Z^*} \ominus Ker\mathbf{T}_{\bar{B}_{Z^*}\frac{h}{\bar{h}}}$, we proved that $dim\mathcal{M} = deg(U_h)$. ■

Now let us consider all possible combinations of $deg(U_h)$ distinct points from $\{z^*_1, \ldots, z^*_g\}$ or, equivalently the set $\mathbf{Z}$ of all positive divisors Z of degree $degU_h$, such that $Z < Z^*$. For each such divisor Z let us consider the finite Blaschke product B_Z, for which Z is a divisor of zeros.

Proposition 5.2. *There exists at least one divisor $Z \in \mathbf{Z}$, for which the norm of Hankel operator on $H^2_{+,\kappa+\kappa_0}$ with symbol $B_Z\frac{\bar{h}}{h}$ is strictly less then* 1*, i.e.* $|\mathbf{H}_{B_Z\frac{\bar{h}}{h}}| < 1$.

Proof. Suppose on contrary that for all such divisors Z, $|\mathbf{H}_{B_Z\frac{\bar{h}}{h}}| = 1$. Then the same reasoning as above in the proof of Proposition 5.1 gives us that for the complementary divisor $Z^* - Z$ of degree $g - degU_h$ (for which $B_{Z^*} = B_Z B_{Z^*-Z}$), $\mathbf{T}_{B_Z\frac{\bar{h}}{h}} = \mathbf{T}_{\bar{B}_{Z^*-Z}}\mathbf{T}_{B_{Z^*}\frac{\bar{h}}{h}}$ and, hence

$$\mathbf{T}_{B_{Z^*}\frac{\bar{h}}{h}}\left(H^2_{+,\kappa+\kappa_0}\right)\bigcap K^{\kappa-\kappa_0+\kappa^*}_{Z^*-Z} \neq \{0\}, \tag{11}$$

where $K^{\kappa-\kappa_0+\kappa^*}_{Z^*-Z} \subset K^{\kappa-\kappa_0+\kappa^*}_{Z^*}$, $dimK^{\kappa-\kappa_0+\kappa^*}_{Z^*-Z} = g - degU_h$, is spanned by the reproducing kernels at points of $Z^* - Z$. But we already know that

$$\mathbf{T}_{B_{Z^*}\frac{\bar{h}}{h}}\left(H^2_{+,\kappa+\kappa_0}\right)\bigcap K^{\kappa-\kappa_0+\kappa^*}_{Z^*} = \mathcal{M},\ dim\mathcal{M} = degU_h.$$

Therefore our assumption implies that for all divisors Z, $degZ = degU_h, 0 < Z < Z^*$, $K^{\kappa-\kappa_0+\kappa^*}_{Z^*-Z} \cap \mathcal{M} \neq \{0\}$.

But this is impossible due to the following simple Lemma

Lemma 5.1. *Let $\mathcal{H}$ be n-dimensional vector space with the basis $B = \{v_1, \dots, v_n\}$ and let $\mathbf{K}_k$, $k < n$, be the family of all k-dimensional subspaces of $\mathcal{H}$, each of which is spanned by some k vectors from B. then for any $n-k$-dimensional subspace $\mathcal{M} \subset \mathcal{H}$, there exists at least one subspace $\mathcal{K} \in \mathbf{K}_k$, such that $\mathcal{M} \cap \mathcal{K} = \{0\}$.*

Proof. Without loss of generality we can assume that $\mathcal{H} = \mathbb{R}^n$ and B is a standard basis in $\mathbb{R}^n$. Let $A = \{a_{jl}\}^{n\ \ ,n-k}_{j=1,l=1}$ be $n \times (n-k)$ matrix, whose columns form the basis in $\mathcal{M}$. Clearly $rankA = n - k$. Let $1 \le j_1 < j_2 < \dots < j_{n-k} \le n$ be the numbers of $n - k$ linear independent rows of A. Then the k-dimensional space $\mathcal{K}$ from $\mathbf{K}_k$ which is defined by the condition that all its vectors have zeros at $j_1, j_2, \dots, j_{n-k}$ coordinate positions is the desired one. Indeed, if $\mathcal{M} \cap \mathcal{K} \neq= \{0\}$ then there exists a vector $\vec{c} \in \mathbb{R}^{n-k}$ such that for nondegenerate $(n-k) \times (n-k)$ matrix $A_{n-k} = \{a_{j_s,l}\}^{n-k}_{s,l=1}$, $A_{n-k}\vec{c} = \mathbf{0}$. ■

All we need now is to apply this Lemma to the case $\mathcal{H} = K^{\kappa-\kappa_0+\kappa^*}_{Z^*}, n = g,\ k = g - degU_h$ and B is a basis of reproducing kernels at the points of Z^*. ■

