Regions free of resonance poles for Schrödinger operators with non-smooth potentials

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Abstract

An elementary method for finding the region free of resonance poles is presented. The method involves estimating time derivatives of solutions to Schrödinger's Equation. These estimates have certain implications regarding the region of analyticity of the reduced resolvent operator. The method also yields some generalizations to results already known about the location of resonance poles.

1 INTRODUCTION

Consider the equation

$$\Delta u + k^2 u - V(x)u = 0, \tag{1}$$

where $|V(x)|$ is in some sense small as $x \to \infty$ in R^N (e.g. V has compact support). We are interested in solutions $u(x,k)$ which are asymptotically of the form $e^{ik.x}$ ($k = (k_1, k_2, \ . \ . \ , k_N)$) in the sense that $u(x,k) - e^{ik.x}$ satisfies the Sommerfield Radiation Condition as $x \to \infty$. These solutions are of interest because one can do generalised Fourier expansions with them [2], [8]. Further, they are important in Scattering Theory, and for certain potentials V, the solutions are meromorphic functions of k whose poles coincide with the poles of the scattering matrix. The location of these poles is of interest for many reasons. For example, they determine the type of local smoothing that certain solutions of the Schrödinger equation

$$i\ u_t = \Delta\ u - V(x)\ u$$

have [10]. As another example, they determine the form of asymptotic expansions for large time of solutions of the wave equation

$$u_{tt} = \Delta\ u - V(x)\ u.$$

In this paper, we investigate the region of analyticity of the function $u(x, k)$ and, in so doing, obtain bounds for the location of the poles. Results of this type are not new and the main purposes of this paper are simply to present an elementary method for obtaining such results and to generalise what is already known (we prove results for non-smooth potentials that belong to various weighted L^pspaces). We mention some early results on this subject.

In [7], A. G. Ramm showed that if $N = 3$ and $V \in C^1(R^3)$ has compact support, then there are constants $c > 0$, b such that $u(x, k)$ is analytic in

$$\Omega_{b,c} = \{k = \sigma + i\tau : \ \tau \geq b - c \ln |\sigma|\}.$$

P. D. Lax and R. S. Phillips (1971) [4] showed that if N is odd and V is bounded, measurable and has compact support, then there are constants $c > 0, b$ such that $u(x, k)$ is analytic in $\Omega_{b,c}$. We note that Ramm, in his proof, uses the resolvent kernel for equation (1) with $V \equiv 0$ and solves (1) for non-zero V using

$$u(x, k) = -\frac{1}{4\pi} \int \frac{e^{i|k||x-y|}}{|x-y|} V(y) u(y, k) \, dy + e^{ik.x}.$$

Lax and Phillips use their scattering theory to solve the problem. Thus, their proof is limited to odd space dimensions. The method used in this paper is a generalisation of Ramm's method in the sense that it is proved by establishing an estimate for a suitable analytic continuation of the resolvent.. Vainberg [11] has established such estimates for second order elliptic differential operators with C^∞ variable coefficients

In Theorem 2 and Corollary 3, we use an elementary method based on a smoothing property of the solution operator for the Schrödinger equation (see the proof of Lemma 1 for this smoothing property) to obtain the same result for all space dimensions. The corollary shows that we can even obtain the result for potentials in $L^p(R^N)$, where $p \geq 2$ and $p > N$. Further, we can dispense with the assumption that V has compact support and obtain similar results for potentials decaying in certain ways at infinity. The main idea in the proof is to obtain certain analytic continuations for the resolvent $R(\Delta, -k^2) = (\Delta + k^2)^{-1}$, valid on certain spaces of functions, and thus solve the equation

