THE ALPERIN AND DADE CONJECTURES FOR THE SIMPLE CONWAY'S THIRD GROUP

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ABSTRACT. This paper is part of a program to study Alperin's weight conjecture and Dade's conjecture on counting ordinary characters in blocks for several finite groups. The classifications of radical subgroups and certain radical chains and their local structures of the simple Conway's third group have been obtained by using the computer algebra system CAYLEY. The Alperin weight conjecture and the Dade final conjecture have been confirmed for the group.

Introduction

Let G be a finite group, p a prime and B a p-block of G. Alperin in [1] conjectured that the number of B-weights should equal the number of irreducible Brauer characters of B. Dade in [6] has presented a conjecture exhibiting the number of ordinary irreducible characters of a fixed height in B, in terms of an alternating sum of similar integers for p-blocks of some local subgroups of the group G. By Dade [6], his final conjecture needs only be verified for finite non-abelian simple groups and is equivalent to the ordinary conjecture whenever a finite group has a trivial Schur multiplier and outer automorphism group. In this paper we verify the Alperin weight conjecture and the Dade ordinary conjecture, and so the final one, for the simple Conway's third group.

Most of the calculations were carried out using the CAYLEY computer system [4]. In Section 1, we fix our notation and state the two conjectures. In Section 2, we classify radical subgroups, determine their local structures and verify the Alperin weight conjecture. In Section 3, we do some cancellations in the alternating sum when $p = 2$ or 3, and then determine radical chains and their local structures. In the last Section, we verify the Dade conjecture.

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1. The Alperin and Dade Conjectures

Let R a p-subgroup of a finite group G. Then R is *radical* if $O_p(N(R)) = R$, where $O_p(N(R))$ is the maximal normal p-subgroup of the normalizer $N(R) = N_G(R)$. Denote by $\mathrm{Irr}(G)$ the set of all irreducible ordinary characters of G, and let $\mathrm{Blk}(G)$ be the set of p-blocks, $B \in \mathrm{Blk}(G)$ and $\varphi \in \mathrm{Irr}(N(R)/R)$. The pair (R, φ) is called a *B-weight* if φ has p-defect 0 and $B(\varphi)^G = B$ (in the sense of Brauer), where $B(\varphi)$ is the block of $N(R)$ containing φ. A weight is always identified with its G-conjugates. Let $\mathcal{W}(B)$ be the number of B-weights, and $\ell(B)$ the number of irreducible Brauer characters of B. Alperin conjectured that $\mathcal{W}(B) = \ell(B)$ for each $B \in \mathrm{Blk}(G)$. If a defect group D of B is cyclic, the Alperin conjecture follows by Theorem 9.1 of [6]. Thus we may suppose D is non-cyclic.

Given a p-subgroup chain

$$C : P_0 < P_1 < \cdots < P_n \tag{1.1}$$

of a finite group G, define $|C| = n$, $C_k : P_0 < P_1 < \cdots < P_k$, $C(C) = C_G(P_n)$, and

$$N(C) = N_G(C) = N(P_0) \cap N_G(P_1) \cap \cdots \cap N_G(P_n). \tag{1.2}$$

The chain C is said to be *radical* if it satisfies the following two conditions:

(a) $P_0 = O_p(G)$ and (b) $P_k = O_p(N(C_k))$

for $1 \leq k \leq n$. Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical p-chains of G. For $B \in \mathrm{Blk}(G)$ and integer $d \geq 0$, let $\mathrm{k}(N(C), B, d)$ be the number of characters in the set

$$\mathrm{Irr}(N(C), B, d) = \{\psi \in \mathrm{Irr}(N(C)) : B(\psi)^G = B,\ \mathrm{d}(\psi) = d\},$$

where $\mathrm{d}(\psi)$ is the defect of ψ (see [6, (5.5)] for the definition). Dade in [6] gives the following conjecture.

Ordinary Conjecture. *If $O_p(G) = 1$ and B is a p-block of G with defect $\mathrm{d}(B) > 0$, then for any integer $d \geq 0$,*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|}\, \mathrm{k}(N(C), B, d) = 0, \tag{1.3}$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits in $\mathcal{R}$.

2. Radical Subgroups and Weights

Let $\Phi(G, p)$ be a set of representatives for conjugacy classes of radical subgroups of G. For $H, K \leq G$, we write $H \leq_G K$ if $x^{-1}Hx \leq K$; and write $H \in_G \Phi(G, p)$ if $x^{-1}Hx \in \Phi(G, p)$ for some $x \in G$. We shall follow the notation of [5]. In particular, $p_+^{1+2\gamma}$

is an extra special group of order $p^{1+2\gamma}$ with exponent p or plus type according as p is odd or even. If X and Y are groups, we use $X.Y$ and $X:Y$ to denote an extension and a split extension of X by Y, respectively. Given $n \in \mathbf{N}$, we use E_{p^n} or simply p^n to denote the elementary abelian group of order p^n, $\mathbf{Z}_n$ or simply n to denote the cyclic group of order n, and D_{2n} to denote the dihedral group of order $2n$.

Let G be the simple Conway's third group Co_3. Then

$$|G| = 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 13 \cdot 23,$$

and we may suppose $p \in \{2, 3, 5\}$, since both conjectures hold for a block with a cyclic defect group by [6].

We denote by $\mathrm{Irr}^0(H)$ the set of ordinary irreducible characters of p-defect 0 of a finite group H. Given $R \in \Phi(G, p)$, let $C(R) = C_G(R)$ and $N = N_G(R)$. If $B_0 = B_0(G)$ is the principal p-block of G, then by [3, (1.3)],

$$\mathcal{W}(B_0) = \sum_R |\,\mathrm{Irr}^0(N/C(R)R)|, \tag{2.1}$$

where R runs over the set $\Phi(G, p)$ such that the p-part $|C(R)R/R|_p$ of $|C(R)R/R|$ is 1. The character table of $N/C(R)R$ can be created by CAYLEY, so that we can find the number $|\,\mathrm{Irr}^0(N/C(R)R)|$.

The following proposition follows by Lemmas 5.7 and 5.14 of [7].

(2A). *The non-trivial radical* 5*-subgroups* R *of* Co_3 (*up to conjugacy*) *are*

R	$C(R)$	N	$\|\,\mathrm{Irr}^0(N/C(R)R)\|$
5	$5 \times A_5$	$(5 \times A_5).4$	
5_+^{1+2}	5	$5_+^{1+2}.(3 \times 8).2$	18.