Renumerating, if necessary, the zeroes on Ω_+ of harmonic measure $d\eta_a$ we can assume that for the divisor $Z_h = z^*_1 + z^*_2 + \dots + z^*_{degU_h}$ the norm of Hankel operator on $H^2_{+,\kappa+\kappa_0}$ with symbol $B_{Z_h}\frac{\bar{h}}{h}$ is lees then 1,

$$|\mathbf{H}_{B_{Z_h}\frac{\bar{h}}{h}}| < 1 \quad and \quad \mathcal{M} \cap K^{\kappa+\kappa_0}_{Z^*-Z_h} = \{0\} \tag{11}$$

Proposition 5.3. *For the divisor Z_h from the Proposition 5.2 the norm of Hankel operator $\mathbf{H}_{\overline{B_{Z_h}}\frac{h}{\bar{h}}}$, acting on a space $H^2_{+,\kappa-\kappa_0+\kappa_h}$ is strictly less then one.*

Proof. Together with (8), (11) immediately implies that $Ker\mathbf{T}_{B_{Z_h}\frac{\bar{h}}{h}} = \{0\}$ and hence

$$\mathbf{T}_{\bar{B}_{Z_h}\frac{h}{\bar{h}}}\left(H^2_{+,\kappa-\kappa_0+\kappa_h}\right) = H^2_{+,\kappa+\kappa_0},$$

where $\kappa_h = \Re\varphi(Z_h - (degU_h)b_0)$ is a character of finite Blaschke product B_{Z_h}.

Let us show that $Ker\mathbf{T}_{\bar{B}_{Z_h}\frac{h}{\bar{h}}} = 0$ on $H^2_{+,\kappa-\kappa_0+\kappa_h}$.

Note that $K^{\kappa-\kappa_0+\kappa^*}_{Z^*} = K^{\kappa-\kappa_0+\kappa^*}_{Z^*-Z_h} \oplus B_{Z^*-Z_h}K^{\kappa-\kappa_0+\kappa_h}_{Z_h}$. Here $K^{\kappa-\kappa_0+\kappa_h}_{Z_h} \subset K^{\kappa-\kappa_0+\kappa_h}_{Z^*}$, $dimK^{\kappa-\kappa_0+\kappa_h}_{Z_h} = degU_h$, is spanned by the reproducing kernels at points of Z_h. Now ,since $dim\mathcal{M} = dimK^{\kappa-\kappa_0+\kappa_h}_{Z_h} = degU_h$ using second part of (11) we see that

$$P_{B_{Z^*-Z_h}K^{\kappa-\kappa_0+\kappa_h}_{Z_h}}\mathcal{M} = B_{Z^*-Z_h}K^{\kappa-\kappa_0+\kappa_h}_{Z_h}.$$

Indeed, otherwise we can find nonzero element from $\mathcal{M} \cap K^{\kappa-\kappa_0+\kappa^*}_{Z^*-Z_h}$. Using the representation $\mathbf{T}_{B_{Z_h}\frac{\bar{h}}{h}} = \mathbf{T}_{\overline{B}_{Z^*-Z_h}}\mathbf{T}_{B_{Z^*}\frac{\bar{h}}{h}}$ we see that

$$P^{\kappa-\kappa_0+\kappa_h}_{+}\bar{B}_{Z^*-Z_h}\mathcal{M} = P_{K^{\kappa-\kappa_0+\kappa_h}_{Z_h}}\bar{B}_{Z^*-Z_h}\mathcal{M} = K^{\kappa-\kappa_0+\kappa_h}_{Z_h} \subset \mathbf{T}_{B_{Z_h}\frac{\bar{h}}{h}}\left(H^2_{+,\kappa+\kappa_0}\right).$$

Hence $Ker\mathbf{T}_{\bar{B}_{Z_h}\frac{h}{\bar{h}}} \perp K^{\kappa-\kappa_0+\kappa_h}_{Z_h}$. Therefore any element from $Ker\mathbf{T}_{\bar{B}_{Z_h}\frac{h}{\bar{h}}}$ has a form $B_{Z_h}f$, with $f \in H^2_{+,\kappa-\kappa_0}$. But this means that $f \in Ker\mathbf{T}_{\frac{h}{\bar{h}}}$ and so $fh/\bar{h} \in B_{Z^*}H^2_{-,\kappa+\kappa_0-\kappa^*}$. But then, multiplying the both side of this inclusion by $\bar{h}$, we obtain that fh simultaneously belongs to $H^1_{+,\kappa}$ and $H^1_{-,\kappa} + \mathfrak{M}_\kappa$. Thus $f = 0$ and $Ker\mathbf{T}_{\bar{B}_{Z_h}\frac{h}{\bar{h}}} = \{0\}$.