$$u(x, k) = R(\Delta, -k^2) V u + e^{ik.x} \tag{2}$$

2 THE METHOD

We first establish some analytic continuation results for the resolvent operator $(\lambda + iH)^{-1}$ of $-iH = \frac{i}{2}\Delta$ (we consider Δ as an unbounded operator on $L^2(R^N)$ with domain $H^2(R^N)$). This is done in Lemma 1, where we use a technique introduced by Jensen [3] to derive smoothing results for the solution operator of the Schrödinger equation. For this, we define $H_a = H + a.p$ where $a \in R^N$ and $p_j = -i\frac{\partial}{\partial x_j}$. It is easy to check that

$$e^{-ia.x}e^{-itH}e^{ia.x} = e^{-itH_a}e^{-ita^2/2}$$

and on $H^1(R^N)$,

$$\frac{\partial}{\partial a_j}e^{-itH_a} = t\frac{\partial}{\partial x_j}e^{-itH_a}.$$

Provided that $(|x_j|+1)f \in L^2$, one may differentiate $e^{-ia.x}e^{-itH}e^{ia.x}f$ with respect to a_j and the estimate obtained may be combined with the equation above (with $a = 0$) to obtain an estimate for $t\frac{\partial}{\partial x_j}e^{-itH}f$. Thus, smoothing results for the free Schrödinger operator are easily derived. Jensen extends the technique to Schrödinger equations with potentials, and the author derives various other smoothing results for Schrödinger equations with potentials in [10]. Similar smoothing results appear in [5] and [9]. The interested reader could consult the references in [10] to see other smoothing results. We note that it is well known in case (1) below that the region of analyticity is the whole complex plane with the non-positive part of the real axis removed. However, a simple proof of our weaker statement is included here to make this paper self-contained (and because it is a simple by-product of the analysis).

Lemma 1 *Let $R(\lambda)$ denote the resolvent operator $(\lambda + iH)^{-1}$ of $-iH = \frac{i}{2}\Delta$. Considering a function w as a multiplication operator on $L^2(R^N)$, let $S(\lambda) = wR(\lambda)w$.*

1. *If $w = \chi$, the characteristic function of a bounded set in R^N, then $S(\lambda)$ has an analytic continuation into a region $\{\lambda : \mathrm{Re}\lambda > -m\mathrm{Im}\lambda\}$, where $m > 0$. Further, for each $\rho > 0,\ \theta > 0$, there are positive constants $c_1(\theta),\ c_2(\theta,\rho)$ such that for*

$$\lambda \in T_{\theta,1} = \{\lambda : \mathrm{Re}\lambda \geq -c_1|\mathrm{Im}\lambda|^{1/2}\log|\mathrm{Im}\lambda|,\ |\mathrm{Im}\lambda| > 1\} \cup \{\lambda : \mathrm{Re}\lambda > \rho\},$$

   we have $||S(\lambda)|| < c_2|\lambda|^{\theta-1/2}$.

2. *If $w(x) = \exp[-r\ \Sigma_{i=1}^n |x_i|^\alpha]$, where $\alpha > 1,\ r > 0$, then $S(\lambda)$ has an analytic continuation into a region $\{\lambda : \mathrm{Re}\lambda > -m|\mathrm{Im}\lambda|\}$, where $m > 0$. Further, for each $\rho > 0,\ \theta > 0$, there are positive constants $c_1(\theta),\ c_2(\theta,\rho)$ such that for*

$$\lambda \in T_{\theta,2} = \{\lambda : \mathrm{Re}\lambda \geq -c_1|\mathrm{Im}\lambda|^{1/2}(\log|\mathrm{Im}\lambda|)^{\frac{\alpha-1}{\alpha}},\ |\mathrm{Im}\lambda| > 1\} \cup \{\lambda :\ \mathrm{Re}\lambda > \rho\},$$

   we have $||S(\lambda)|| < c_2|\lambda|^{\theta-1/2}$.

