Quasiuniform Hyperspaces Of Compact Subsets*

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Abstract

Let $(X,\mathcal{U})$ be a quasiuniform space, $\mathcal{K}(X)$ be the family of all nonempty compact subsets of $(X,\mathcal{U})$. In this paper, the notion of compact symmetry for $(X,\mathcal{U})$ is introduced, and relationships between the Bourbaki quasiuniformity and the Vietoris topology on $\mathcal{K}(X)$ are examined. Furthermore we establish that for a compactly symmetric quasiuniform space $(X,\mathcal{U})$ the Bourbaki quasiuniformity $\mathcal{U}_*$ on $\mathcal{K}(X)$ is complete if and only if $\mathcal{U}$ is complete. This theorem generalizes the well-known Zenor-Morita theorem for uniformisable spaces to the quasiuniform setting.

1 Introduction

In general topology, one basic problem is to construct new spaces from old spaces. Then we study what kinds of properties possessed by the old spaces can be preserved by the new spaces. Hyperspaces are such objects constructed via existing spaces. The theory of hyperspaces begins with the well-known Hausdorff metric. Michael [13] has made major contributions to this field. Nowaday, it not only is a basic structure in general topology but also has many applications in other branches of Mathematics, such as mathematical economics, fractals, Banach space geometry, convex analysis and optimization (see [1], [9] and [17]).

A *quasiuniformity* on a (nonempty) set X is a filter $\mathcal{U}$ on $X \times X$ such that (a) each member of $\mathcal{U}$ contains $\Delta(X)$, where $\Delta(X)$ is the diagonal of X, and (b) if $U \in \mathcal{U}$, then $V^2 \subseteq U$ for some $V \in \mathcal{U}$. The pair $(X,\mathcal{U})$ is called a *quasiuniform space* [7]. By $\tau(\mathcal{U})$,

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we denote the topology induced by $\mathcal{U}$ on X and if $\tau(\mathcal{U}) = \tau$ for some topology τ on X, then $\mathcal{U}$ is said to be *compatible* with τ. If $\mathcal{U}$ is a quasiuniformity on X, then $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ is also a quasiuniformity on X and is called the *conjugate* of $\mathcal{U}$. Furthermore, a quasiuniformity $\mathcal{U}$ is a uniformity provided $\mathcal{U} = \mathcal{U}^{-1}$.

Let $(X,\mathcal{U})$ be a quasiuniform space, and let $\mathcal{P}_0(X)$ (resp. $\mathcal{K}(X)$) be the collection of all nonempty (resp. nonempty compact) subsets of $(X,\mathcal{U})$. On one hand, we can introduce a hyper-topology on $\mathcal{P}_0(X)$ in the following way. For a finite family $\mathcal{G}$ of nonempty subsets of $(X,\mathcal{U})$, define $\mathcal{G}^\wedge = \{A \in \mathcal{P}_0(X) : A \subseteq \cup\mathcal{G}$ and $A \cap G \neq \emptyset$ for each $G \in \mathcal{G}\}$. The *Vietoris topology* (or *finite topology*) on $\mathcal{P}_0(X)$, denoted by $2^{\tau(\mathcal{U})}$, is generated by taking $\{\mathcal{G}^\wedge : \mathcal{G}$ is a finite family of $\tau(\mathcal{U})\}$ as a base [13]. On the other hand, it is also possible to introduce a hyper-quasiuniformity on $\mathcal{P}_0(X)$. For each $U \in \mathcal{U}$, set

$$H(U) = \{(A,B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : B \subseteq U(A) \text{ and } A \subseteq U^{-1}(B)\}.$$

Then $\{H(U) : U \in \mathcal{U}\}$ is a base for a quasiuniformity on $\mathcal{P}_0(X)$, which is called the *Bourbaki quasiuniformity* induced by $\mathcal{U}$ and this quasiuniformity is denoted by $\mathcal{U}_*$ [2], [11]. If $\mathcal{U}$ is a uniformity on X, then $\mathcal{U}_*$ is nothing but the Hausdorff uniformity on $\mathcal{P}_0(X)$ induced by $\mathcal{U}$. Recently, Künzi and Ryser [11] investigated quasiuniform properties of $(\mathcal{P}_0(X),\mathcal{U}_*)$. The main purpose of this paper is to study some properties of the quasiuniform hyperspace $(\mathcal{K}(X),\mathcal{U}_*)$, where we use $\mathcal{U}_*$ in the subspace $\mathcal{K}(X)$ instead of $\mathcal{U}_*|_{\mathcal{K}(X)}$. In Section 2, relationships between the Bourbaki quasiuniformity and the Vietoris topology on $\mathcal{K}(X)$ for a given quasiuniform space $(X,\mathcal{U})$ are examined. In contrast to the uniform case, we observe that the Vietoris topology is always coarser than the topology of the Bourbaki quasiuniformity on $\mathcal{K}(X)$, and they do not coincide in general. To reconcile the Bourbaki quasiuniformity and the Vietoris topology on $\mathcal{K}(X)$, notions of compact symmetry and small-set symmetry for a quasiuniform space are used.