Now, since

$$\mathbf{T}_{\bar{B}_{Z_h}}\frac{h}{\bar{h}}\left(H^2_{+,\kappa-\kappa_0+\kappa_h}\right) = H^2_{+,\kappa+\kappa_0} \quad and \quad Ker\mathbf{T}_{\bar{B}_{Z_h}\frac{h}{\bar{h}}} = \{0\}, \tag{12}$$

by Banach's theorem on inverse operator, $\mathbf{T}_{\bar{B}_{Z_h}\frac{h}{\bar{h}}}$ has a bounded inverse operator. But then

$$\inf_{\substack{f\in H^2_{+,\kappa-\kappa_0+\kappa_h}\\ \|f\|_2=1}} \|\mathbf{T}_{\bar{B}_{Z_h}\frac{h}{\bar{h}}}f\|_2 > 0,$$

and, finally, using (8) we see that

$$|\mathbf{H}_{\bar{B}_{Z_h}\frac{h}{\bar{h}}}| < 1. \tag{13}$$

■

5.3.

Proposition 5.4. *Let U_h be the divisor from Proposition 4.2, then for $k = degU_h|_u$, $u \in \Gamma$, the expression $h/(z-u)^k$ is locally square summable at u.*

Proof. Due to (12) $Ker\mathbf{T}_{\bar{b}_a\bar{B}_{Z_h}\frac{h}{\bar{h}}} \neq \{0\}$, and let $q \in H^2_{+,\kappa-\kappa_0+\kappa_h}$ be nonzero function from this kernel with inner-outer factorization $q = kv$, where k is an outer function and v is an inner function. Now repeating the above arguments used to get (9), we have

$$\frac{B_{Z^*}}{b_a}\overline{kv'} = \frac{1}{b_aB_{Z_h}}\frac{h}{\bar{h}}kv,$$

where v' is an inner function. This expression implies that

$$B_{Z^*}\overline{hkv'} = \frac{hkv}{B_{Z_h}}. \tag{14}$$

In this case the left-hand side of this equality belongs to $H^1_{-,\kappa} + \mathfrak{M}_\kappa + k_a^\kappa$ and the right-hand side belongs to $H^1_{+,\kappa} + \mathfrak{M}_\kappa$. Again the application of analog of Riesz brother's theorem gives us that this function belongs to $\mathfrak{M}_\kappa + k_a^\kappa$. From (14) we observe that

$$\psi = \frac{hkv}{B_{Z_h}} \in L_\kappa((v)_+ + J((v')_+) + U_h - Z_h - J(Z^*)),$$

here $(v)_+$ and $(v')_+$ are divisors of zeros of inner functions v and v'. Clearly $deg(v)_+ + deg(v')_+ \le g$ and all other zeros of ψ lies on Γ. Thus $(\psi) = (v)_+ + J((v')_+) + U_h - Z_h - J(Z^*) + U_\psi$, where U_ψ nonnegative divisor contained totally in Γ, $degU_\psi + deg(v)_+ + deg(v')_+ = g$. By Lemma 3.2 we can find the positive divisor U^* of degree g contained totally in Γ, such that there exists a character-automorphic function f with divisor on $(f) = U^* - \{U_\psi + (v)_+ + J((v')_+)\}$. Then since $1/h$ is locally square summable at all points on $\Gamma \setminus U_h$, the function $\tilde{\psi} = B_{Z_h} f\psi$ is from the space $L_\nu(U_h + U^* - J(Z_h) - J(Z^*))$ for some character ν, $\bar{\tilde{\psi}} = B_{Z^*}B_{Z_h}\psi$ and therefore, by (14) and the Proposition 4.2, $h^* = \tilde{\psi}/h = B_{Z_h} f\psi/h = kvf = \frac{B_{Z_h}\psi}{h} f$ is square summable on Γ and belongs to $H^2_{+,\nu-\kappa_0}$. The function h^* is outer, since it has no zeros in Ω_+ and its product with the outer function h is an outer function. Thus

$$\frac{\overline{h^*}}{h^*} B_{Z^*} = \bar{B}_{Z_h}\frac{h}{\bar{h}}, \tag{15}$$

and by (13)

$$|\mathbf{H}_{\frac{\overline{h^*}}{h^*}B_{Z^*}}| < 1. \tag{15$'$}$$

The last inequality, as we already checked, is equivalent to the nonzero angle condition in $L^2(|h^*|^2 d\eta_a, \Gamma)$ between the subspaces $Clos_{L^2(|h^*|^2 d\eta_a)}\mathcal{B}_-^{\kappa-\nu+\kappa_h}$ and $Clos_{L^2(|h^*|^2 d\eta_a)}\mathcal{B}_+^{\kappa-\nu+\kappa_h}$. Using Proposition 4.2 we get that there exists the positive divisor $U = U_{h^*}$ of degree less or equal to g, contained totally in Γ, such that $1/h^*$ is locally square summable at all points $\Gamma \setminus U_{h^*}$, $1/h^*$ is not square summable at the points of U_{h^*}, but the expression $(z-u)^k/h^*$ is still square summable for $k = gedU_{h^*}|_u$, and k is the minimal nonnegative integer with this property. Now, since $\tilde{\psi}/h^* = h$ is square summable on Γ, by Remark 4.2, $0 \le U_{h^*} \le U_h + U^*$.