3. *If $w(x) = \exp[-r\ \Sigma_{i=1}^n |x_i|]$, where $r > 0$, then $S(\lambda)$ has an analytic continuation into a region $\{\lambda : \mathrm{Re}\lambda > -m|\mathrm{Im}\lambda|\} \cap \{\lambda : \mathrm{Re}\lambda > -l|\mathrm{Im}\lambda|^{1/2}\}$, where $m > 0,\ l > 0$. Further, for each $\rho > 0,\ \theta > 0$, there are positive constants $c_1(\theta),\ c_2(\theta,\rho)$ such that for*

$$\{\lambda \in T_{\theta,3} = \lambda : \mathrm{Re}\lambda \geq -c_1|\mathrm{Im}\lambda|^{1/2},\ |\mathrm{Im}\lambda| > 1\} \cup \{\lambda : \mathrm{Re}\lambda > \rho\},$$

   we have $||S(\lambda)|| < c_2|\lambda|^{\theta-1/2}$.

Proof. Since the Laplacian is self adjoint, $R(\lambda)$ exists for all $\mathrm{Re}\lambda > 0$. We first show that $\chi R(\lambda)\chi$ has an analytic continuation into part of the left half plane. This result is obtained by estimating the time derivatives of (in this case) $\chi e^{-iHt}\chi$. Without loss of generality, we suppose that χ is the characteristic function of the cube $|x_i| \le R$. By the discussion above, we have

$$\frac{\partial^\alpha}{\partial a^\alpha} e^{-itH_a} = (t\frac{\partial}{\partial x})^\alpha e^{-itH_a}. \tag{3}$$

But $e^{-itH_a} = e^{-ia.x}e^{-itH}e^{ia.x}e^{ia^2t/2}$, showing that $\chi e^{-itH_a}\chi$ is an entire function of a taking values in the bounded operators on L^2. Thus, by Cauchy's Theorem,

$$\chi \frac{\partial^\alpha}{\partial a^\alpha}|_{a=0} e^{-itH_a}\chi = \frac{1}{(2\pi i)^N}\int_C \frac{\alpha!\chi e^{-ib.x}e^{-itH}e^{ib.x}e^{ib^2t/2}\chi}{b_1^{1+\alpha_1}b_2^{1+\alpha_2}\ldots b_N^{1+\alpha_N}}\, db. \tag{4}$$

Here, $b = (b_1, b_2, \ldots, b_N)$ and C is the contour in C^N consisting of the Cartesian product of N positively oriented circles, each of which has radius μ and center the origin. Thus, we obtain

$$||\chi \frac{\alpha}{a^\alpha}|_{a=0} e^{-itH}\chi|| \le \frac{\alpha!}{\mu^{|\alpha|}} e^{N\mu R + \mu^2 t/2} \tag{5}$$

where $\mu > 0$ is arbitrary. Thus, by (3) and (5),

$$||\chi \frac{\partial^\alpha}{\partial x^\alpha} e^{-itH}\chi|| \le \frac{\alpha!}{|\mu t|^{|\alpha|}} e^{N\mu R + \mu^2 t/2}. \tag{6}$$

But

$$\Delta^m = (\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2})^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}},$$

and it is easily shown by induction that

$$\sum_{|\alpha|=m} \frac{(2\alpha)!}{\alpha!} \le (\frac{5}{2})^{N-1}\frac{(2m)!}{m!},$$

so we obtain

$$||\frac{d^m}{dt^m}\chi e^{-itH}\chi|| \le (\frac{5}{2})^{N-1}\frac{(2m)!}{|2\mu t|^{2m}} e^{N\mu R + \mu^2 t/2} = A_m t^{-2m}.$$