(2B). *The non-trivial radical* 3*-subgroups* R *of* Co_3 (*up to conjugacy*) *are*

R	$C(R)$	N	$\|\,\mathrm{Irr}^0(N/C(R)R)\|$
3	$3 \times L_2(8):3$	$S_3 \times L_2(8):3$	
3_+^{1+4}	3	$3_+^{1+4}\!:4S_6$	4
3^5	3^5	$3^5\!:(2 \times M_{11})$	2
S	3	$S:(2 \times SD_{2^4})$	14,

where $S \in \mathrm{Syl}_3(\mathrm{Co}_3)$ *and* SD_{2^4} *is a semidihedral group of order* 2^4.

Proof. Given $i \in \{1, 2, 3\}$, let M_i be a maximal subgroup of $G = \mathrm{Co}_3$ such that $M_1 \simeq S_3 \times L_2(8){:}\,3$, $M_2 \simeq 3_+^{1+4} : 4S_6$ and $M_3 \simeq 3^5\!:(2 \times M_{11})$. If T is an elementary 3-subgroup, then by [7, p.73], $N(T) \leq M_i$ for some i. If R is a non-trivial radical 3-subgroup of G, then $\Omega_1(Z(R))$ is elementary abelian and $N(R) \leq N(\Omega_1(Z(R)))$. Thus we may suppose $R \in \Phi(M_i, 3)$ for some i and $N(R) = N_{M_i}(R)$.

Let $3 = O_3(M_1)$ and $S' \in \mathrm{Syl}_3(M_1)$. Then

$$\Phi(S_3 \times L_2(8){:}\,3, 3) = \{3, S'\}$$

and $N_{M_1}(S') \neq N(S')$. Thus we may suppose $3 \in \Phi(G, 3)$.

Let $3_+^{1+4} = O_3(M_2)$ and $3^5 = O_3(M_3)$. Then

$$\Phi(M_2, 3) = \{3_+^{1+4}, S\} \quad \text{and} \quad \Phi(M_3, 3) = \{3^5, S\},$$

and moreover, $N_{M_i}(R) = N(R)$ for each $R \in \Phi(M_i, 3)$. Thus we may suppose $\Phi(M_i, 3) \subseteq \Phi(G, 3)$ for $i = 2, 3$ and each non-trivial element of $\Phi(G, 3)$ is given (up to conjugacy) by (2B). In addition, $|\operatorname{Irr}^0(4S_6)| = 4$. Since $|\operatorname{Irr}^0(M_{11})| = 1$, it follows that $|\operatorname{Irr}^0(M_3/3^5)| = 2$. If $Q \in \mathrm{Syl}_3(M_{11})$, then $N_{M_{11}}(Q)/Q \simeq SD_{2^4}$ (cf. [3, (2A)]). Since $|N(S)/S| = 2^5$, it follows that $N(S) \simeq S{:}\,(2 \times SD_{2^4})$ and $|\operatorname{Irr}^0(N(S)/S)| = 14$.

(2C). *The non-trivial radical* 2*-subgroups* R *of* Co_3 (*up to conjugacy*) *are*

R	$C(R)$	$N/C(R)R$	$\lvert\operatorname{Irr}^0(N/C(R)R)\rvert$
2	$2.S_6(2)$	1	
2^*	$2^* \times M_{12}$	1	
2^2	$2^2 \times S_5$	3	
2^3	$2^3 \times S_3$	F_7^3	
2^4	2^4	A_8	1
$2^2.2^4$	2^2	S_6	1
2_+^{1+6}	2	$L_3(2)$	1
$2^3.2^4$	2^3	$L_3(2)$	1
$2^2.2^6$	2^2	$F_{3^2}^2.S_3$	1
$2.2^3.2^5$	2	S_3	1
$(2.2^3.2^5)^*$	2	S_3	1
$2.2^4.2^4$	2	S_3	1
$(2.2^4.2^4)^*$	2	S_3	1
S	2	1	$1,$

where $S \in \mathrm{Syl}_2(\mathrm{Co}_3)$, H^* *denotes a non-conjugate subgroup of* Co_3 *which is isomorphic to* H, *and* $F_{n^a}^m$ *denotes the Frobenius group with kernel* E_{n^a} *and complement* Z_m.

Proof. For $1 \leq i \leq 6$, let M_i be the maximal subgroup of G such that $M_1 \simeq 2.S_6(2)$, $M_2 \simeq 2^4.A_8$, $M_3 \simeq 2 \times M_{12}$, $M_4 \simeq 2^2.[2^7.3^2].S_3$, $M_5 \simeq A_4 \times S_5$ and $M_6 \simeq S_3 \times L_2(8){:}\,3$.

If $1 \neq R \in \Phi(G,2)$, then $N(R) \leq N(W)$, where $W = \Omega_1(Z(R))$. Thus $N(R) \leq N(A)$ for some $2A$-pure or $2B$-pure elementary abelain 2-group A (see [7, p.68]). It follows by Lemmas 5.8, 5.9 and 5.10 of [7] that $N(A) \leq M_i$ for $1 \leq i \leq 5$, except when A is a $2B$-pure and $A \simeq 2^3$, in which case $N(A) \leq M_6$ (see the remark after Lemma 5.10 of [7]). Thus $N(R) \leq_G M_i$ for some i, and we may suppose $R \in \Phi(M_i, 2)$ with $N(R) = N_{M_i}(R)$.

(1) Let $2 = O_2(M_1)$. Then

$$\Phi(M_1,2) = \{2, 2^2.2^4, 2_+^{1+6}, 2^2.2^6, 2.2^3.2^5, (2.2^3.2^5)^*, (2.2^4.2^4)^*, S\}, \tag{2.2}$$

where $S \in \mathrm{Syl}_2(G)$. Moreover, $N_{M_1}(R) = N(R)$ for each $R \in \Phi(M_1,2)\backslash\{2^2.2^6\}$.

(2) Let $2^4 = O_2(M_2)$. Then

$$\Phi(M_2,2) = \{2^4, 2_+^{1+6}, 2^3.2^4, 2^2.2^6, 2.2^3.2^5, (2.2^3.2^5)^*, (2.2^4.2^4)^*, S\}, \tag{2.3}$$

and moreover, $N_{M_2}(R) = N(R)$ for each $R \in \Phi(M_2,2)\backslash\{2^2.2^6\}$.