Let us show that $U_{h^*} = U^*$. Since by the definition of divisors U_h and U_{h^*} the numbers $k_{h^*} = gedU_{h^*}|_u$ and $k_h = gedU_h|_u$ are the smallest nonnegative integers for which respectively the expressions $(z-u)^{k_{h^*}}/h^*$ and $(z-u)^{k_h}/h$ are locally square summable at u and since, by the definitions of function $\tilde{\psi} = hh^*$ and divisor U^*, $k_{h^*}, k_h \le k_u = deg(U_h + U^*)|_u = deg(\tilde{\psi})|_u$, using Holder inequality for sufficiently small interval X_u, containing the point u, we have

$$\infty = \int_{X_u} \frac{1}{|z-u|} d\eta_a(z) = \int_{X_u} \frac{h}{|z-u|^{k_u - k_{h^*}}} \frac{|z-u|}{h}^{k_u - k_{h^*} - 1} d\eta_a(z) \le$$

$$C_u \left\{ \int_{X_u} \left| \frac{(z-u)^{k_{h^*}}}{h^*} \right|^2 d\eta_a(z) \right\}^{1/2} \left\{ \int_{X_u} \left| \frac{(z-u)^{k_u - k_{h^*}}}{h(z-u)} \right|^2 d\eta_a(z) \right\}^{1/2},$$

where C_u is a constant depending on the derivative of $\tilde{\psi}$ of order k_u at u. This means that $(z-u)^{k_u-k_{h^*}-1}/h$ is not locally square summable at u, and therefore $k_h \geq k_u - k_{h^*}$ or, equivalently, $k_h + k_{h^*} \geq k_u$. Summing up the last inequality over all $u \in \Gamma, deg(\tilde{\varphi})|_u > 0$, we get

$$degU_h + degU_{h^*} \geq degU_h + g.$$

But we know that $degU_{h^*} \leq g$ and so $degU_{h^*} = g$, which means that we should have equality $degU_h + degU_{h^*} = degU_h + g$ and, therefore moreover we should have equality $k_h + k_{h^*} = k_u$ at every $u \in \Gamma$. But the last quality is nothing but

$$degU_h|_u + degU_{h^*}|_u = degU_h|_u + degU^*|_u,$$

thus $U^* = U_{h^*}$.

Moreover the above reasoning has shown us that in addition to the statement of Proposition 4.2 not only $(z-u)^{k_h}/h$ is locally square summable at $u \in \Gamma$, but also $h/(z-u)^{k_h} \sim C_u(z-u)^{k_{h^*}}/h^*$ is locally square summable at u. ■

Corollary 5.1. *For any character-automorphic meromorphic on $\hat{\Omega}$ function f, whose divisor on Γ is exactly equal to U_h, the both functions h/f and f/h are square summable on Γ.*

6. Necessary and sufficient conditions on the weight.

6.1. Now we are ready to obtain the necessary condition on the weight $w = |h|^2$. Let U_h be the divisor of "bad behaviour" of function h and Z_h the corresponding subdivisor of Z^* of degree equal to $degU_h$, discussed in Propositions 5.2 and 5.3. Then, by Lemma 3.2 we can find a divisor Q of degree g, contained totally inside Ω_+, such that for certain $\nu \in \mathbb{R}^g/\mathbb{Z}^g$ there exists ν-automorphic meromorphic on $\hat{\Omega}$ function ϕ with the divisor $(\phi) = U_h + J(Z^* - Z_h) - J(Q)$. Note that ϕ itself is an outer function from $H^\infty_{+,\nu}$ and, by Corollary 5.1, h/ϕ and ϕ/h are both outer functions from $H^2_{+,\kappa_0-\nu}$ and $H^2_{+,\nu-\kappa_0}$ respectively. Recall that by B_Q we denote the finite character-automorphic Blaschke product with the divisor of zeroes equal to Q and corresponding to the character $\kappa_Q = 2\Re\varphi(q - gb_0)$.

Proposition 6.1. *If the angle in $L_2(|h|^2 d\eta_a)$ between D^κ_- and D^κ_+ is nonzero then for any character-automorphic meromorphic on $\hat{\Omega}$ function ψ with divisor on Γ exactly equal to U_h the weight $|h/\psi|^2$ should satisfy the $A_2'(\Gamma)$ Muckenhoupt condition, i.e.*

$$\sup_{\lambda \in \Omega_+} \int_\Gamma \left|\frac{h}{\psi}\right|^2 d\eta_\lambda \cdot \int_\Gamma \left|\frac{\psi}{h}\right|^2 d\eta_\lambda < \infty. \tag{16}$$