But $A_m/A_{m-1} = \frac{2m(2m-1)}{2\mu^2}$, so

$$
\begin{aligned}
||\frac{d^m}{dt^m} t^p \chi e^{-itH} \chi|| &\leq t^p A_m t^{-2m} + p t^{p-1} A_{m-1} t^{-2m+2} \begin{pmatrix} m \\ 1 \end{pmatrix} + p(p-1) A_{m-2} t^{-2m+4} t^{p-2} \begin{pmatrix} m \\ 2 \end{pmatrix} \\
&\quad + \ \ldots \ + p(p-1)(p-2)\ldots(p-m+1) t^{p-m} A_0 \\
&\leq A_m t^{p-2m} (1 + \frac{pm}{2m(2m-1)} 2\mu^2 t + \frac{p(p-1)m(m-1)}{2!2m(2m-1)(2m-2)(2m-3)} (2\mu^2 t)^2 \\
&\quad + \ \ldots \ + \frac{p(p-1)\ldots(p-m+1)}{(2m)!} (2\mu^2 t)^m) \\
&\leq A_m t^{p-2m} (1 + \frac{p}{2m} \frac{2\mu^2 t}{1!} + \frac{p(p-1)}{2m(2m-1)} \frac{(2\mu^2 t)^2}{2!} + \ \ldots \\
&\quad + \frac{p(p-1)\ldots(p-m+1)}{2m(2m-1)\ldots(m+1)} \frac{(2\mu^2 t)^m}{m!}) \\
&\leq A_m t^{p-2m} e^{2\mu^2 t} \max_{0 \leq q \leq \min(m,p)} \frac{p!(2m-q)!}{(p-q)!(2m)!} \\
&\leq A_m t^{p-2m} e^{2\mu^2 t} \begin{cases} 1, & p \leq 2m \\ \frac{m!p!}{(2m)!(2p)!}, & p \geq 2m \end{cases} \qquad (7)
\end{aligned}
$$

But since for $\mathrm{Re}\lambda > 0$, $R(\lambda) = \int_0^\infty e^{-\lambda t} \, e^{-itH} \, dt$, we obtain

$$
\chi \frac{d^n R(\lambda)}{d\lambda^n} \chi = \int_0^\infty (-t)^n e^{-\lambda t} \, \chi e^{-itH} \chi \, dt.
$$

Letting $t_0 > 0$ be arbitrary for now, we integrate this by parts m times (where $2m \geq n$), break the integral up into two integrals over $[0, t_0]$ and$[t_0, \infty)$ respectively, and then integrate the latter by parts once again to obtain

$$
\begin{aligned}
||\chi \frac{d^n R(\lambda)}{d\lambda^n} \chi|| \ &\leq \ \frac{1}{|\lambda|^m} \int_0^{t_0} e^{-t\mathrm{Re}\lambda} ||\frac{d^m}{dt^m} t^n \chi e^{-itH} \chi|| \, dt \\
&\quad + \frac{1}{|\lambda|^{m+1}} \int_{t_0}^\infty e^{-t\mathrm{Re}\lambda} ||\frac{d^{m+1}}{dt^{m+1}} t^n \chi e^{-itH} \chi|| \, dt \\
&\quad + \frac{1}{|\lambda|^{m+1}} e^{-t_0 \mathrm{Re}\lambda} ||\frac{d^m}{dt^m}|_{t=t_0} t^n \chi e^{-itH} \chi||. \qquad (8)
\end{aligned}
$$

We consider the two cases n even or odd separately. Suppose that n is even. Setting $m = n/2$ in estimate (8) and using estimate (7), we obtain (with $\mathrm{Re}\lambda > \frac{5}{2}\mu^2$)

$$
||\chi \frac{d^n R(\lambda)}{d\lambda^n} \chi|| \leq (\frac{5}{2})^{N-1} n! (2\mu^2)^{-n/2} |\lambda|^{-n/2} e^{N\mu R} M(n, \lambda, \mu),
$$

where

$$
M(n, \lambda, \mu) \ = \ \int\limits_0^{t_0} e^{(5\mu^2/2 - \mathrm{Re}\lambda)t} \, dt + \frac{1}{|\lambda|} \int\limits_{t_0}^{\infty} \frac{(n+1)(n+2)}{2\mu^2 t^2} e^{(5\mu^2/2 - \mathrm{Re}\lambda)t} \, dt + \frac{1}{|\lambda|} e^{(5\mu^2/2 - \mathrm{Re}\lambda)t_0}
$$