(3) Let $2^* = O_2(M_3)$. Then 2^* is generated by a $2B$-element. By [3, (2D)], $\Phi(M_{12},2) = \{1, \mathrm{Z}_2, E_4, E_4.E_8, 2_+^{1+4}, Q\}$, where $Q \in \mathrm{Syl}_2(M_{12})$. Thus

$$\Phi(M_3,2) = \{2^*, 2^2, 2^3, 2 \times 2_+^{1+4}, 2^3.2^3, 2^3.2^3.2\}, \tag{2.4}$$

where $2^2 = 2^* \times \mathrm{Z}_2$, $2^3 = 2^* \times E_4$, $2^3.2^3 = 2^* \times E_4.E_8$, and $2^3.2^3.2 = 2 \times Q$. Moreover, $N_{M_3}(R) \neq N(R)$ for $R \in \Phi(M_3,2)\backslash\{2^*\}$.

(4) Let $2^2.2^6 = O_2(M_4)$. Then

$$\Phi(M_4,2) = \{2^2.2^6, (2.2^3.2^5)^*, 2.2^4.2^4, (2.2^4.2^4)^*, S\}, \tag{2.5}$$

and moreover, $N_{M_4}(R) = N(R)$ for all $R \in \Phi(M_4,2)$.

(5) Let $2^2 = O_2(M_5)$. Since $\Phi(S_5,2) = \{1, \mathrm{Z}_2, E_4, D_8\}$, it follows that

$$\Phi(M_5,2) = \{2^2, 2^3, (2^4)^*, 2^2 \times D_8\}, \tag{2.6}$$

and moreover, $N_{M_5}(R) \neq N(R)$ for each $R \in \Phi(M_5,2)\backslash\{2^2\}$.

(6) Finally, if $\mathrm{Z}_2 = O_2(S_3)$ and $Q \in \mathrm{Syl}_2(M_6)$, then

$$\Phi(M_6,2) = \{\mathrm{Z}_2, 2^3, Q\}$$

and moreover, $N_{M_6}(R) \neq N(R)$ unless $R = 2^3$.

Thus the nontrivial radical subgroups are given by (2C), and their normalizer and centralizer are obtained by CAYLEY.

Denote by $D(B)$ a defect group of a block B, $\operatorname{Irr}(B)$ the set of irreducible ordinary characters of B, and $\mathrm{k}(B) = |\operatorname{Irr}(B)|$.

(2D). *Let $G = \mathrm{Co}_3$ and let $\operatorname{Blk}^0(G,p)$ be the set of p-blocks with a non-trivial defect group.*

(a) *If $p = 5$ or 3, then $\operatorname{Blk}^0(G,p) = \{B_0, B_1\}$ such that $D(B_1) \simeq \mathrm{Z}_p$. In the notation of* [5, p.135]

$$\operatorname{Irr}(B_1) = \begin{cases} \{\chi_5, \chi_{12}, \chi_{29}, \chi_{35}, \chi_{39}\} & \text{if } p = 5, \\ \{\chi_{31}, \chi_{32}, \chi_{36}\} & \text{if } p = 3. \end{cases}$$

(b) *If $p = 2$, then $\operatorname{Blk}^0(G,2) = \{B_0, B_1, B_2\}$ such that $D(B_1) =_G 2$, $D(B_2) =_G 2^3$ and $\ell(B_2) = 5$. In the notation of* [5, p.135], $\operatorname{Irr}(B_1) = \{\chi_{33}, \chi_{34}\}$ *and*

$$\operatorname{Irr}(B_2) = \{\chi_6, \chi_7, \chi_{18}, \chi_{19}, \chi_{29}, \chi_{32}, \chi_{38}, \chi_{39}\}.$$

Moreover,

$$\ell(B_1) = \begin{cases} 4 & \text{if } p = 5, \\ 2 & \text{if } p = 3, \\ 1 & \text{if } p = 2. \end{cases} \qquad \ell(B_0) = \begin{cases} 18 & \text{if } p = 5, \\ 20 & \text{if } p = 3, \\ 10 & \text{if } p = 2. \end{cases} \tag{2.7}$$

Proof. If $B \in \operatorname{Blk}(G,p)$ is non-principal with $D = D(B)$, then $\operatorname{Irr}^0(C(D)D/D)$ has a non-trivial character, so by (2A), (2B) and (2C), $D \in_G \{5, 3, 2, 2^3\}$. Moreover, for each such D, $|\operatorname{Irr}^0(C(D)D/D)| = 1$, so G has exactly one block with a defect group D.

Using the method of central characters on elements of types, 2B, 3C and 5A, we have $\operatorname{Irr}(B)$ given as in (2D). If $D(B)$ is cyclic, then $\ell(B)$ is the number of B-weights. Thus $\ell(B_1)$ is given as (2.7).

Suppose $p = 2$ and $B = B_2$. Then $\mathrm{k}(B) = 8$ and $D(B) =_G 2^3$. Let $K = C(2^*) = 2^* \times M_{12}$, and let $b \in \operatorname{Blk}(K)$ such that $b^G = B$. Then $D(b) \simeq 2^3$ and $b = b_0 \times b_1$, where b_0 is the principal block of 2^* and $b_1 \in \operatorname{Blk}(M_{12})$ with $D(b_1) \simeq E_4$. Now M_{12} has exactly one block b_1 with a defect group E_4 and by [3, (2D)], $\ell(b_1) = 3$. Thus K has exactly one block b such that $b^G = B$, and moreover, $\ell(b) = 3$. As shown on page 72 of [7], 2^3 is $2B$-pure, so that $\mathrm{k}(B) = \ell(B) + \ell(b)$ and $\ell(B) = 5$.

Finally, let $\ell(G)$ be the number of p-regular conjugacy G-classes. Then $\ell(B_0)$ can be calculated by the following equation due to Brauer,

$$\ell(G) = \bigcup_{B \in \operatorname{Blk}^0(G,p)} \ell(B) + |\operatorname{Irr}^0(G)|.$$

This completes the proof.

(2E). *Let $G = \mathrm{Co}_3$ and B a p-block of G with a non-cyclic defect group. Then the number of B-weights is the number of irreducible Brauer characters of B.*

Proof. If $B = B_0$, then (2E) follows by (2.1) and (2A)-(2D). If $p = 2$ and $B = B_2$, then by [3, (1.4)] and (2C), the number of B-weights is 5, since $|\operatorname{Irr}(F_7^3)| = 5$.

3. Radical Chains

The notation and terminology of Sections 1 and 2 are continued in this section. Let $G = \mathrm{Co}_3$, $C \in \mathcal{R}(G)$ and $N(C) = N_G(C)$.

