Collaring Theorems and the Volumes of Symmetric Hyperbolic n-Manifolds

F.W. Gehring, T.H.Marshall, G.J. Martin

Abstract

Let Γ be a discrete subgroup of orientation preserving isometries of hyperbolic n-space, and let $\mathcal{F}$ be a collection of finite subgroups of Γ, for which any two distinct fixed point sets of groups in $\mathcal{F}$ are disjoint. Under some minor additional hypotheses we prove that, for some $r > 0$, the tubular neighbourhoods of radius r about these sets, called collars, are also pairwise disjoint. As an application we show that if F is a finite subgroup of Γ which has a precisely invariant fixed point set of dimension d and such that the orbit of any point not in the fixed point set contains at least p points, then the radius r is bounded below by a value depending only on n, d and p, and which goes to infinity as p does.

We use these results to obtain lower bounds for volumes of hyperbolic n–manifolds whose symmetry groups have nonempty fixed point sets.

1 Introduction

In this paper we prove collaring theorems for elements of finite order in discrete nonelementary Möbius groups in all dimensions. Such theorems have been proven to be useful in investigating the geometry of 3–dimensional Möbius groups and their orbit spaces, hyperbolic 3–manifolds and orbifolds, see [BM], [Me], [GM 1–3], and we present some applications here too.

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An essential feature of collaring theorems is that they bound the distance between the fixed point sets of conjugate finite subgroups of a Möbius group acting on hyperbolic n–space. A suitable collar about the fixed point set of finite subgroup F of a Möbius group Γ projects into the orbit space via the action of F. The action of F on a collar about its fixed point set is much easier to understand than the action of Γ on $\mathbb{H}^n$ and from a few simple observations one can obtain various geometric information, such as volume and diameter bounds, about the orbit space of Γ, (a hyperbolic n–manifold or orbifold). A related application concerns the automorphism group of a hyperbolic n–fold. One can obtain geometric information about a finite volume hyperbolic n–manifold by lifting its symmetry group to the universal cover and studying the size of collars about the fixed point sets of the lifts of symmetries.

The basic idea behind our proofs is a quite simple, but important, geometric observation concerning the hyperbolic sphere kissing problem. In Euclidean geometry, for each dimension there is a bound on the number of spheres of given radius r, all tangent to some other sphere of radius r. Scaling shows that this number is actually independent of r. In hyperbolic geometry there is no such uniform bound. However, in order to have a large number of spheres of hyperbolic radius r tangent to a given sphere S_0 of the same radius, r must be large. In our applications, the kissing spheres must be tangent to a torus imbedded in S_0, giving even better lower bounds for r.

In dimension two, the Euclidean kissing constant is 6. Our geometric proofs give insight into the well known dichotomy between geometric constraints on Kleinian groups containing elliptics of order $p \leq 6$ and those with elliptics of order $p \geq 7$. Indeed this dichotomy persists in higher dimensions when the elliptic fixed point set has codimension 2.

As an application we give volume bounds for *fairly symmetric* hyperbolic n–manifolds. This term was coined by Kuiper in his paper [Ku] where he constructed examples of fairly symmetric hyperbolic 4-manifolds. For us a fairly symmetric hyperbolic n–manifold will be a hyperbolic n–manifold which admits an isometric finite cyclic symmetry which preserves orientation and has nonempty fixed point set (Kuiper allows the orientation reversing case). Many of our results also hold for finite (not necessarily cyclic) groups. In this setting Mostow's celebrated rigidity theorem [Mo] asserts that any (topological) symmetry is homotopic to an isometric symmetry. The existence of fixed points is a homotopy invariant, and so the question becomes essentially topological.

By the Kazdan-Margulis Theorem [KM], there is a lower bound ν_n on the co–volume of any discrete group of hyperbolic isometries of hyperbolic n-space. If M_p^n is a hyperbolic n-manifold with a symmetry $\langle f \rangle$ of order p, then $Q_p^n = M_p^n / \langle f \rangle$ is a hyperbolic n-orbifold and

$$\nu_n \leq \mathrm{Vol}(Q_p^n) = \mathrm{Vol}(M_p^n / \langle f \rangle) = \mathrm{Vol}(M_p^n)/p, \tag{1}$$