Recall that a filter $\mathcal{F}$ on a uniform space $(X,\mathcal{U})$ is called a *Cauchy filter* if for each $U \in \mathcal{U}$ there exists an $F \in \mathcal{F}$ such that $F \times F \subseteq U$. A uniform space $(X,\mathcal{U})$ is called *complete* if every Cauchy filter on $(X,\mathcal{U})$ converges [6]. Moreover, a Tychonoff space is said to be *Dieudonné complete* if it has a compatible complete uniformity, equivalently, the finest compatible uniformity is complete. In 1970, by using the inverse limit approach, Zenor [18] proved that for a Tychonoff space (X,τ) the hyperspace $(\mathcal{K}(X), 2^\tau)$ is Dieudonné complete if and only if (X, τ) is Dieudonné complete. Morita [14] improved Zenor's result by showing that for a uniform space $(X,\mathcal{U})$ (a) the Hausdorff uniformity $\mathcal{U}_*$ is compatible with the Vietoris topology $2^{\tau(\mathcal{U})}$, and (b) $(\mathcal{K}(X),\mathcal{U}_*)$ is complete if and only if $(X,\mathcal{U})$ is complete.

Motivated by the above work of Zenor and Morita, we study the completeness of the quasiuniform hyperspace $(\mathcal{K}(X),\mathcal{U}_*)$ of a given quasiuniform space $(X,\mathcal{U})$ in Section 3. It is proved that for a compactly symmetric quasiuniform space $(X,\mathcal{U})$ the hyperspace $(\mathcal{K}(X),\mathcal{U}_*)$ is complete if and only if $(X,\mathcal{U})$ is complete. This is a quasiuniform analogy of the Zenor-Morita theorem for uniformisable spaces. We also give an example to show that the condition "compact symmetry" in this result cannot be removed or weakened to "local symmetry". Throughout the paper, $\mathbf{N}$ denotes the set of all natural numbers and $\omega = \mathbf{N} \cup \{0\}$. The topological spaces and quasiuniform spaces are not assumed to satisfy any separation axioms, except those explicitly stated. For other undefined concepts about quasiuniform spaces, see [7].

2 The Bourbaki quasiuniformity on $\mathcal{K}(X)$

In this section, we discuss relationships between the Bourbaki quasiuniformity and the Vietoris topology on $\mathcal{K}(X)$ for a given quasiuniform space $(X,\mathcal{U})$. Some sufficient conditions under which the Bourbaki quasiuniformity is compatible with the Vietoris topology are provided.

Proposition 2.1 *Let $(X,\mathcal{U})$ be a quasiuniform space. Then $2^{\tau(\mathcal{U})} \subseteq \tau(\mathcal{U}_*)$ on $\mathcal{K}(X)$.*

Proof: Let $\mathcal{G}$ be a finite family of open subsets of $(X,\mathcal{U})$ and let $A \in \mathcal{G}^\wedge$. Since A is compact, we can choose a $U_0 \in \mathcal{U}$ such that $U_0(A) \subseteq \cup\mathcal{G}$. Pick a point $x_G \in A \cap G$, and then choose a $U_G \in \mathcal{U}$ such that $U_G(x_G) \subseteq G$ for each $G \in \mathcal{G}$. Let $U = U_0 \cap (\cap_{G\in\mathcal{G}} U_G)$. We wish to show $H(U)(A) \subseteq \mathcal{G}^\wedge$. Let $B \in H(U)(A)$. Obviously, $B \subseteq U(A) \subseteq \cup\mathcal{G}$. On the other hand, for each $G \in \mathcal{G}$, $x_G \in U^{-1}(B)$ implies that $\emptyset \neq B \cap U(x_G) \subseteq B \cap G$. It follows that $B \in \mathcal{G}^\wedge$. Therefore $2^{\tau(\mathcal{U})} \subseteq \tau(\mathcal{U}_*)$ on $\mathcal{K}(X)$. $\square$

Example 2.2 Generally, the Bourbaki quasiuniformity may not be compatible with the Vietoris topology on $\mathcal{K}(X)$. Let $X = \omega$, and define $U_n = \Delta(X) \cup \{(0, x) : x \in X, x \geq n\}$

for each $n \in \mathbf{N}$. Let $\mathcal{U}$ be the quasiuniformity on X generated by $\{U_n : n \in \mathbf{N}\}$. Then $(X,\mathcal{U})$ is a compact quasiuniform space. Since any $2^{\tau(\mathcal{U})}$-neighbourhood of X contains a finite subset of $\mathbf{N}$, $H(U_n)(X)$ cannot include any $2^{\tau(\mathcal{U})}$-neighbourhood of X for each $n \in \mathbf{N}$. It follows that $\tau(\mathcal{U}_*) \neq 2^{\tau(\mathcal{U})}$ on $\mathcal{K}(X)$.