(3A). *Follow the notation of* (2A). *The radical* 5*-chains* C *of* G (*up to conjugacy*) *are*

C	$N(C)$	C	$N(C)$
$C(1) : 1$	G	$C(2) : 1 < 5$	$(5 \times A_5).4$
$C(3) : 1 < 5 < 5^2$	$(5 \times F_5^2).4$	$C(4) : 1 < 5_+^{1+2}$	$5_+^{1+2}.24.2$,

where $5^2 \in \mathrm{Syl}_5(5 \times A_5)$.

Proof. It is straightforward.

In the notation of (2B), define the radical 3-chains $C(i)$ for $1 \le i \le 6$ as follows:

$$
\begin{array}{ll}
C(1) : 1 & C(2) : 1 < 3 \\
C(3) : 1 < 3 < S' & C(4) : 1 < 3_+^{1+4} \\
C(5) : 1 < 3^5 < S & C(6) : 1 < 3^5,
\end{array}
\tag{3.1}
$$

where $S' \in \mathrm{Syl}_3(S_3 \times L_2(8){:}\,3)$.

(3B). (a) *Let* $\mathcal{R}^0(G)$ *be the* G*-invariant subfamily of* $\mathcal{R}(G)$ *such that* $\mathcal{R}^0(G)/G = \{C(i) : 1 \le i \le 6\}$. *Then*

$$
\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|}\, \mathrm{k}(N(C), B_0, d) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|}\, \mathrm{k}(N(C), B_0, d)
$$

for all integers $d \ge 0$.

(b) *Suppose* C *is a chain given by* (3.1). *Then*

C	$N(C)$	C	$N(C)$
$C(1)$	G	$C(2)$	$S_3 \times L_2(8) : 3$
$C(3)$	$S'.2^2$	$C(4)$	$3_+^{1+4} : 4S_6$
$C(5)$	$S : (2 \times SD_{2^4})$	$C(6)$	$3^5 : (2 \times M_{11})$.

Proof. Let $C : 1 < S$ and $C' : 1 < 3_+^{1+4} < S$. Then $N(C) = N(C') = N(S)$, and we can delete C and C' in the sum (1.3). The rest of the proof is clear from the proof of (2B).

Following the notation of (2C), we define radical 2-chains: $C(i)$ for $1 \le i \le 16$ as follows:

	$C(1): 1$	$C(2): 1 < 2$
	$C(3): 1 < 2 < 2^2.2^6$	$C(4): 1 < 2^2.2^6$
	$C(5): 1 < 2^4 < 2_+^{1+6}$	$C(6): 1 < 2^4$
	$C(7): 1 < 2^4 < 2^2.2^6$	$C(8): 1 < 2^4 < 2^2.2^6 < (2.2^3.2^5)^*$
	$C(9): 1 < 2^* < 2^3.2^3$	$C(10): 1 < 2^*$
	$C(11): 1 < 2^* < 2 \times 2_+^{1+4}$	$C(12): 1 < 2^* < 2^3 < (2^4)^*$
	$C(13): 1 < 2^* < 2^3$	$C(14): 1 < 2^* < 2^3.2^3 < 2^3.2^3.2$
(3.2)	$C(15): 1 < 2^3 < (2^4)^*$	$C(16): 1 < 2^3,$

where $2^3.2^3, 2 \times 2_+^{1+4}, 2^3.2^3.2 \in \Phi(M_3, 2)$, and $(2^4)^* \in \mathrm{Syl}_2(N(2^3))$. We have the following proposition:

(3C). (a) *Let $\mathcal{R}^0(G)$ be the G-invariant subfamily of $\mathcal{R}(G)$ such that $\mathcal{R}^0(G)/G = \{C(k) : k = 1, 2, \ldots, 16\}$. Then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B, d) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B, d)$$

for all integers $d \ge 0$ and for each block B with a non-cyclic defect group.

(b) *Let C be a chain given by* (3.2). *Then*

C	$N(C)$	C	$N(C)$
$C(1)$	Co_3	$C(2)$	$2.S_6(2)$
$C(3)$	$2^2.2^6.F_{3^2}^2.2$	$C(4)$	$2^2.2^6.F_{3^2}^2.S_3$
$C(5)$	$2_+^{1+6}.L_3(2)$	$C(6)$	$2^4.A_8$
$C(7)$	$2^2.2^6.S_3.S_3$	$C(8)$	$(2.2^3.2^5)^*.S_3$
$C(9)$	$2^3.2^3.S_3$	$C(10)$	$2^* \times M_{12}$
$C(11)$	$2 \times 2_+^{1+4}.S_3$	$C(12)$	$2^2 \times A_4$
$C(13)$	$2 \times A_4 \times S_3$	$C(14)$	$2^3.2^3.2$
$C(15)$	$(2^4)^*.F_7^3$	$C(16)$	$(2^3 \times S_3).F_7^3$

Proof. (a) Suppose C' is a radical 2-chain such that

$$C' : 1 < P_1' < \ldots < P_m'. \tag{3.3}$$

Let $C \in \mathcal{R}(G)$ be given by (1.1) with $P_1 \in \Phi(G, 2)$.

Case (1). Let $R \in \Phi(M_1, 2) \backslash \{2, 2^2.2^6\}$. Define G-invariant subfamilies $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$ of $\mathcal{R}(G)$, such that

$$\begin{aligned} \mathcal{M}^+(R)/G &= \{C' \in \mathcal{R}/G : P_1' = R\}, \quad \text{and} \\ \mathcal{M}^0(R)/G &= \{C' \in \mathcal{R}/G : P_1' = 2, P_2' = R\}. \end{aligned} \tag{3.4}$$

For $C' \in \mathcal{M}^+(R)$ given by (3.3), the chain

$$g(C') : 1 < 2 < P'_1 = R < P'_2 < \ldots < P'_m \tag{3.5}$$

is a chain in $\mathcal{M}^0(R)$ and $N(C') = N(g(C'))$. For any $B \in \mathrm{Blk}(G)$ and for any integer $d \geq 0$,

$$\mathrm{k}(N(C'), B, d) = \mathrm{k}(N(g(C')), B, d). \tag{3.6}$$

In addition, g is a bijection between $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$. So we may suppose

$$C \notin \bigcup_{R \in \Phi(M_1,2)\setminus\{2,2^2.2^6\}} (\mathcal{M}^+(R) \cup \mathcal{M}^0(R)).$$

In particular, $P_1 \notin \{2^2.2^4, 2_+^{1+6}, 2.2^3.2^5, (2.2^3.2^5)^*, (2.2^4.2^4)^*, S\}$, and if $P_1 = 2$, then $C =_G C(2)$ or $P_2 = 2^2.2^6$.