$$\begin{aligned}
&\leq \; t_0 + \frac{(n+1)(n+2)}{2\mu^2|\lambda|t_0^{1+\theta}} \int_0^\infty e^{(5\mu^2/2-\mathrm{Re}\lambda)t}\, dt + \frac{1}{|\lambda|}, \qquad \text{(where} \quad \theta > 0)\\
&= \; t_0 + \frac{1}{|\lambda|} + \frac{(n+1)(n+2)}{2\mu^2|\lambda|t_0^{1+\theta}} \frac{\Gamma(\theta)}{(\mathrm{Re}\lambda - 5\mu^2/2)^\theta}.
\end{aligned}$$

Choosing $t_0^{2+\theta} = \frac{\Gamma(\theta)}{2\mu^2|\lambda|(\mathrm{Re}\lambda - 5\mu^2/2)^\theta}$ and restricting $\mathrm{Re}\lambda \geq 7\mu^2/2$ yields

$$\begin{aligned}
M(n,\lambda,\mu) \;&\leq\; (n+1)(n+2)C(\theta)(|\lambda|^{-1} + |\lambda|^{-\frac{1}{2}+\frac{\theta/2}{2+\theta}}(\mu^2)^{-\frac{1}{2}-\frac{\theta/2}{2+\theta}})\\
&\leq\; 2(n+1)(n+2)C(\theta)|\lambda|^{-\frac{1}{2}+\frac{\theta/2}{2+\theta}}(\mu^2)^{-\frac{1}{2}-\frac{\theta/2}{2+\theta}}
\end{aligned}$$

because $|\lambda| \geq \mathrm{Re}\lambda \geq 7\mu^2/2$. A similar estimate is easily obtained for n odd (one puts $m = (n-1)/2$). Thus, we have proved that for each $\mu > 0$ and each $\delta \in (0, \frac{1}{2})$ there exists a constant C_δ depending only on δ such that for all $\mathrm{Re}\lambda \geq 7\mu^2/2$

$$||\chi \frac{d^n R(\lambda)}{d\lambda^n} \chi|| \leq C_\delta e^{N\mu R}(n+2)!(2\mu^2)^{-n/2}|\lambda|^{-n/2}|\lambda|^{-1/2+\delta}(\mu^2)^{-1/2-\delta}. \tag{9}$$

The Taylor series at $\lambda = 7\mu^2/2 + i\tau$ thus has a radius of convergence of at least $(2\mu^2|\tau|)^{1/2}$. Letting $\mu = \frac{\epsilon}{NR}\log|\mathrm{Im}\lambda|$, where $\epsilon > 0$ and $|\mathrm{Im}\lambda| > 1$, we see that $\chi R(\lambda)\chi$ has an analytic continuation into the set

$$\{\lambda : |\mathrm{Re}\lambda - \frac{7}{2}(\frac{\epsilon}{NR}\log|\mathrm{Im}\lambda|)^2| \leq (\frac{\epsilon}{NR}\log|\mathrm{Im}\lambda|)|\mathrm{Im}\lambda|^{1/2},\ |\mathrm{Im}\lambda| > 1\},$$

and in this set $||\chi R(\lambda)\chi|| \leq C(\delta,\epsilon)|\lambda|^{-1/2+\delta+\epsilon}$. Thus we have proved the following: For each $\theta > 0$ there exist positive constants $C_1(\theta)$ and $C_2(\theta)$ such that $\chi R(\lambda)\chi$ has an analytic continuation into the set

$$S_\theta = \{\lambda : |\mathrm{Re}\lambda| \leq C_1(\theta)(\log|\mathrm{Im}\lambda|)|\mathrm{Im}\lambda|^{1/2},\ |\mathrm{Im}\lambda| > 1\}$$

in which $||\chi R(\lambda)\chi|| \leq C_2(\theta)|\lambda|^{\theta-1/2}$. Further, Estimate (8) with $n = 0$ gives $||\chi R(\lambda)\chi|| \leq C_\delta e^{N\mu R}|\lambda|^{\delta-1/2}(\mu^2)^{-1/2-\delta}$ for $\mathrm{Re}\lambda \geq \frac{7}{2}\mu^2$. This proves the second part of (1).