Remark 2.3 Let (X, τ) be a topological space, and set $U_G = G \times G \cup (X - G) \times X$ for any open subset G of (X, τ). Then $\{U_G : G \in \tau\}$ is a subbase for the so-called *Pervin quasiuniformity* $\mathcal{P}$ on (X, τ) [7]. It is shown in [2] and [12] that $\mathcal{P}_*$ is always compatible with 2^τ on $\mathcal{K}(X)$.

As a natural way to reconcile the Bourbaki quasiuniformity and the Vietoris topology on quasiuniform hyperspaces, we will use some weak forms of symmetry in the original spaces. Recall that a quasiuniform space $(X,\mathcal{U})$ is said to be *locally symmetric* [7] if for each $U \in \mathcal{U}$ and each point $x \in X$, there exists a symmetric $V \in \mathcal{U}$ such that $V^2(x) \subseteq U(x)$. A well-known fact is that each locally symmetric quasiuniform space is regular.

Example 2.4 Let $(X,\mathcal{U})$ be the quasiuniform space defined in Example 2.2. As it is shown in Example 2.2, the Bourbaki quasiuniformity $\mathcal{U}_*$ is not compatible with the Vietoris topology $2^{\tau(\mathcal{U})}$ on $\mathcal{K}(X)$. Furthermore, $(X,\mathcal{U})$ is compact. Now we prove that $(X,\mathcal{U})$ is locally symmetric. For each point $x \in X$ and each $n \in \mathbf{N}$, we wish to find a symmetric $U \in \mathcal{U}$ such that $U^2(x) \subseteq U_n(x)$. If $x \neq 0$, then $U_n(x) = \{x\}$. Choose any integer $m > max\{x, n\}$. It is easy to see that $U = U_m \cup U_m^{-1}$ is a symmetric element of $\mathcal{U}$ with $U^2(x) \subseteq U_n(x)$. If $x = 0$, then $U_n(0) = \{0\} \cup \{n, n+1, ...\}$. Choose an integer $m > n$. Again $U = U_m \cup U_m^{-1} \in \mathcal{U}$ is symmetric such that $U^2(0) \subseteq U_n(0)$. Finally we claim that $(\mathcal{K}(X),\mathcal{U}_*)$ is not locally symmetric. In fact, if $(\mathcal{K}(X),\mathcal{U}_*)$ is locally symmetric, then for any $n \in \mathbf{N}$ there exists a symmetric $\mathcal{W} \in \mathcal{U}_*$ such that $\mathcal{W}^2(X) \subseteq H(U_n)(X)$. Therefore, there is an $m \in \mathbf{N}$ such that $H(U_m^{-1})(X) \subseteq \mathcal{W}(X) \subseteq H(U_n)(X)$. Let $A = \{0, 1, 2, ..., m-1\}$. Since $A \in H(U_m^{-1})(X)$ and $A \notin H(U_n)(X)$, we obtain a contradiction.

Since local symmetry is not well-behaved with respect to the hyperspace $(\mathcal{K}(X),\mathcal{U}_*)$ for a quasiuniform space $(X,\mathcal{U})$ as shown in Example 2.4, we will turn our attention to some other weak forms of symmetry in the sequel.

Definition 2.5 A quasiuniform space $(X,\mathcal{U})$ is called *compactly symmetric* if for each $U \in \mathcal{U}$ and each $C \in \mathcal{K}(X)$, there exists a symmetric $V \in \mathcal{U}$ such that $V^2(x) \subseteq U(x)$ whenever $x \in C$.

Let $(X,\mathcal{U})$ be the quasiuniform space defined in Example 2.2. We already knew that it is compact and locally symmetric. Since any compact and compactly symmetric quasiuniform space must be a uniform space, $(X,\mathcal{U})$ is not compactly symmetric. In the following, we provide a compactly symmetric quasiuniform space which is not a uniform space.

Example 2.6 [4] Let $X = \mathbf{R} \setminus \{0\}$ and define $\mathcal{U}_d$ by the quasi-metric

$$d(x,y) = \begin{cases} min\{y-x,1\}, & \text{if } x < 0 < y; \\ 0, & \text{if } x = y; \\ 1, & \text{otherwise.} \end{cases}$$

Then (X,d) is a locally symmetric and quiet [5] quasimetric space such that $\tau(d)$ and $\tau(d^{-1})$ are the discrete topology on X. Thus $(X,\mathcal{U}_d)$ is compactly symmetric. However $\mathcal{U}_d$ is not a uniformity.