Case (2). Replace R by $2.2^4.2^4 \in \Phi(M_4, 2)$ and 2 by $2^2.2^6 \in \Phi(M_4, 2)$ in the definition (3.4), and repeat the proof above. Then we may suppose $P_1 \neq 2.2^4.2^4$, and if $P_1 = 2^2.2^6$, then $P_2 \neq_G 2.2^4.2^4$. Now $N_{M_1}(2^2.2^6) = N(C(3)) \simeq 2^2.2^6.F_{3^2}^2.2$, so

$$\Phi(2^2.2^6.F_{3^2}^2.2, 2) = \{2^2.2^6, (2.2^3.2^5)^*, (2.2^4.2^4)^*, S\} \subseteq \Phi(M_4, 2),$$

and in addition, $N_{N(C(3))}(R) = N(R) = N_{M_4}(R)$ for $R \in \Phi(2^2.2^6.F_{3^2}^2.2, 2)\setminus\{2^2.2^6\}$.

Let $\Omega = \{(2.2^3.2^5)^*, (2.2^4.2^4)^*, S\} \subseteq \Phi(2^2.2^6.F_{3^2}^2.2, 2)$. Given $Q \in \Omega$, define G-invariant subfamilies $\mathcal{L}^+(Q)$ and $\mathcal{L}^0(Q)$ of $\mathcal{R}(G)$, such that

$$\begin{aligned} \mathcal{L}^+(Q)/G &= \{C' \in \mathcal{R}/G : P'_1 = 2^2.2^6, P'_2 = Q\}, \quad \text{and} \\ \mathcal{L}^0(Q)/G &= \{C' \in \mathcal{R}/G : P'_1 = 2, P'_2 = 2^2.2^6, P'_3 = Q\}. \end{aligned} \tag{3.7}$$

A similar proof to above shows that we may suppose

$$C \notin \bigcup_{Q \in \Omega} (\mathcal{L}^+(Q) \cup \mathcal{L}^0(Q)). \tag{3.8}$$

It follows that we may suppose

$$P_1 \in \{2, 2^*, 2^2, 2^3, 2^4, 2^3.2^4, 2^2.2^6\},$$

and moreover, if $P_1 \in \{2, 2^2.2^6\}$, then $C \in_G \{C(2), C(3), C(4)\}$.

Case (3). Let $\mathcal{M}^+(2^3.2^4)$ and $\mathcal{M}^0(2^3.2^4)$ be given by (3.4) with R replaced by $2^3.2^4$ and 2 by 2^4. Then (3.6) holds for $C' \in \mathcal{M}^+(2^3.2^4)$ and we may suppose $P_1 \neq 2^3.2^4$ and if $P_1 = 2^4$, then $P_2 \neq_G 2^3.2^4$. Since $N(2_+^{1+6}) \simeq 2_+^{1+6}.L_3(2) \leq M_2$, it follows that

$$\Phi(2_+^{1+6}.L_3(2), 2) = \{2_+^{1+6}, 2.2^3.2^5, (2.2^3.2^5)^*, S\} \subseteq \Phi(M_2, 2),$$

and moreover, $N_{N(2_+^{1+6})}(R) = N_{M_2}(R) = N(R)$ for all $R \in \Phi(N(2_+^{1+6}), 2)$. Let $\Omega' = \{2.2^3.2^5, (2.2^3.2^5)^*, S\} \subseteq \Phi(2_+^{1+6}.L_3(2), 2)$, and $W \in \Omega'$. Define G-invariant subfamilies $\mathcal{K}^+(W)$ and $\mathcal{K}^0(W)$ of $\mathcal{R}(G)$, such that

$$\begin{aligned} \mathcal{K}^+(W)/G &= \{C' \in \mathcal{R}/G : P_1' = 2^4, P_2' = W\}, \quad \text{and} \\ \mathcal{K}^0(W)/G &= \{C' \in \mathcal{R}/G : P_1' = 2^4, P_2' = 2_+^{1+6}, P_3' = W\}. \end{aligned} \tag{3.9}$$

A similar proof to above shows that we may suppose

$$C \notin \bigcup_{W \in \Omega'} (\mathcal{K}^+(W) \cup \mathcal{K}^0(W)).$$

Thus if $P_1 = 2^4$, then we may suppose $P_2 \in \{2_+^{1+6}, 2^2.2^6, (2.2^4.2^4)^*\}$, and moreover, if $P_2 =_G 2_+^{1+6}$, then $C =_G C(5)$.

Similarly, $N_{M_2}(2^2.2^6) \simeq 2^2.2^6.S_3.S_3$, and

$$\Phi(2^2.2^6.S_3.S_3, 2) = \{2^2.2^6, (2.2^3.2^5)^*, (2.2^4.2^4)^*, S\},$$

and moreover, $N_{N_{M_2}(2^2.2^6)}(R) = N_{M_2}(R) = N(R)$ for each $R \in \Phi(N_{M_2}(2^2.2^6), 2)\backslash\{2^2.2^6\}$. Replace W by $(2.2^4.2^4)^*$ and 2_+^{1+6} by $2^2.2^6$ in the definition (3.9). We may suppose $P_2 \neq_G (2.2^4.2^4)^*$, and if $P_1 = 2^4$ and $P_2 =_G 2^2.2^6$, then $P_3 \not\simeq 2.2^4.2^4$.

Let C' be the chain $1 < 2^4 < 2^2.2^6 < (2.2^3.2^5)^* < S$, and $g(C') : 1 < 2^4 < 2^2.2^6 < S$. Then $N(C') = N(g(C'))$ and (3.6) holds. It follows that if $P_1 = 2^4$, then $C \in_G \{C(5), C(6), C(7), C(8)\}$.

Case (4). If $P_1 = 2^3$, then $C = C(15)$ or $C(16)$, since $N(2^3) \simeq (2^3 \times S_3).F_7^3$ and $\Phi(N(2^3), 2) = \{2^3, (2^4)^*\}$.

Case (5). Let $C' : 1 < 2^* < S'$ and $g(C') : 1 < 2^* < 2 \times 2_+^{1+4} < S'$, where $2 \times 2_+^{1+4}, S' = 2^3.2^3.2 \in \Phi(M_3, 2)$. Then $N(C') = N(g(C')) = S'$ and (3.6) holds.