For the first part of (1), we return to (9), which shows that for arbitrary $\mu > 0$, the Taylor series of $\chi R(\lambda)\chi$ at $\lambda = 7\mu^2/2 + i\tau$ has a radius of convergence of at least $(2\mu^2|\tau|)^{1/2}$. But clearly the operator is defined for all $\mathrm{Re}\lambda > 0$, so we see that it exists as an analytic continuation into the set

$$\{\lambda : \mathrm{Re}\lambda > -|2\mu^2\mathrm{Im}\lambda|^{1/2} + 7\mu^2/2\}.$$

Since $\mu > 0$ is arbitrary, the analytic continuation actually exists everywhere to the right of the envelope of the parabolic curves $\mathrm{Re}\lambda = -|2\mu^2\mathrm{Im}\lambda|^{1/2} + 7\mu^2/2$, which is easily computed to be $\mathrm{Re}\lambda = -\frac{1}{7}|\mathrm{Im}\lambda|$. This completes the proof of (1).

To prove (2), we study $wR(\lambda)w$ and use the same method that we used to study $\chi R(\lambda)\chi$ above, i.e. we consider $we^{-itH_a}w$ and estimate the derivatives of this operator using a contour integral. The main difference is that instead of estimate (5) we obtain

$$||w\frac{\partial^\beta}{\partial a^\beta}|_{a=0}e^{-itH_a}w|| \leq \frac{\beta!}{\mu^{|\beta|}}\exp[Nr(\alpha-1)(\frac{\mu}{\alpha r})^{\alpha/(\alpha-1)}+\mu^2t/2]. \tag{10}$$

Thus an estimate for the derivatives of $wR(\lambda)w$ similar to (9) is obtained by replacing '$e^{N\mu R}$' by '$\exp[Nr(\alpha-1)(\frac{\mu}{\alpha r})^{\alpha/(\alpha-1)}]$'. The remainder of the proof of (2) is similar to that of (1).

The proof of (3) differs from that of (2) in that now the arbitrary paramater μ is restricted to be in the interval $(0,r]$, and for such μ we have

$$||w\frac{\partial^\beta}{\partial a^\beta}|_{a=0}e^{-itH_a}w|| \leq \frac{\beta!}{\mu^{|\beta|}}\exp[\mu^2t/2].$$

The restriction of μ to $(0,r]$ accounts for the different region of analyticity of $S(\lambda)$, for this region is not the entire envelope of the family of parabolas.

Theorem 2 *Let $V : R^N \to C$ be measurable and suppose that $-iH$ generates a strongly continuous group, where $H = -1/2\Delta + V$, with domain $H^2(R^N)$. Then the following results hold (see the statement of Lemma 1).*

1. *Let χ be the characteristic function of a bounded set $W \subset R^N$ and suppose that V has support in W. Suppose also that there exist constants $C > 0$ and $1/2 > \sigma \geq 0$ such that for all $f \in H^2_{loc}(R^N)$*

$$||Vf||_{L^2} \leq C(||\chi\Delta f||^\sigma_{L^2}||\chi f||^{1-\sigma}_{L^2} + ||\chi f||_{L^2}).$$