Proof. First we note that if the $A_2'(\Gamma)$ condition holds for one certain ψ with $(\psi)|_\Gamma = U_h$, then it holds for all such functions. Indeed, by [7, Theorem 4.2] $A_2'(\Gamma)$ condition on the weight w is equivalent to $A_2(\Gamma)$ condition on w which is the direct analog of the classical Muckenhoupt condition for the unit circle:

$$\sup_I \frac{1}{m(I)} \int_I w d\eta_a \cdot \frac{1}{m(I)} \int_I \frac{1}{w} d\eta_a < \infty, \tag{$A_2(\Gamma)$}$$

where I is an interval on Γ of harmonic measure $\int_I d\eta_a = m(I)$. Now if ψ and ψ' are two character- automorphic meromorphic on $\hat\Omega$ functions, $(\psi)|_\Gamma = (\psi')|_\Gamma = U_h$, then $|\psi'/\psi|$ is continuous on Γ function, bounded away from zero and infinity on Γ, and thus the weights $|h/\psi|^2$ and $|h/\psi'|^2 = |h/\psi|^2|\psi/\psi'|^2$ satisfy or do not satisfy $A_2(\Gamma)$ condition simultaneously. Therefore we will show that (16) holds for the weight $|h/\phi|^2$, where ϕ is the function introduced at the beginning of Sec.6.1. We already know that if the angle under consideration is nonzero then (11) holds. In the same way as it was done in Sec.5, it can be checked easily that (11) is equivalent to nonzero angle condition in $L_2(wd\eta_a)$, $w = |h|^2$, between the subspaces $Clos_{wd\eta_a} H^\infty_{+,\kappa}$ and $Clos_{wd\eta_a} B_{Z^*-Z_h} H^\infty_{-,\kappa-\kappa^*+\kappa_h}$. Indeed,

$$1 > |\mathbf{H}_{\bar B_{Z_h}\frac{h}{\bar h}}| = \sup_{\substack{f_+\in H^2_{+,\kappa+\kappa_0}\\ f_-\in H^2_{-,\kappa-\kappa_0+\kappa_h-\kappa^*}}} \frac{|< B_{Z_h}\frac{\bar h}{h} f_+, B_{Z^*} f_- >|}{\|f_+\|_2\cdot\|f_-\|_2} =$$

$$\sup_{\substack{f_+\in H^2_{+,\kappa+\kappa_0}\\ f_-\in H^2_{-,\kappa-\kappa_0+\kappa_h-\kappa^*}}} \frac{|< B_{Z_h}\frac{f_+}{h} h, B_{Z^*}\frac{f_-}{\bar h} h >|}{\|f_+\|_2\cdot\|f_-\|_2} = \sup_{\substack{f_+\in H^\infty_{+,\kappa}\\ f_-\in H^\infty_{-,\kappa+\kappa_h-\kappa^*}}} \frac{|< f_+ h, B_{Z^*-Z_h} f_- h >|}{\|f_+ h\|_2\cdot\|f_- h\|_2}$$

$$= \cos(Clos_{wd\eta_a} H^\infty_{+,\kappa}, Clos_{wd\eta_a} B_{Z^*-Z_h} H^\infty_{-,\kappa-\kappa^*+\kappa_h}).$$

Now note that the divisor of character-automorphic meromorphic on $\hat\Omega$ function $\bar\phi/\phi = \overline{\phi\circ J}/\phi$ is exactly equal to $(Z^*-Z_h)+J(Q)-J(Z^*-Z_h)-Q$, which in its turn is exactly the divisor of the function $B_{Z^*-Z_h}/B_Q$. Hence $\bar\phi/\phi = B_{Z^*-Z_h}/B_Q$. Clearly $-2\nu = \kappa^*-\kappa_h-\kappa_Q$. Since h/ϕ is an outer function from $H^2_{+,\kappa_0-\nu}$, for any character μ the linear spaces $H^\infty_{+,\mu+\nu}/\phi$ and $H^\infty_{-,\mu-\nu}/\bar\phi$ are contained in $L^2(wd\eta_a)$ and, moreover, are dense in $\mathcal{D}^\mu_+$ and $\mathcal{D}^\mu_-$ respectively. Therefore, we can continue the previous chain of equalities in a following way

$$1 > \cos(Clos_{wd\eta_a} H^\infty_{+,\kappa}, Clos_{wd\eta_a} B_{Z^*-Z_h} H^\infty_{-,\kappa-\kappa^*+\kappa_h}) =$$

$$\sup_{\substack{f_+\in \mathcal{D}^\kappa_+\\ f_-\in \mathcal{D}^{\kappa+\kappa_h-\kappa^*}_-}} \frac{|< f_+ h, B_{Z^*-Z_h} f_- h >|}{\|f_+ h\|_2\cdot\|f_- h\|_2} = \sup_{\substack{f_+\in H^\infty_{+,\kappa+\nu}\\ f_-\in H^\infty_{-,\kappa+\kappa_h-\kappa^*-\nu}}} \frac{|< \frac{f_+}{\phi} h, B_{Z^*-Z_h}\frac{f_-}{\bar\phi} h >|}{\|\frac{f_+}{\phi} h\|_2\cdot\|\frac{f_-}{\bar\phi} h\|_2}$$