Definition 2.7 [8] Let $(X,\mathcal{U})$ be a quasiuniform space and $U \in \mathcal{U}$. A subset B of X is called *U-small* provided that $B \subseteq U(A)$ whenever A is an open subset and $A \cap B \neq \emptyset$. The quasiuniform space $(X,\mathcal{U})$ is called *small-set symmetric* if for each $U \in \mathcal{U}$ the open U-small subsets form a cover of X.

Lemma 2.8 [10] *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $(X,\mathcal{U})$ is small-set symmetric if and only if $\tau(\mathcal{U}^{-1}) \subseteq \tau(\mathcal{U})$.*

Theorem 2.9 *Let $(X,\mathcal{U})$ be a quasiuniform space. If $(X,\mathcal{U})$ is either compactly symmetric or small-set symmetric, then $\mathcal{U}_*$ is compatible with $2^{\tau(\mathcal{U})}$ on $\mathcal{K}(X)$.*

Proof: Following Proposition 2.1, it suffices to show that $\tau(\mathcal{U}_*) \subseteq 2^{\tau(\mathcal{U})}$. To do this, we prove that for each $U \in \mathcal{U}$ and each $A \in \mathcal{K}(X)$, the $\tau(\mathcal{U}_*)$-neighbourhood $H(U)(A)$ of A is also a $2^{\tau(\mathcal{U})}$-neighbourhood of A in the following two cases.

Case 1: Suppose that $(X,\mathcal{U})$ is compactly symmetric. Choose a symmetric $V \in \mathcal{U}$ such that $V^2(x) \subseteq U(x)$ whenever $x \in A$. Since A is compact, there exists a finite subset $A_1 \subseteq A$ such that $A \subseteq \cup_{a \in A_1} int(V(a))$. Let $\mathcal{G} = \{int(V(a)) \colon a \in A_1\}$. Then $\mathcal{G}^\wedge$ is a $2^{\tau(\mathcal{U})}$-open set in $\mathcal{K}(X)$ containing A. For any $B \in \mathcal{G}^\wedge$, $B \subseteq V(A) \subseteq U(A)$. On the other hand, for each point $x \in A$, there exists a point $a \in A_1$ such that $x \in V(a)$. Choose a point $b \in B \cap V(a)$. It follows that $b \in V(a) \subseteq V^2(x) \subseteq U(x)$ which implies that $B \cap U(x) \neq \emptyset$ for each point $x \in A$. Consequently, we have $A \subseteq U^{-1}(B)$ and $\mathcal{G}^\wedge \subseteq H(U)(A)$ which implies that $H(U)(A)$ is a $2^{\tau(\mathcal{U})}$-neighbourhood of A.

Case 2: Suppose that $(X,\mathcal{U})$ is small-set symmetric. Choose a $V \in \mathcal{U}$ such that $V^2 \subseteq U$. By Lemma 2.8, for $x \in X$ we have that $V^{-1}(x)$ is a $\tau(\mathcal{U})$-neighbourhood of x. Since A is compact, there is a finite subset $A_0 \subseteq A$ such that $A \subseteq \cup_{a \in A_0} int(V^{-1}(a))$. Let $\mathcal{G} = \{int(V(A))\} \cup \{int(V(a)) \colon a \in A_0\}$. If $B \in \mathcal{G}^\wedge$, then $B \subseteq V(A) \subseteq U(A)$. On the other hand, for each $B \in \mathcal{G}^\wedge$ and $a \in A_0$, $B \cap V(a) \neq \emptyset$ implies $a \in V^{-1}(B)$. It follows that $A \subseteq V^{-2}(B) \subseteq U^{-1}(B)$. Hence $A \in \mathcal{G}^\wedge \subseteq H(U)(A)$. Therefore, we have $\tau(\mathcal{U}_*) \subseteq 2^{\tau(\mathcal{U})}$ in either case. □

Corollary 2.10 [14] *For a uniform space $(X,\mathcal{U})$, $\mathcal{U}_*$ is compatible with $2^{\tau(\mathcal{U})}$ on $\mathcal{K}(X)$.*

The following result shows that for a quasiuniform space $(X,\mathcal{U})$ both compact symmetry and small-set symmetry are preserved by the hyperspace $(\mathcal{K}(X),\mathcal{U}_*)$.