   Then there are constants $c > 0$, b such that there exists a solution $u(x,k)$ of Equation 2 which is an analytic function of k in

$$\Omega_{b,c} = \{k = \sigma + i\tau : \ \tau \geq b - c\ln|\sigma|\}.$$

2. *Let $w(x) = \exp[-r\ \Sigma^n_{i=1}|x_i|^\alpha]$, where $\alpha \geq 1$ and $r > 0$. Suppose that for fixed constants $C > 0$ and $1/2 > \sigma > 0$ we have*

$$||w^{-1}Vf|| \leq C(||w\Delta f||^\sigma||wf||^{1-\sigma} + ||wf||)$$

   for all f in H^2_{loc} such that wf and $w\Delta f$ belong to L^2. Then there are constants $c > 0$, b such that there exists a solution $u(x,k)$ of Equation 2 which is an analytic function of k in

$$\Omega_{b,c} = \begin{cases} \{k = \sigma + i\tau : \ \tau \geq b - c(\ln|\sigma|)^{(\alpha-1)/\alpha}\}, & \alpha > 1 \\ \{k : \ \mathrm{Im}k \geq -c, \ |k| \geq b\}, & \alpha = 1 \end{cases}$$

Proof. We only present the proof of (1), and note that the proof of (2) is almost identical to that of (1). Assume that the hypotheses for (1) hold. Then statement (1) of Lemma 1 follows.

Further, since and $||S(\lambda)|| < c_2|\lambda|^{\theta-1/2}$ for $\lambda \in T_{\theta,1}$, it is easy to see that there exists a constant $c_3(\theta)$ such that for $\lambda \in T_{\theta,1}$,

$$||\chi\Delta R(\lambda)\chi|| \leq C_3(\theta)|\lambda|^{\theta+1/2}.$$

Consider now the operator $VR(\lambda)\chi = V\chi R(\lambda)\chi$. By the preceding estimates and our hypotheses on V, for $\lambda \in T_{\theta,1}$

$$\begin{aligned} ||V\chi R(\lambda)\chi|| &\leq Cc((|\lambda|^{1/2+\theta})^{\sigma}(|\lambda|^{-1/2+\theta})^{1-\sigma} + |\lambda|^{-1/2+\theta}) \\ &= Cc(|\lambda|^{\theta+\sigma-1/2} + |\lambda|^{-1/2+\theta}), \end{aligned}$$

where c depends on only c_2 and c_3. We choose $0 < \theta_0 < 1/2 - \sigma$ and conclude that for $d > 0$ sufficiently large, $VR(\lambda)\chi$ is a contraction on $L^2(R^N)$ for all $\lambda \in T_{\theta_0,1} \cap \{\lambda : \ |\lambda| > d\}$.

We now solve equation (2) for $u(x,k)$. With $k^2 = 2i\lambda$, it is clear that we can equivalently study the equation

$$u(x,k) = 2iR(\lambda)Vu + e^{ik.x}. \tag{11}$$

Iterating this equation displays the solution

$$u(x,k) = e^{ik.x} + 2iR(\lambda)(\sum_{n=0}^{\infty}(V\chi R(\lambda)\chi)^n)Ve^{ik.x}. \tag{12}$$

We pause to verify that (12) makes sense for $\lambda \in T_{\theta_0,1} \cap \{\lambda : |\lambda| > d\}$. First, $Ve^{ik.x}$ is in $L^2(R^N)$ because V has support in W. Further, $\sum_{n=0}^{\infty}(V\chi R(\lambda)\chi)^n$ converges to an operator on L^2 that is analytic on $T_{\theta_0,1} \cap \{\lambda : |\lambda| > d\}$. The range of this operator is contained in the set of functions that have support in W. Thus, by (2) of Lemma 1, $R(\lambda)\chi$ also has an analytic continuation into $\{\lambda : \mathrm{Re}\lambda > -m|\mathrm{Im}\lambda|\}$, as a bounded operator from L^2 into $L^2_w(R^N) = \{f : \ f/w \in L^2\}$, which has norm given by $||f||_w = ||f/w||_{L^2}$. Finally, $R(\lambda)$ is defined on such functions and takes values in $L^2_w(R^N)$.