which implies that the volume growth is at least linear with respect to the order of the symmetry. It is well known that this linear growth is best possible when $n = 2$ or $n = 3$. The constant ν_n is unknown in any dimension except 2, and in general there are only exceedingly poor estimates for this number. A refinement of this analysis is to consider Wang's finiteness theorem [W]. This theorem (a nice application of Weil's strong rigidity theorem and Chabauty's convergence theorem) asserts that for $n \geq 4$, the set of volumes of hyperbolic n-orbifolds (co–volumes of lattices in $\mathbb{H}^n$) is discrete in the real numbers. By Selberg's Theorem each such group has a finite index subgroup which is torsion free, we therefore see that $\mathrm{Vol}(Q_p^n) \to \infty$ as $p \to \infty$. Thus for $n \geq 4$,

$$\mathrm{Vol}(M_p^n) \geq a_n(p)\, p, \tag{2}$$

and $a_n(p) \to \infty$ as $p \to \infty$. From Wang's finiteness theorem it appears one cannot conclude more about $a_n(p)$. We show that $a_n(p)$ grows at least as fast as a positive power of p, except when the symmetry has a one-dimensional fixed point set.

Let $\mu_0 = 1$, $\mu_1 = 0$ and define for $n \geq 2$

$$\mu_n = \min\{\mathrm{Vol}(M) : M \text{ is a hyperbolic n-manifold}\} \tag{3}$$

Clearly $\mu_n \geq \nu_n > 0$ for $n \neq 1$. If n is even, the Chern-Gauss-Bonnet theorem for a closed hyperbolic n-manifold M (constant curvature equal to -1) gives

$$\mathrm{Vol}_n(M) = \omega_n |\chi(M)| \tag{4}$$

where ω_n denotes the volume of the unit $n-1$-sphere and $\chi(M)$ is the Euler characteristic of M, a nonzero integer. In particular this gives $\mu_n \geq \omega_n$ for even n. Equality is known to hold in the case $n = 4$ [K, R § 11.1].

It is a general fact that the fixed point set of an isometry is componentwise totally geodesic and embedded. Thus, by Smith Theory, the fixed point set is a hyperbolic submanifold of even codimension. We prove a number of theorems concerning the rate of volume growth in terms of the order of the symmetry and the dimension and volume of its fixed point set. Given this data we give quite explicit estimates for the volume growth. For instance we prove the following theorem (a consequence of Theorem 22 below).

Theorem *Let M be a closed hyperbolic n-manifold with a symmetry $\langle f \rangle$ of order p with codimension two fixed point set. Then,*

$$\mathrm{Vol}(M) \geq \frac{2\,\pi\,\mu_{n-2}}{(n-1)} \left[\left(\frac{p}{2\pi} \right)^{n-1} - 1 \right] \tag{5}$$

In other applications we restrict our attention to the most important case of prime order symmetries. Our methods work in a more general setting, but there are technical difficulties which we prefer not to consider here. Another result (Theorem 23 below) is

Theorem *Let M be a closed hyperbolic n-manifold with a symmetry $\langle f \rangle$ of prime order p with $d = \dim(\mathrm{fix}(f))$. Then, if p is sufficiently large,*

$$\mathrm{Vol}(M) \geq \frac{\omega_{n-d}}{(n-1)} \mu_d \left[\frac{n-d}{32\pi^2} \left(\frac{p\beta_{(n-d)/2}}{\Delta_{(n-d)/2}} \right)^{4/(n-d)} - 1 \right]^{(n-1)/2} \tag{6}$$

where ω_k, β_k, and Δ_k denote respectively the volume of a unit $(k-1)$-sphere, the volume of a unit k-ball, and the greatest possible density of a lattice packing of balls in $\mathbb{R}^k$, (Note $n-d$ is even).

Notice that whenever $d \neq 1$, $\mathrm{Vol}_d(\mathrm{fix}(f))$ is bounded below by the volume of the smallest volume hyperbolic manifold, and the rate of growth of volume with respect to order is asymptotic to a constant times $p^{2(n-1)/(n-d)}$. Notice too that if n is even, so too is d and we may use (4) to find the volume of the fixed point set. Then as another application, consider a fairly symmetric hyperbolic 4-manifold. For a disjoint collection of hyperbolic surfaces $X = \cup X_i$ we let $\chi(X) = \Sigma_i \chi(X_i)$. Since $\Delta_2 = \pi/\sqrt{12}$, [CS], $\omega_4 = 2\pi^2$ and $\beta_2 = \pi$, we find,

Corollary *Let M^4 be a hyperbolic 4-manifold with a symmetry $\langle f \rangle$ of order p. If the fixed point set of f has dimension 2. Then*

$$\mathrm{Vol}(M) \geq \frac{4\pi^2}{3} |\chi(\mathrm{fix}(f))| \left[\left(\frac{p}{2\pi} \right)^3 - 1 \right] \tag{7}$$