Theorem 2.11 *Let $(X,\mathcal{U})$ be a quasiuniform space. Then $(\mathcal{K}(X),\mathcal{U}_*)$ is compactly (resp. small-set) symmetric if and only if $(X,\mathcal{U})$ is compactly (resp. small-set) symmetric.*

Proof: The necessity is trivial, since both small-set symmetry and compact symmetry are hereditary properties. To prove the sufficiency, we first assume that $(X,\mathcal{U})$ is compactly symmetric. To show that $(\mathcal{K}(X),\mathcal{U}_*)$ is compactly symmetric, let $\mathcal{A}$ be a nonempty compact subset of $(\mathcal{K}(X),\mathcal{U}_*)$ and $U \in \mathcal{U}$. By Therem 2.9, $\tau(\mathcal{U}_*) = 2^{\tau(\mathcal{U})}$ on $\mathcal{K}(X)$, so $\mathcal{A}$ is compact in $(\mathcal{K}(X), 2^{\tau(\mathcal{U})})$. It follows from [13, Theorem 2.5.2] that $\cup\mathcal{A}$ is compact in $(X,\mathcal{U})$. Since $(X,\mathcal{U})$ is compactly symmetric, there exists a symmetric $V \in \mathcal{U}$ such that $V^2(x) \subseteq U(x)$ whenever $x \in \cup\mathcal{A}$. It follows that $(H(V))^2(A) \subseteq H(U)(A)$ for all $A \in \mathcal{A}$. Therefore $(\mathcal{K}(X),\mathcal{U}_*)$ is compactly symmetric.

Now suppose that $(X,\mathcal{U})$ is small-set symmetric. Let $A \in \mathcal{K}(X)$ and let $U \in \mathcal{U}$. Consider the $\tau(\mathcal{U}_*{}^{-1})$-neighbourhood of A, $(H(U))^{-1}(A) = \{B \in \mathcal{K}(X)\colon A \subseteq U(B), B \subseteq U^{-1}(A)\}$ (note that $(H(U))^{-1} = H(U^{-1})$). Choose a $W \in \mathcal{U}$ such that $W^2 \subseteq U$. Since $(X,\mathcal{U})$ is small-set symmetric, for each $x \in X$ there is $V_x \in \mathcal{U}$ with $V_x^2(x) \subseteq W^{-1}(x)$. Thus $(V_x \cap W)^2(x) \subseteq W^{-1}(x)$ for all $x \in X$. By the compactness of A, there is a finite subset A' of A such that $A \subseteq \cup_{x\in A'}(V_x \cap W)(x)$. Let $V = \cap_{x\in A'}(V_x \cap W)$. We shall show that $H(V)(A) \subseteq H(U^{-1})(A)$. Let $B \in H(V)(A)$. Then $B \subseteq V(A)$ and $A \subseteq V^{-1}(B)$. Take any point $a \in A$. Since $A \subseteq V^{-1}(B)$, $(a,b) \in V$ for some point $b \in B$. Moreover, there is a point $x \in A'$ such that $a \in (V_x \cap W)(x)$. Hence $b \in (V_x \cap W)^2(x) \subseteq W^{-1}(x)$. It follows that $a \in W^2(b) \subseteq U(b)$. We have shown that $A \subseteq U(B)$. Now take any point $b \in B$. Since $B \subseteq V(A)$, $(a,b) \in V$ for some point $a \in A$. There is a point $x \in A'$ such that $a \in (V_x \cap W)(x)$. Hence $b \in (V_x \cap W)^2(x) \subseteq W^{-1}(x)$. Thus $a \in W^2(b) \subseteq U(b)$. We have shown that $B \subseteq U^{-1}(A)$. Therefore $B \in H(U^{-1})(A)$. We conclude that $(\mathcal{K}(X),\mathcal{U}_*)$ is small-set symmetric. $\square$

In the next paragraph we establish a relationship between small-set symmetry and compact symmetry.

Proposition 2.12 *Let $(X,\mathcal{U})$ be a quasiuniform space such that $\tau(\mathcal{U})$ is a k-topology. If $(X,\mathcal{U})$ is compactly symmetric, then it is small-set symmetric.*

Proof: Let $G \in \tau(\mathcal{U}^{-1})$ be a nonempty subset. Consider an arbitrary $K \in \mathcal{K}(X)$. Then we have $G \cap K \in \tau(\mathcal{U}^{-1}|_K)$. Let $k \in G \cap K$ be a point. Choose a $V \in \mathcal{U}$ such that $V^{-1}(k) \cap K \subseteq G \cap K$. By compact symmetry, there exists a symmetric $W \in \mathcal{U}$ such that $W^2(k) \subseteq V(k)$ whenever $k \in K$. For each $k' \in W(k) \cap K$, we have $k \in W(k') \subseteq V(k')$. Thus $k' \in V^{-1}(k) \cap K$. It follows that $W(k) \cap K \subseteq V^{-1}(k) \cap K$. Consequently, $G \cap K \in \tau(\mathcal{U}|_K)$. Since $(X,\tau(\mathcal{U}))$ is a k-space, G is open in $(X,\mathcal{U})$. Therefore, $\tau(\mathcal{U}^{-1}) \subseteq \tau(\mathcal{U})$. By Lemma 2.8, $(X,\mathcal{U})$ is small-set symmetric. $\square$