$$= \sup_{\substack{f_+\in H^\infty_{+,\kappa+\nu}\\ f_-\in H^\infty_{-,\kappa+\kappa_h-\kappa^*-\nu}}} \frac{|< \frac{h}{\phi} f_+, B_{Z^*-Z_h}\frac{\phi}{\bar\phi} f_- \frac{h}{\phi} >|}{\|f_+\frac{h}{\phi}\|_2\cdot\|f_-\frac{h}{\phi}\|_2} = \sup_{\substack{f_+\in H^\infty_{+,\kappa+\nu}\\ f_-\in H^\infty_{-,\kappa+\kappa_h-\kappa^*-\nu}}} \frac{|< \frac{h}{\phi} f_+, B_Q f_- \frac{h}{\phi} >|}{\|f_+\frac{h}{\phi}\|_2\cdot\|f_-\frac{h}{\phi}\|_2}$$

$$\geq \sup_{\substack{f_+\in H^\infty_{+,\kappa+\nu}\\ f_-\in \bar B_Q H^\infty_{-,\kappa+\nu}}} \frac{|< \frac{h}{\phi} f_+, B_Q f_- \frac{h}{\phi} >|}{\|f_+\frac{h}{\phi}\|_2\cdot\|f_-\frac{h}{\phi}\|_2} = \sup_{\substack{f_+\in H^\infty_{+,\kappa+\nu}\\ f_-\in H^\infty_{-,\kappa+\nu}}} \frac{|< \frac{h}{\phi} f_+, f_- \frac{h}{\phi} >|}{\|f_+\frac{h}{\phi}\|_2\cdot\|f_-\frac{h}{\phi}\|_2}$$

But the right-hand side of the last expression, which, as we now see, is strictly less then 1, is exactly equal to the cosine of an angle in $L_2(|\frac{h}{\phi}|^2 d\eta_a)$ between subspaces $Clos_{L_2(|\frac{h}{\phi}|^2 d\eta_a)} H^\infty_{-,\kappa+\nu}$ and $Clos_{L_2(|\frac{h}{\phi}|^2 d\eta_a)} H^\infty_{+,\kappa+\nu}$. Thus this angle is nonzero. Now we can apply Proposition 4.2 to the weight $|\frac{h}{\phi}|^2$. Since, as we remarked at the beginning of this section, by Corollary 5.1, both functions h/ϕ and ϕ/h are square summable, the divisor of "bad behaviour" of this weight is zero, Therefore case (1) of the alternative from proposition 4.2 takes place and this weight satisfies the $A_2'(\Gamma)$ Muckenhoupt condition. ■

6.2. Note that (see (7′) and statement (1) of corollary 4.2) if the weight $|h|^2$ provides the nonzero angle between subspaces $\mathcal{D}_-^\kappa$ and $\mathcal{D}_+^\kappa$, then necessarily (7′) holds. This condition is equivalent to the following one

$$r^{-\kappa}[a - U_h] = 0, \tag{17}$$

since dividing any differential from $\mathfrak{N}_{-\kappa}(Z^* + J(Z^*) - U_h - J(a))$ by harmonic measure $d\eta_a$,which is meromorphic ($\mathbf{0}$-automorphic) differential on $\hat{\Omega}_+$ with the divisor $(d\eta_a) = Z^* + J(Z^*) - a - J(a)$, we will get the function from $L_{-\kappa}(a - U_h)$.

We will show that conditions (16) and (17) together are also sufficient for our problem.

Proposition 6.2. *If an outer function $h \in H^2_{+,\kappa_0}$ is such that there exists a character-automorphic meromorphic on $\hat{\Omega}$ function ψ with nonnegative divisor $U = (\psi)|_\Gamma$ on Γ, $0 \le degU \le g$ and*

$$r^{-\kappa}[a - U] = 0, \quad \kappa \in \mathbb{R}^g/\mathbb{Z}^g,$$
$$\sup_{\lambda \in \Omega_+} \int_\Gamma \left|\frac{h}{\psi}\right|^2 d\eta_\lambda \cdot \int_\Gamma \left|\frac{\psi}{h}\right|^2 d\eta_\lambda < \infty, \tag{18}$$

then the angle in $L_2(wd\eta_a), w = |h|^2$, between the subspaces $\mathcal{D}_-^\kappa = Clos_{L_2(wd\eta_a)} H^\infty_{-,\kappa}$ and $\mathcal{D}_+^\kappa = Clos_{L_2(wd\eta_a)} H^\infty_{+,\kappa}$ is nonzero.

Proof.