Since $N_{M_3}(2^2) = 2 \times N_{M_{12}}(\mathrm{Z}_2) \simeq 2^2 \times S_5$ (see [3, (2D)]), it follows that

$$\Phi(2^2 \times S_5, 2) = \{2^2, 2^3, (2^4)^*, 2^2 \times D_8\}.$$

Let $\Omega^* = \{2^3, (2^4)^*, 2^2 \times D_8\} \subseteq \Phi(N_{M_3}(2^2), 2)$. Then we may suppose $\Omega^* \subseteq \Phi(M_5, 2)$. For each $R \in \Omega^*$, $R = 2^4 \times R'$ for some $R' \leq S_5$, $N_{N_{M_3}(2^2)}(R) \simeq 2^2 \times N_{S_5}(R')$ and $N_{M_5}(R) \simeq A_4 \times N_{S_5}(R')$. Replace Q by a group Q' in Ω^*, $2^2.2^6$ by 2^2 and 2 by 2^* in the definition (3.7). If $C' \in \mathcal{L}^+(Q')$ and P is the final subgroup of C', then $P = 2^2 \times P'$ and $N(C') = 2^2 \times N'$, where P' and N' are some subgroups of S_5. Let $g(C')$ be a chain of $\mathcal{L}^0(Q')$ defined similar to (3.5). Then P is also the final subgroup of $g(C')$ and $N(g(C')) \simeq A_4 \times N'$. Note that both 2^2 and A_4 have exactly 4 irreducible characters of defect 2, and all of them lie in their principal block. It follows that we may suppose

(3.8) holds for $Q' \in \Omega^*$. Similarly, if $C' : 1 < 2^* < 2^2$ and $g(C') : 1 < 2^2$, then $N(g(C')) \simeq A_4 \times S_5$ and $N(C') \simeq 2^2 \times S_5$. Thus (3.6) still holds. Hence if $P_1 \in \{2^*, 2^2\}$, then $C \in_G \{C(i) : i = 9, \ldots, 14\}$.

(b) The proof of (b) follows easily by that of (a) above or (2C).

4. The Proof of Dade's Conjecture for Co_3

The notation and terminology of Sections 2 and 3 are continued in this section.

(4A). *Let B be a p-block of the simple Conway group $G = \mathrm{Co}_3$ with a defect positive defect. If p is odd, then B satisfies the ordinary conjecture of Dade.*

Proof. We may suppose $p = 5$ or 3, and $B = B_0$.

Suppose $p = 5$ and let $C = C(2)$, $C' = C(3)$. Then $N(C) \simeq (5 \times A_5).4$ and $N(C') \simeq (5 \times F_5^2).4$. The principal blocks of $N(C)$ and $N(C')$ both have exactly 20 irreducible characters of height 0, so that

$$\mathrm{k}(N(C), B_0, d) = \mathrm{k}(N(C'), B_0, d).$$

for all integers $d \geq 0$. The degrees of irreducible characters of $N(C(4)) \simeq 5_+^{1+2}.24.2$ is given by Table 1.

Table 1. The degrees of characters of $\mathrm{Irr}(5_+^{1+2}.24.2)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}
1	1	1	1	1	1	1	1	2	2	2	2	2	2
ξ_{15}	ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}		
2	2	2	2	20	20	20	20	24	24	40	40		

It follows by [5, p.135] and (2D) that

$$\mathrm{k}(G, B_0, d) = \mathrm{k}(N(C(4))), B_0, d) = \begin{cases} 20 & \text{if } d = 3, \\ 6 & \text{if } d = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus (4A) follows when $p = 5$.

Suppose $p = 3$. Let $C = C(2)$ and $C' = C(3)$ given by (3.1). The degrees of characters in $\mathrm{Irr}(N(C))$ and $\mathrm{Irr}(N(C'))$ are given by Tables 2 and 3.

TABLE 2. The degrees of characters of $\mathrm{Irr}(S_3 \times L_2(8):3)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}	ξ_{16}	ξ_{17}
1	1	1	1	1	1	2	2	2	7	7	7	7	7	7	8	8
ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}	ξ_{29}	ξ_{30}	ξ_{31}	ξ_{32}	ξ_{33}	
8	8	8	8	14	14	14	16	16	16	21	21	27	27	42	54	

TABLE 3. The degrees of characters of $\mathrm{Irr}(S'.2^2)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}
1	1	1	1	1	1	1	1	1	1	1	1	2	2	2
ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}	ξ_{29}	ξ_{30}
2	2	2	2	2	2	2	2	2	4	4	4	6	6	12

It follows that

$$\mathrm{k}(N(C(2)), B_0, d) = \mathrm{k}(N(C(3)), B_0, d) = \begin{cases} 27 & \text{if } d = 4, \\ 3 & \text{if } d = 3, \\ 0 & \text{otherwise.} \end{cases}$$

The degrees of irreducible characters of $N(C(5)) \simeq S\colon(2 \times SD_{2^4})$ and $N(C(6))) \simeq 3^5\colon(2 \times M_{11})$ are given as follows:

TABLE 4. The degrees of characters of $\mathrm{Irr}(S\colon(2 \times SD_{2^4}))$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}	ξ_{16}	ξ_{17}	ξ_{18}
1	1	1	1	1	1	1	1	2	2	2	2	2	2	4	4	4	4
ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}	ξ_{29}	ξ_{30}	ξ_{31}	ξ_{32}	ξ_{33}	ξ_{34}	ξ_{35}	ξ_{36}
4	4	4	4	8	8	8	8	8	8	16	16	16	16	18	18	18	18
ξ_{37}	ξ_{38}	ξ_{39}	ξ_{40}	ξ_{41}	ξ_{42}	ξ_{43}	ξ_{44}	ξ_{45}	ξ_{46}	ξ_{47}	ξ_{48}	ξ_{49}	ξ_{50}	ξ_{51}	ξ_{52}		
32	36	36	36	72	72	72	72	72	72	72	72	72	72	72	72		

TABLE 5. The degrees of characters of $\mathrm{Irr}(3^5\colon(2 \times M_{11}))$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}
1	1	10	10	10	10	10	10	11	11	16	16	16	16
ξ_{15}	ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}
22	22	44	44	45	45	55	55	198	198	220	220	220	220
ξ_{29}	ξ_{30}	ξ_{31}	ξ_{32}	ξ_{33}	ξ_{34}	ξ_{35}	ξ_{36}	ξ_{37}					
220	220	220	352	440	880	880	880	880					

It follows that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 33 & \text{if } d = 7, \\ \alpha & \text{if } d = 5, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(5), C(6)\}$ and $\alpha = 19$ or 4 according as $C = C(5)$ or $C(6)$. Finally, consider $C = C(4)$. The degrees of $N(C) \simeq 3_+^{1+4}{:}\,4S_6$ are given by Table 6.