Example 2.13 Generally, a compactly symmetric quasiuniform space may not be small-set symmetric. Let $X = \omega \cup \{p\}$ be equipped with its subspace topology induced by $\beta\omega$, where $p \in \beta\omega \setminus \omega$. Equip X with its Pervin quasiuniformity $\mathcal{P}$. Since X is regular, $\mathcal{P}$ is locally symmetric. But $(X,\mathcal{P})$ is not small-set symmetric, since $\tau(\mathcal{P}^{-1})$ is discrete. If

$K \subseteq X$ is compact, then K is finite. Otherwise, K is compact in $\beta\omega$. But any infinite closed subset of $\beta\omega$ has cardinality 2^c. This is a contradiction. Clearly each locally symmetric quasiuniform space in which compact subsets are finite is compactly symmetric. Hence $(X, \mathcal{P})$ is compactly symmetric, but not small-set symmetric.

3 The completeness of $(\mathcal{K}(X), \mathcal{U}_*)$

In this section, we extend the well-known Zenor-Morita theorem for uniformisable spaces to the quasiuniform setting by using the notion of compact symmetry. Firstly we recall some definitions relating to the completeness of a quasiuniform space.

Definition 3.1 [7] A filter $\mathcal{F}$ on a quasiuniform space $(X,\mathcal{U})$ is called a *Cauchy filter* if for each $U \in \mathcal{U}$ there exists a point $x \in X$ such that $U(x) \in \mathcal{F}$. A quasiuniform space $(X,\mathcal{U})$ is called *complete* (*convergence complete*) if every Cauchy filter on $(X,\mathcal{U})$ has a cluster point (converges).

Clearly, convergence completeness implies completeness, and the two notions coincide in the class of locally symmetric quasiuniform spaces [7, Corollary 3.9], therefore in the class of compactly symmetric quasiuniform spaces. For convenience, we shall use certain convergent nets instead of filters. To this end, we recall the following definition.

Definition 3.2 [15] A net $\{x_\alpha : \alpha \in D\}$ on $(X,\mathcal{U})$ is called an *$\mathcal{N}$-Cauchy net* if for each $U \in \mathcal{U}$ there exists a point $x \in X$ such that $\{x_\alpha : \alpha \in D\}$ is eventually in $U(x)$.

Lemma 3.3 *A quasiuniform space $(X,\mathcal{U})$ is convergence complete if and only if every $\mathcal{N}$-Cauchy net in $(X,\mathcal{U})$ is convergent.*

Corollary 3.4 *A compactly symmetric quasiuniform space $(X,\mathcal{U})$ is complete if and only if every $\mathcal{N}$-Cauchy net in $(X,\mathcal{U})$ is convergent.*

Theorem 3.5 *Let $(X,\mathcal{U})$ be a compactly symmetric quasiuniform space. Then $(\mathcal{K}(X),\mathcal{U}_*)$ is complete if and only if $(X,\mathcal{U})$ is complete.*

Proof: Suppose that $(\mathcal{K}(X),\mathcal{U}_*)$ is complete. Let $\{x_\alpha\colon \alpha \in D\}$ be an $\mathcal{N}$-Cauchy net in $(X,\mathcal{U})$. Clearly, $\{\{x_\alpha\}\colon \alpha \in D\}$ is an $\mathcal{N}$-Cauchy net in $(\mathcal{K}(X),\mathcal{U}_*)$, so it has a cluster point $C \in \mathcal{K}(X)$. Fix any point $x_0 \in C$. It easily follows that $\{x_\alpha\colon \alpha \in D\}$ clusters to x_0 in $(X,\mathcal{U})$. Hence $(X,\mathcal{U})$ is complete.

Conversely, suppose that $(X,\mathcal{U})$ is complete. Let $\{C_\alpha\colon \alpha \in D\}$ be an $\mathcal{N}$-Cauchy net in $(\mathcal{K}(X),\mathcal{U}_*)$. For each $\alpha \in D$, define $F_\alpha = \cup\{C_\beta : \beta \in D, \beta \geq \alpha\}$. Let $\mathcal{F} = fil\{F_\alpha\colon \alpha \in D\}$ and let $\mathcal{F}'$ be an ultrafilter containing $\mathcal{F}$. We show that $\mathcal{F}'$ is a Cauchy ultrafilter on $(X,\mathcal{U})$. Let $U \in \mathcal{U}$, and choose $V \in \mathcal{U}$ such that $V^2 \subseteq U$. Since $\{C_\alpha\colon \alpha \in D\}$ is an $\mathcal{N}$-Cauchy net, there are $B \in \mathcal{K}(X)$ and $\alpha_V \in D$ such that $C_\alpha \subseteq V(B)$ and $B \subseteq V^{-1}(C_\alpha)$ for all $\alpha \geq \alpha_V$. By the compactness of B, there is a finite subset B' of B such that $B \subseteq \cup_{b\in B'} V(b)$. Consequently $F_{\alpha_V} = \cup_{\beta\geq\alpha_V} C_\alpha \subseteq V(B) \subseteq \cup_{b\in B'} V^2(b) \subseteq \cup_{b\in B'} U(b)$. Since $\mathcal{F}'$ is an ultrafilter and $F_{\alpha_V} \in \mathcal{F}'$, $U(b) \in \mathcal{F}'$ for some $b \in B'$. Thus $\mathcal{F}'$ is a Cauchy ultrafilter on $(X,\mathcal{U})$. Hence it converges to some point in X. Therefore $C \neq \emptyset$, where $C = \cap_{F\in\mathcal{F}} \overline{F}$.