Note first that, just by definition of Hankel operator, the nonzero angle condition we want to prove is also equivalent to the nonzero angle in $L_2(d\eta_a)$ between subspaces $H^2_{-,\kappa-\kappa_0}$ and $\frac{\bar{h}}{h} H^2_{+,\kappa+\kappa_0}$. Using Lemma 3.2, in the same way as it was done in the proof of Proposition 6.1 we can find ν-automorphic meromorphic on Γ function ϕ, such that $(\phi) = U + J(Z) - J(Q)$, where Z is any fixed nonnegative subdivisor of critical Green's divisor Z^* of degree $degZ = g - degU$ and Q is a positive divisor of degree g contained totally in Ω_+. Clearly ϕ is an outer function from $H^2_{+,\nu}$, $\bar{\phi}/\phi = B_Z/B_Q$, $(\phi)|_\Gamma = (\psi)|_\Gamma = U$ and again, by the reasoning from the proof of Proposition 5.1, the weight $|h/\phi|^2$ satisfies the $A_2'(\Gamma)$ Muckenhoupt condition. Let $\tilde{h} = h/\phi$, then $\tilde{h}$ is an outer function from $H^2_{+,\kappa_0-\nu}$ and the norm of the hankel operator $\mathbf{H}_{\frac{\bar{\tilde{h}}}{\tilde{h}}}$ is strictly less then one on $H^2_{+,\mu}$ for any character μ. In particular $|\mathbf{H}_{\frac{\bar{\tilde{h}}}{\tilde{h}}}| < 1$ on $H^2_{+,\kappa+\kappa_0}$, which means that the angle in $L_2(d\eta_a)$ between the subspaces $H^2_{-,\kappa-\kappa_0} + \mathfrak{M}_{\kappa-\kappa_0}$ and $\frac{\bar{\tilde{h}}}{\tilde{h}} H^2_{+,\kappa+\kappa_0}$ is nonzero. But $\bar{h}/h = \{\bar{\tilde{h}}/\tilde{h}\}\{\bar{\phi}/\phi\} = \{B_Z \bar{\tilde{h}}\}/\{B_Q \tilde{h}\}$. thus, in terms of $\tilde{h}$, we want to show that the angle in $L_2(d\eta_a)$ between the subspaces $H^2_{-,\kappa-\kappa_0}$ and $\dfrac{B_Z}{B_Q}\dfrac{\bar{\tilde{h}}}{\tilde{h}} H^2_{+,\kappa+\kappa_0-\kappa_Z+\kappa_Q}$ is nonzero. But $H^2_{+,\kappa+\kappa_0-\kappa_Z+\kappa_Q} = K_Q^{\kappa+\kappa_0-\kappa_Z+\kappa_Q} \oplus B_Q H^2_{+,\kappa+\kappa_0-\kappa_Z}$ and therefore $\dfrac{B_Z}{B_Q} H^2_{+,\kappa+\kappa_0-\kappa_Z+\kappa_Q} = \dfrac{B_Z}{B_Q} K_Q^{\kappa+\kappa_0-\kappa_Z+\kappa_Q} \oplus B_Z H^2_{+,\kappa+\kappa_0-\kappa_Z}$. But we already know that the angle between $H^2_{-,\kappa-\kappa_0}$ and $B_Z \dfrac{\bar{\tilde{h}}}{\bar{\tilde{h}}} H^2_{+,\kappa+\kappa_0-\kappa_Z}$ is nonzero and the sum of these subspaces $H^2_{-,\kappa-\kappa_0} \dot{+} B_Z \dfrac{\bar{\tilde{h}}}{\tilde{h}} H^2_{+,\kappa+\kappa_0-\kappa_Z}$ is direct. Thus, if the angle, we are interested in, is zero due to the finite dimension of the subspace $\dfrac{B_Z}{B_Q} K_Q^{\kappa+\kappa_0-\kappa_Z+\kappa_Q}$, there exist the functions $f_- \in H^2_{-,\kappa-\kappa_0}$ and $f_+ \in H^2_{+,\kappa+\kappa_0-\kappa_Z+\kappa_Q}$, such that $f_- = \dfrac{B_Z}{B_Q} \frac{\bar{\tilde{h}}}{\tilde{h}} f_+$. Therefore

$$\frac{f_-}{\bar{\tilde{h}}} = \frac{B_Z f_+}{\tilde{h} B_Q}. \tag{19}$$

But since the weight $|\tilde{h}|^2$ satisfy the Muckenhoupt condition for any character μ the "Riesz" projection P_+^μ is a bounded operator in $L_2(|\tilde{h}|^2 d\eta_a)$ (can be extended as a bounded operator from the dense subset in $L_2(d\eta_a) \cap L_2(|\tilde{h}|^2 d\eta_a)$ to the whole space $L_2(|\tilde{h}|^2 d\eta_a)$). Therefore the operator $P^\mu_{K_Q} = P_+^\mu - B_Q P_+^{\mu-\kappa_Q} \bar{B}_Q$ is also bounded in $L_2(|\tilde{h}|^2 d\eta_a)$. Thus for any $\mu \in \mathbb{R}^g/\mathbb{Z}^g$,

$$Clos_{L_2(|\tilde{h}|^2 d\eta_a)} H^\infty_{+,\mu} = \frac{H^2_{+,\mu+\kappa_0-\nu}}{\tilde{h}} = K_Q^\mu \dot{+} B_Q \frac{H^2_{+,\mu+\kappa_0-\nu-\kappa_Q}}{\tilde{h}}, \tag{20}$$

where the right-hand sum in (20) is direct.