TABLE 6. The degrees of characters of $\mathrm{Irr}(3_+^{1+4}{:}\,4S_6)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}	ξ_{16}	ξ_{17}	ξ_{18}
1	1	1	1	5	5	5	5	5	5	5	5	8	8	9	9	9	9
ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}	ξ_{29}	ξ_{30}	ξ_{31}	ξ_{32}	ξ_{33}	ξ_{34}	ξ_{35}	ξ_{36}
10	10	10	10	16	16	16	16	18	18	20	20	72	72	72	72	80	80
ξ_{37}	ξ_{38}	ξ_{39}	ξ_{40}	ξ_{41}	ξ_{42}	ξ_{43}	ξ_{44}	ξ_{45}	ξ_{46}	ξ_{47}	ξ_{48}	ξ_{49}	ξ_{50}	ξ_{51}	ξ_{52}	ξ_{53}	ξ_{54}
80	80	90	90	90	90	160	160	160	160	162	162	180	180	288	288	320	360

It follows by [5, p.134] that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 33 & \text{if } d = 7, \\ \beta & \text{if } d = 5, \\ 2 & \text{if } d = 3, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(1), C(4)\}$ and $\beta = 4$ or 19 according as $C = C(1)$ or $C(4)$. Thus Dade's conjecture holds when $p = 3$.

(4B). *Let B be a 2-block of the simple Conway group $G = \mathrm{Co}_3$ with defect $\mathrm{d}(B) \geq 1$. Then B satisfies the ordinary conjecture of Dade.*

Proof. (1) Let $C(i)$ be given by (3.2). If $b^G = B_2$ for some $b \in \mathrm{Blk}(N(C(i)))$, then $i \in \{1, 10, 13, 16\}$. Let $C = C(13)$ and $C' = C(16)$. Then $N(C) \simeq 2 \times A_4 \times S_3$ and $N(C') \simeq (2^3 \times S_3).F_7^3$ both have exactly two blocks, principal block b_0 and b_2, where $b_2^G = B_2$. In addition, the degrees of characters in $\mathrm{Irr}(N(C'))$ are given by

TABLE 7. The degrees of characters of $\mathrm{Irr}((2^3 \times S_3).F_7^3)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}
1	1	1	1	1	1	2	2	2	3	3	3	3	6
ξ_{15}	ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}				
6	7	7	7	7	7	7	14	14	14				

It follows that

$$\mathrm{k}(N(C(13)), B, d) = \mathrm{k}(N(C(16)), B, d) = \begin{cases} \alpha & \text{if } d = 4, \\ \beta & \text{if } d = 3, \\ 0 & \text{otherwise,} \end{cases}$$

where $(\alpha, \beta) = (16, 0)$ or $(0, 8)$ according as $B = B_0$ or B_2. Since $N(C(10)) \simeq 2^* \times M_{12}$, it follows by James [8, Theorem 8.2] and (2D) that $\mathrm{k}(G, B_2, d) = \mathrm{k}(N(C(10)), B_2, d) = 8$ or 0 according as $d = 3$ or $d \neq 3$. This proves (4B) when $B = B_2$.

(2) Suppose $B = B_0$. The degrees of irreducible characters of $N(C(9)) \simeq 2^3.2^3.S_3$ and $N(C(14)) \simeq 2^3.2^3.2$ are given in Tables 8 and 9.

TABLE 8. The degrees of characters of $\mathrm{Irr}(2^3.2^3.S_3)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}
1	1	1	1	1	1	1	1	2	2	2	2	3	3
ξ_{15}	ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}
3	3	3	3	3	3	6	6	6	6	6	6	6	6

TABLE 9. The degrees of characters of $\mathrm{Irr}(2^3.2^3.2)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}	ξ_{16}
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}	ξ_{29}	ξ_{30}	ξ_{31}	ξ_{32}
2	2	2	2	2	2	2	2	2	2	2	2	4	4	4	4

It follows that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 16 & \text{if } d = 7, \\ 12 & \text{if } d = 6, \\ \gamma & \text{if } d = 5, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(9), C(14)\}$ and $\gamma = 0$ or 4 according as $C = C(9)$ or $C(14)$. The degrees of irreducible characters of $N(C(11)) \simeq 2 \times 2_+^{1+4}.S_3$ are given in the following table:

TABLE 10. The degrees of characters of $\mathrm{Irr}(2 \times 2_+^{1+4}.S_3)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}
1	1	1	1	2	2	3	3	3	3	3	3	3	3
ξ_{15}	ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}		
3	3	3	3	4	4	4	4	6	6	8	8		

If $C = C(10)$ or $C(11)$, then by Table 10 and James [8, Theorem 8.2],

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 16 & \text{if } d = 7, \\ 4 & \text{if } d = 6, \\ \eta & \text{if } d = 5, \\ 2 & \text{if } d = 4, \\ 0 & \text{otherwise,} \end{cases}$$

where $\eta = 0$ or 4 according as $C = C(10)$ or $C(11)$. Now $N(C(15)) \simeq (2^4)^*.F_7^3$ has 16 irreducible characters of height 0, and hence $\mathrm{k}(N(C(15)), B_0, d) = \mathrm{k}(N(C(12)), B_0, d) = 16$ or 0 according as $d = 4$ or $d \neq 4$. It follows that

$$\sum_{i=9}^{16} (-1)^{|C(i)|} \mathrm{k}(N(C(i)), B_0, d) = 0. \tag{4.1}$$

The degrees of irreducible characters of $N(C(3)) \simeq 2^2.2^6.F_{3^2}^2.2$ and $N(C(4)) \simeq 2^2.2^6.F_{3^2}^2.S_3$ are given by Tables 11 and 12.