Now we prove that C is compact. Let $\mathcal{H}$ be a filter on C and let $\mathcal{G} = fil\{W(H)\cap F\colon H \in \mathcal{H}, W \in \mathcal{U}, F \in \mathcal{F}\}$ (note that $\mathcal{G}$ is well-defined because $H \subseteq C$ for each $H \in \mathcal{H}$ and $C = \cap_{F\in\mathcal{F}} \overline{F}$). Let $\mathcal{G}'$ be an ultrafilter containing $\mathcal{G}$. We have shown that for each $U \in \mathcal{U}$ there are $B \in \mathcal{K}(X)$ and $\alpha_V \in D$ (where $V^2 \subseteq U$) such that $F_{\alpha_V} \subseteq V(B) \subseteq \cup_{b\in B'} U(b)$ (where B' is some finite subset of B), so $F_{\alpha_V} \cap V(H) \subseteq V(B) \subseteq \cup_{b\in B'} U(b)$ for any $H \in \mathcal{H}$. Since $F_{\alpha_V} \cap V(H) \in \mathcal{G}'$, we have $U(b) \in \mathcal{G}'$ for some $b \in B'$. Thus $\mathcal{G}'$ is a Cauchy ultrafilter on $(X,\mathcal{U})$. So it converges to some point $y_0 \in X$. Since $\mathcal{F} \subseteq \mathcal{G}$, $y_0 \in C$. We see that $\mathcal{H}$ clusters to y_0. Let $U \in \mathcal{U}$ and $H \in \mathcal{H}$. Choose a symmetric $V \in \mathcal{U}$ such that $V^2(y_0) \subseteq U(y_0)$. Since $\mathcal{G}$ clusters to y_0, there is $z \in V(H) \cap V(y_0)$. By the symmetry of V, $x \in V^2(y_0) \subseteq U(y_0)$ for some $x \in H$. Therefore y_0 is a cluster point of $\mathcal{H}$. We conclude that C is compact.

Finally, the net $\{C_\alpha : \alpha \in D\}$ is convergent to C in $(\mathcal{K}(X),\mathcal{U}_*)$. For each $U \in \mathcal{U}$, choose a symmetric $V \in \mathcal{U}$ such that $V^3(x) \subseteq U(x)$ whenever $x \in C$. Again, since $\{C_\alpha : \alpha \in D\}$ is $\mathcal{N}$-Cauchy, there are an $\alpha_0 \in D$ and an $E \in \mathcal{K}(X)$ such that $(E, C_\alpha) \in H(V)$ for all $\alpha \geq \alpha_0$. Since $V(e) \cap C_\alpha \neq \emptyset$ for each $e \in E$ and each $\alpha \geq \alpha_0$, $\{F \cap V(e) : F \in \mathcal{F}\}$ is contained in a Cauchy filter on $(X,\mathcal{U})$. Thus, we have $\emptyset \neq \cap_{F\in\mathcal{F}} \overline{F \cap V(e)} \subseteq C \cap \overline{V(e)} \subseteq C \cap V^2(e)$. It follows that $e \in V^2(C)$ for each $e \in E$. Hence $E \subseteq V^2(C)$ and $C_\alpha \subseteq V(E) \subseteq V^3(C) \subseteq U(C)$. Since $C \subseteq \overline{F_\alpha} \subseteq \overline{V(E)} \subseteq V^2(E)$, for each point $x \in C$ there must be a point $y \in E$

such that $x \in V^2(y)$. On the other hand, for each $\alpha \in D$ and $\alpha \geq \alpha_0$, there exists a point $x_\alpha \in C_\alpha$ with $y \in V(x_\alpha)$. Hence $x_\alpha \in V^3(x) \subseteq U(x)$. It follows that $x \in U^{-1}(x_\alpha)$ and $C \subseteq U^{-1}(C_\alpha)$ for all $\alpha \in D$ and $\alpha \geq \alpha_0$. Therefore $(C, C_\alpha) \in H(U)$. Combining the previous arguments, $(\mathcal{K}(X), \mathcal{U}_*)$ is complete. □