Now applying the decomposition (20) to $\mu = \kappa - \kappa_Z + \kappa_Q + \nu$ and function $f_+/\tilde{h}$ from (19), we get that

$$\frac{f_+}{\tilde{h}} = k_Q + B_Q \frac{\tilde{f}_+}{\tilde{h}},$$

where $k_Q \in K_Q^{\kappa+\nu-\kappa_Z+\kappa_Q}$ and $\tilde{f}_+ \in H^2_{+,\kappa+\kappa_0-\kappa_Z}$. Now we can rewrite (19) in the following form

$$\frac{B_Z}{B_Q} k_Q = \frac{f_-}{\bar{\tilde{h}}} - B_Z \frac{\tilde{f}_+}{\tilde{h}}. \tag{21}$$

Note that $\frac{f_-}{\bar{\tilde{h}}} \in H^1_{-,\kappa+\nu}$, $\frac{\tilde{f}_+}{\tilde{h}} \in H^1_{+,\kappa+\nu-\kappa_Z}$ and $\dfrac{B_Z}{B_Q} k_Q \in H^2_{-,\kappa+\nu} + \mathfrak{M}_{\kappa+\nu} + K_Z^{\kappa+\nu}$. Hence $\dfrac{B_Z}{B_Q} k_Q$ is "orthogonal" to $B_Z H^\infty_{+,\kappa+\nu-\kappa_Z}$ and, by analog of Riesz brothers theorem, (21) implies that $\tilde{f}_+ = 0$. This means that $\frac{B_Z}{B_Q} k_Q = \frac{f_-}{\bar{\tilde{h}}} \in H^2_{-,\kappa+\nu}$. But from [8] we know that k_Q is a character-automorphic meromorphic on $\hat{\Omega}$ function with the divisor $(k_Q) \geq J(a) - J(Q) - J(Z^*)$. Hence $(\frac{B_Z}{B_Q} k_Q) \geq J(a) + Z - Q - J(Z^*) - J(Z)$ and, moreover, since $\frac{B_Z}{B_Q} k_Q \in H^2_{-,\kappa+\nu}$,

$$\left(\frac{B_Z}{B_Q} k_Q\right) \geq (a) + Z - Q.$$

Hence, multiplying $\frac{B_Z}{B_Q}k_Q$ by $\frac{1}{\phi}$, we will get the function χ from $L^{\kappa}(J(a)-U)$, but then $\overline{\chi \circ J} \in L^{-\kappa}(a-U)$, which contradicts the first condition from (18). This contradiction proves that the angle under consideration is nonzero. Reversing the above reasoning we see that the fulfilment of the first condition from (18) is essential for the nonzero angle, i.e. if this condition does not hold, even if the weight $|h/\phi|^2$ satisfy the $A_2'(\Gamma)$ condition, the angle is zero. ■

Now combining the Propositions 6.1 and 6.2 we get the following necessary and sufficient condition on the weight:

Theorem 6.1. *Let $d\mu$ be the finite positive measure on Γ then the subspaces $\mathcal{D}_-^{\kappa} = Clos_{L_2(d\mu)}B_-^{\kappa}$ and $\mathcal{D}_+^{\kappa} = Clos_{L_2(d\mu)}B_+^{\kappa}$ are at nonzero angle in $L_2(d\mu)$ if and only if*

1. *Measure $d\mu$ is absolutely continuous with respect to harmonic measure $d\eta_a$, $d\mu = wd\eta_a,\ w \in L_1(\Gamma),\ \log w \in L_1(\Gamma)$*
2. *There exists such character-automorphic meromorphic on $\hat{\Omega}$ function ψ with nonnegative divisor on Γ, $(\psi)|_\Gamma = U_w$, $0 \le degU_w \le g$, that conditions (18) hold.*

Since $A_2'(\Gamma)$ condition is equivalent to $A_2(\Gamma)$ condition and $A_2(\Gamma)$ condition is the local one, i.e. it should be fulfilled at any point of Γ for sufficiently small intervals of Γ containing this point, we can reformulate the previous theorem in a following way:

Theorem 6.2. *Let $d\mu$ be the finite positive measure on Γ then the subspaces $\mathcal{D}_-^{\kappa} = Clos_{L_2(d\mu)}B_-^{\kappa}$ and $\mathcal{D}_+^{\kappa} = Clos_{L_2(d\mu)}B_+^{\kappa}$ are at nonzero angle in $L_2(d\mu)$ if and only if*

1. *Measure $d\mu$ is absolutely continuous with respect to harmonic measure $d\eta_a$, $d\mu = wd\eta_a,\ w \in L_1(\Gamma),\ \log w \in L_1(\Gamma)$*
2. *There exists a nonnegative divisor U_w of points on Γ, $0 \le degU_w \le g$, such that at any point $u \in \Gamma$ the weight $w/|z-u|^{2k}$, $k = degU_w|_u$, is locally the Muckenhoupt one.*

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