While if p is sufficiently large and prime and $\mathrm{fix}(f)$ *comprises* $N \geq 1$ *points, then*

$$\mathrm{Vol}(M) \geq \frac{2\pi^2 N}{3}\left(\frac{p\sqrt{3}}{4\pi^2} - 1\right)^{3/2} \approx \frac{Np^{3/2}}{3^{1/4}4\pi}. \tag{8}$$

2 Notation and Definitions

We begin by introducing some basic notions concerning hyperbolic geometry and Möbius groups. Denote by $\mathbb{R}^n$ n-dimensional Euclidean space and $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \infty$ the one point compactification. The usual basis vectors are $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. The upper half space of $\mathbb{R}^n$ is

$$\mathbb{H}^n = \{x : (x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_n > 0\} \tag{9}$$

and the unit ball is $\mathbf{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$. In the upper half space, the Riemannian metric $\mathrm{d}s^2 = |\mathrm{d}x|^2/x_n^2$ is complete and has constant curvature equal to -1 and gives rise to the hyperbolic metric. A *Möbius transformation* of $\overline{\mathbb{R}}^n$ is the finite composition of reflections in hyperspheres and hyperplanes. We denote the group of all Möbius transformations of $\overline{\mathbb{R}}^n$ by $\mathrm{Möb}(n)$. This is precisely the group of all conformal and anticonformal mappings of space. For the basic properties of hyperbolic spaces and Möbius transformations we refer to the books of Ahlfors [A], Beardon [B] and Ratcliffe [R]. The boundary $\partial\mathbb{H}^n$ of $\mathbb{H}^n$ (in $\overline{\mathbb{R}}^n$) is $\overline{\mathbb{R}}^{n-1}$. Thus the group of Möbius transformations of $\overline{\mathbb{R}}^n$ which fix $\mathbb{H}^n$ is naturally identified with $\mathrm{Möb}(n)$. There is another model for hyperbolic geometry and discrete groups of Möbius transformations which we will have occasion to use. Namely the open unit ball $\mathbf{B}^n$ with the metric $\mathrm{d}s^2 = (1 - |x|^2)^{-2}|\mathrm{d}x|^2$. Throughout this paper we let $\rho(x, y)$ denote the *hyperbolic distance* between a pair of points $x, y \in \mathbb{H}^n$ or $x, y \in \mathbf{B}^n$. In this latter model we recall that for $x \in \mathbf{B}^n$,

$$\rho(0, x) = \log\frac{1 + |x|}{1 - |x|}. \tag{10}$$

If $\psi : \mathbf{B}^n \to \mathbf{B}^n$ is a Möbius transformation, because of the formula [R, § 4.5]

$$\frac{|\psi(x) - \psi(y)|^2}{(1 - |\psi(x)|^2)(1 - |\psi(y)|^2)} = \frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2))} \tag{11}$$

we see that ψ acts as an isometry in the hyperbolic metric. Here we will only be concerned with the *orientation preserving* subgroup $\text{Möb}^+(n)$ of index two in $\text{Möb}(n)$. Let Γ be a subgroup of $\text{Möb}^+(n)$. We think of Γ as acting as a group of hyperbolic isometries of $\mathbb{H}^n$ (or $\mathbf{B}^n$). We say that Γ is *discrete* if the identity is isolated in Γ. Discreteness here is equivalent to the requirement that Γ acts *properly discontinuously* in $\mathbb{H}^n$. That is for all compact K in $\mathbb{H}^n$, $g(K) \cap K = \emptyset$ except for finitely many $g \in \Gamma$. Notice that Γ need not act discontinuously in $\partial\mathbb{H}^n$. The order of an element g of a discrete Möbius group Γ is

$$\text{ord}(g) = \inf\{p : p \geq 2 \text{ and } g^p = \text{Identity}\} \tag{12}$$

The fixed point set is

$$\text{fix}(g) = \{x \in \overline{\mathbb{H}^n} : g(x) = x\}. \tag{13}$$

We denote the cardinality of a set E by $|E|$. Then the elements of a discrete Möbius group are classified as

- elliptic : $\text{ord}(g) < \infty$
- parabolic : $\text{ord}(g) = \infty$ and $|\text{fix}(g)| = 1$
- loxodromic : $\text{ord}(g) = \infty$ and $|\text{fix}(g)| = 2$