Corollary 3.6 [14] *Let $(X,\mathcal{U})$ be a uniform space. Then $(\mathcal{K}(X),\mathcal{U}_*)$ is complete if and only if $(X,\mathcal{U})$ is complete.*

Corollary 3.7 [18] *Let (X,τ) be a Tychonoff space. Then $(\mathcal{K}(X), 2^\tau)$ is Dieudonné complete if and only if (X,τ) is Dieudonné complete.*

Example 3.8 The condition "compact symmetry" in Theorem 3.5 cannot be weakened to "local symmetry". Let $X = \omega$, and define $d\colon X \times X \to \mathbf{R}^+$ as follows

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 1/y, & \text{if } x = 0, y > 0; \\ 1/x, & \text{if } 0 < x < y; \\ 1, & \text{otherwise.} \end{cases}$$

Then (X, d) is a compact quasi-metric space. Let $\mathcal{U}_d$ be the quasiuniformity induced by d. Note that 0 is the only nonisolated point in X and that the conjugate topology of $\mathcal{U}_d$ is discrete. By a criterion of Fletcher and Lindgren [7, Proposition 2.23], it is straightforward to check that $(X,\mathcal{U}_d)$ is locally symmetric. For each $n \in \mathbf{N}$, set $K_n = \{1, 2, ..., n\}$ and $A_n = \{0, n+1\} \cup K_n$. Then $K_m \in H(U_{1/n})(A_n)$ whenever $m > n$, where $U_{1/n} = \{(x, y) : d(x, y) < 1/n\}$. It follows that $\{K_n : n \in \mathbf{N}\}$ is an $\mathcal{N}$-Cauchy sequence in $(\mathcal{K}(X), (\mathcal{U}_d)_*)$. However $\{K_n : n \in \mathbf{N}\}$ has no cluster point. Therefore $(\mathcal{K}(X), (\mathcal{U}_d)_*)$ is not complete.

Let $(X,\mathcal{U})$ be a quasiuniform space. A filter $\mathcal{F}$ on $(X,\mathcal{U})$ is called *right K-Cauchy* if for each $U \in \mathcal{U}$ there is an $F \in \mathcal{F}$ such that $U^{-1}(x) \in \mathcal{F}$ for all $x \in F$. The space $(X,\mathcal{U})$ is called *right K-complete* if every right K-Cauchy filter on $(X,\mathcal{U})$ is convergent [16]. Recall that a filter $\mathcal{F}$ on $(X,\mathcal{U})$ is called *stable* if for each $U \in \mathcal{U}$, $\cap_{F\in\mathcal{F}} U(F)$ belongs to $\mathcal{U}$. In [11], Künzi and Ryser successfully generalized the Isbell-Burdick's theorem as follows: For a quasiuniform space $(X,\mathcal{U})$ the Bourbaki quasiuniformity $\mathcal{U}_*$ on $\mathcal{P}_0(X)$ is right K-complete if and only if each stable filter on $(X,\mathcal{U})$ has a cluster point. In light of this result, it is likely to conjecture that $(\mathcal{K}(X),\mathcal{U}_*)$ is right K-complete if and only if $(X,\mathcal{U})$ is right K-complete. Unfortunately, the following example shows that this conjecture is false.

Example 3.9 Let $X = \mathbf{R}$ (where $\mathbf{R}$ denotes the reals), and define

$$U_n = \{(x,y) : x \in X, y < -2^n\} \cup \{(x,y) : x \in X, x - 2^{-n} < y < x + 2^{-n}\}$$

for each $n \in \omega$. Let $\mathcal{U}$ be the quasiuniformity on X generated by $\{U_n : n \in \omega\}$. Let $\{x_\alpha : \alpha \in D\}$ be a right K-Cauchy net in $(X,\mathcal{U})$. Then there exists an $\alpha_0 \in D$ such that $\overline{\{x_\alpha : \alpha \geq \alpha_0\}}$ is bounded from above. Since any subset of X which is closed and bounded from above in $(X,\mathcal{U})$ is compact, $\{x_\alpha : \alpha \in D\}$ is convergent in $(X,\mathcal{U})$. Therefore $(X,\mathcal{U})$ is right K-complete. For each $n \in \omega$, set $K_n = (-\infty, n]$. Then it is ready to check that $\{K_n : n \in \omega\}$ is a right K-Cauchy sequence in $(\mathcal{K}(X),\mathcal{U}_*)$. But $\{K_n : n \in \omega\}$ is not convergent in $(\mathcal{K}(X),\mathcal{U}_*)$. In fact, suppose that $K_n \to C \in \mathcal{K}(X)$. Then $K_n \subseteq U_1(C)$ for n large enough. Since C is bounded from above, we obtain a contradiction.

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