A simple computation shows that the image of the set $\{\lambda : |\lambda| > d\} \cap T_{\theta_0,1}$ under the transformation $k^2 = 2i\lambda$ contains $\{k : \mathrm{Im}k > 0, \mathrm{Re}k > b - c\log(\mathrm{Im}k)\} \cup \{k : \mathrm{Re}k > 0, \mathrm{Im}k > b - c\log(\mathrm{Re}k)\}$ for some $b > 0$, $c > 0$. This is not the full region $\Omega_{b,c}$. But everything that we have done with the Schrödinger equation applies equally well to its conjugate equation with solution operator e^{itH}. The region obtained is the reflection of the set above in the imaginary k axis. The union of these two sets contains $\Omega_{b,c}$.

Corollary 3 *Let V be a real-valued potential with compact support and suppose that $p > N$, $p \geq 2$, and $V \in L^p$. Then there are constants $c > 0, b$ such that $u(x,k)$ is an analytic function of k in $\{\Omega_{b,c} = k = \sigma + i\tau : \tau \geq b - c\ln|\sigma|\}$.*

Proof. We easily verify that V satisfies the hypotheses of Theorem 2.

Corollary 4 *.Let $g(x) = r(|x_1|^{\alpha} + |x_2|^{\alpha} + \ \ldots \ + |x_N|^{\alpha})$, where and $\alpha \geq 1$. Suppose that V is a real-valued potential and that $Ve^{2g} \in L^p(R^N)$, where $p \geq 2$ and $p > N$. Then there are constants $c > 0, b$ such that is an analytic function of k in*

$$\Omega_{b,c} = \begin{cases} k = \sigma + i\tau : \ \tau \geq b - c(\ln|\sigma|)^{(\alpha-1)/\alpha}, & \alpha > 1 \\ k : \ \mathrm{Im}k \geq -c, \ |k| \geq b, & \alpha = 1 \end{cases} .$$

Remark:In [7] Ramm states a result similar to this for the case $\alpha = 1$ and with $N = 3$ (i.e. three space dimensions), and the result is proved for the case $N = 3$ in [1] by C. L. Dolph, J. B. McLeod and D. Thoe.

References

[1] C. L. DOLPH, J. B. MCLEOD, D. THOE, The Analytic Continuation of the Resolvent Kernel and Scattering Operator Associated with the Schrödinger Operator, Journal of Mathematical Analysis and Applications 16, (1966), 311-332.

[2] T. IKEBE, Eigenfunction Expansions Associated with the Schroedinger Operators and their Applications to Scattering Theory, Arch. Rat. Mech. Anal. 5 (1960), 1 - 34.

[3] A. JENSEN, Commutator Methods and a Smoothing Property of the Schrödinger Evolution Group, Math. Z. 191 (1986), 53 - 59.

[4] P. D. LAX AND R. S. PHILLIPS, A Logarithmic Bound on the Location of the Poles of the Scattering Matrix, Arch. Rat, Mech. Anal. 40, (1971).

[5] W. LITTMAN AND S. W. TAYLOR, Smoothing Evolution Equations and Boundary Control Theory, J. Analyse Math., 59 (1992), 117-131.

[6] A. PAZY, "Semigroups of Linear Operators and Applications to Partial Differential Equations", Springer-Verlag, Berlin, (1983).

[7] A. G. RAMM, Regions Free of Resonance Poles, Soviet-Dokl. 11, No. 2, (1966) 114-116.

[8] D. W. THOE, Eigenfunction Expansions Associated with the Schroedinger Operators in $R^n, n \geq 4$, Arch. Rat. Mech. Anal. 26 (1967), 335 - 356.

[9] S. W. TAYLOR, Gevrey Solutions of Euler-Bernoulli Type Equations, J. Differential Equations 92 (1991), 331-359.

[10] S. W. TAYLOR, Gevrey Smoothing Properties of the Schrödinger Evolution Group in Weighted Sobolev Spaces, Journal of Mathematical Analysis and Applications 194 (1995), 14 - 38.

[11] B. R. Vainberg, On a point Source in an Inhomogeneous Medium. Math. USSR Sbornik, 23(1) (1974), 123-148.