TABLE 11. The degrees of characters of $\mathrm{Irr}(2^2.2^6.F_{3^2}^2.2)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}	ξ_{16}	ξ_{17}	ξ_{18}
1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4	4	6
ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}	ξ_{29}	ξ_{30}	ξ_{31}	ξ_{32}	ξ_{33}	ξ_{34}	ξ_{35}	ξ_{36}
6	6	6	8	8	8	8	8	8	9	9	9	9	9	9	9	9	12
ξ_{37}	ξ_{38}	ξ_{39}	ξ_{40}	ξ_{41}	ξ_{42}	ξ_{43}	ξ_{44}	ξ_{45}	ξ_{46}	ξ_{47}	ξ_{48}	ξ_{49}	ξ_{50}	ξ_{51}	ξ_{52}	ξ_{53}	ξ_{54}
12	12	12	12	16	16	16	16	16	18	18	18	18	24	24	24	24	48

TABLE 12. The degrees of characters of $\mathrm{Irr}(2^2.2^6.F_{3^2}^2.S_3)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}
1	1	1	1	2	2	2	2	4	6	6	9	9	9	9
ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}	ξ_{29}	ξ_{30}
9	9	9	9	12	12	12	12	18	18	18	18	18	18	24
ξ_{31}	ξ_{32}	ξ_{33}	ξ_{34}	ξ_{35}	ξ_{36}	ξ_{37}	ξ_{38}	ξ_{39}	ξ_{40}	ξ_{41}	ξ_{42}	ξ_{43}	ξ_{44}	ξ_{45}
24	24	24	27	27	27	27	36	36	36	36	36	48	72	72

If $C = C(3)$ or $C(4)$, then

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 16 & \text{if } d = 10, \\ 12 & \text{if } d = 9, \\ 10 & \text{if } d = 8, \\ \alpha & \text{if } d = 7, \\ \beta & \text{if } d = 6, \\ 0 & \text{otherwise,} \end{cases}$$

where $(\alpha, \beta) = (10, 6)$ or $(6, 1)$ according as $C = C(3)$ or $C = C(4)$. By (2C), $N(C(5)) \simeq 2_+^{1+6}.L_3(2)$ and $N(C(6)) \simeq 2^4.A_8$.

TABLE 13. The degrees of characters of $\mathrm{Irr}(2^4.A_8)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}
1	7	14	15	20	21	21	21	28	35	45	45	45	45
ξ_{15}	ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}			
56	64	70	90	105	105	105	120	210	315	315			

By Tables 13 and [2, Table 10],

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 16 & \text{if } d = 10, \\ 4 & \text{if } d = 9, \\ 2 & \text{if } d = 8, \\ x & \text{if } d = 7, \\ y & \text{if } d = 6, \\ 1 & \text{if } d = 4, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(5), C(6)\}$ and $(x, y) = (6, 1)$ or $(2, 0)$ according as $C = C(5)$ or $C(6)$.

TABLE 14. The degrees of characters of $\mathrm{Irr}(2^2.2^6.S_3.S_3)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}
1	1	1	1	2	2	2	2	3	3	3	3	4	6	6
ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}	ξ_{29}	ξ_{30}
6	6	9	9	9	9	9	9	9	9	12	12	12	12	12
ξ_{31}	ξ_{32}	ξ_{33}	ξ_{34}	ξ_{35}	ξ_{36}	ξ_{37}	ξ_{38}	ξ_{38}	ξ_{39}	ξ_{40}				
18	18	18	18	24	24	36	36	36	36	36				

The degrees of irreducible characters of $N(C(8)) \simeq 2.2^3.2^5.S_3$ are given by Table 15.

TABLE 15. The degrees of characters of $\mathrm{Irr}((2.2^3.2^5).S_3)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}
1	1	1	1	2	2	3	3	3	3	3	3	3	3	3
ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}	ξ_{29}	ξ_{30}
3	3	3	4	4	4	4	6	6	6	6	6	6	6	6
ξ_{31}	ξ_{32}	ξ_{33}	ξ_{34}	ξ_{35}	ξ_{36}	ξ_{37}	ξ_{38}	ξ_{39}	ξ_{40}	ξ_{41}	ξ_{42}	ξ_{43}	ξ_{44}	ξ_{45}
6	6	8	8	8	8	12	12	12	12	12	12	16	24	24

It follows that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 16 & \text{if } d = 10, \\ 12 & \text{if } d = 9, \\ 10 & \text{if } d = 8, \\ u & \text{if } d = 7, \\ v & \text{if } d = 6, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(7), C(8)\}$, $(u, v) = (2, 0)$ or $(6, 1)$ according as $C = C(7)$ or $C(8)$.

TABLE 16. The degrees of characters of $\mathrm{Irr}(2.S_6(2))$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}
1	7	8	15	21	21	27	35	35	48	56	64	64	70	84
ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}	ξ_{29}	ξ_{30}
105	105	105	112	112	120	120	168	168	189	189	189	210	210	216
ξ_{31}	ξ_{32}	ξ_{33}	ξ_{34}	ξ_{35}	ξ_{36}	ξ_{37}	ξ_{38}	ξ_{39}	ξ_{40}	ξ_{41}	ξ_{42}	ξ_{43}		
280	280	280	315	336	378	405	420	448	512	512	560	720		

It follows by Table 15 and [5, p.135] that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 16 & \text{if } d = 10, \\ 4 & \text{if } d = 9, \\ 2 & \text{if } d = 8, \\ a & \text{if } d = 7, \\ b & \text{if } d = 6, \\ 3 & \text{if } d = 4, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(1), C(2)\}$ and $(a, b) = (6, 1)$ or $(10, 6)$ according as $C = C(1)$ or $C(2)$. Thus (4B) follows by

$$\sum_{j=1}^{8} (-1)^{|C(j)|} \mathrm{k}(N(C(j)), B_0, d) = 0.$$

References

1. J. L. Alperin, *Weights for finite groups*, The Arcata Conference on Representations of Finite Groups, Proc. of Symposia in Pure Math. **47**(1987), 369-379.
2. Jianbei An, *The Alperin and Dade conjectures for the simple Held group*, J. Algebra (to appear).
3. Jianbei An and M. Conder, *The Alperin and Dade conjectures for the simple Mathieu groups*, Comm. Algebra **23**(1995), 2797-2823.
4. J. J. Cannon, *An introduction to the group theory language CAYLEY*, Computational Group Theory, Academic Press, New York, 1984, pp.145-183.
5. J. H. Conway, R. T. Curtis, S. P. Parker and R. A. Wilson, *An ATLAS of finite groups*, Clarendon Press, Oxford, 1985.
6. E. Dade, *Counting characters in blocks, I*, Invent. Math. **109** (1992), 187-210.
7. L. Finkelstein, *The maximal subgroups of Conway's group C_3 and McLaughlin's group*, J. Algebra **25**(1973), 58-89.
8. G. D. James, *The modular characters of the Mathieu groups*, J. Algebra **27** (1973), 57-111.