If g is parabolic or loxodromic, then necessarily $\text{fix}(g)$ lies in $\partial\mathbb{H}^n$. We denote the orbit $\{g(x) : g \in \Gamma\}$ of a point x under Γ by $\Gamma(x)$. The limit set $L(\Gamma)$ of a discrete Möbius group Γ is then the set of accumulation points of the orbit of an arbitrary point $x \in \mathbb{H}^n$ [R, §12.1]. We say that a discrete Möbius group Γ is *elementary* if $|L(\Gamma)| \leq 2$, *nonelementary* otherwise. If Γ is nonelementary, then $L(\Gamma)$ is a closed, uncountable, perfect Γ-invariant subset of $\partial\mathbb{H}^n$. Two equivalent characterizations that we will use are that Γ is elementary if and only if it contains an abelian subgroup of finite index, and if and only if it has a finite orbit in $\overline{\mathbb{H}^n}$. In dimension 3 discrete nonelementary subgroups of $\text{Möb}^+(3)$ are called a *Kleinian* groups and in dimension 2 such groups are called *Fuchsian* groups.

We say a discrete subgroup Γ of $\text{Möb}^+(n)$ is co–compact if Γ is discrete and the orbit space $\mathbb{H}^n/\Gamma$ is compact. This is equivalent to the requirement that $\text{Vol}(\mathbb{H}^n/\Gamma) < \infty$ and Γ has no parabolics. Co–compact groups are finitely generated and nonelementary.

We briefly review some facts concerning the fixed point set of an elliptic Möbius transformation g. In the ball model $\mathbf{B}^n$ g is conjugate (by some other Möbius transformation) to an orthogonal rotation. Every such rotation in $O(n)$ is conjugate, by another rotation, to a rotation whose matrix is in Jordan block form, that is it comprises diagonal blocks, $B(\theta_1), B(\theta_2), \ldots, B(\theta_k)$, where, $B(0) = (1)$, $B(\pi) = (-1)$ and, for other θ in $[0, 2\pi)$, $B(\theta)$ is the 2×2 rotation matrix,

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{14}$$

The values $\theta_1, \theta_2, \ldots, \theta_k$ are the *rotation angles* of M. Since Γ is discrete, each θ_i is a rational multiple of π, say $\theta_i = 2\pi p_i/q_i$, where p_i and q_i are relatively prime. With each Jordan block $B(\theta_i)$ $(\theta_i \neq 0, \pi)$ is associated a two dimensional hyperbolic plane $\Pi(\theta_i)$ which is invariant under g. Restricted to such a plane $\Pi(\theta_i)$, g acts as a rotation of finite order q_i with $q_i|\mathrm{ord}(g)$ and hence $\mathrm{ord}(g) = \mathrm{lcm}(q_i)$, and if $p = \mathrm{ord}(g)$ is prime, then $q_i = p$. The fixed point set of g is

$$\mathrm{fix}(g) = E^q \cap \mathbf{B}^n, \tag{15}$$

where the eigenvalue 1 of g has multiplicity q and E^q is the corresponding eigenspace. In the induced metric, $\mathrm{fix}(g)$ is totally geodesic and naturally isometric to a q-dimensional hyperbolic space. If g is orientation preserving then the eigenvalue -1 has even multiplicity, whence $q = n - 2k \geq 0$ for some integer $k \geq 1$.

The *reduced order* of g is

$$\text{r-ord}(g) = \min\{q_i : q_i > 1, i = 1, 2, \ldots [\frac{n}{2}]\} \tag{16}$$

In dimension two and three $\text{r-ord}(g) = \mathrm{ord}(g)$. It is clear that if p is prime and if f is elliptic of order p, then $\text{r-ord}(f) = p$. The reduced order is the minimum value of the order of g restricted to an invariant hyperbolic subspace which g does not pointwise fix. Thus

$$\text{r-ord}(g) = \min\{|\{g^n(x) : n \geq 0\}| : x \in \mathbb{H}^n \text{ and } g(x) \neq x\} \tag{17}$$

More generally for a finite subgroup F of Γ we set

$$\text{r-ord}(F) = \min\{|\{f(x) : f \in F\}| : x \in \mathbb{H}^n \text{ and } F(x) \neq x\} \tag{18}$$

Let E be a subset of $\mathbb{H}^n$. Then the *collar* of radius r about E is the set

$$C(E;r) = \{x \in \mathbb{H}^n : \rho(x,E) < r\}. \tag{19}$$

We say that a set X is *precisely invariant* under the group Γ if for each $g \in \Gamma$ either

$$g(X) = X \text{ or } g(X) \cap X = \emptyset. \tag{20}$$

A *hyperbolic n-manifold* M is a complete, oriented, n-dimensional manifold locally isometric to $\mathbb{H}^n$. That is M admits a complete metric of constant negative curvature equal to -1. It is a classical fact [Wo] that $M = \mathbb{H}^n/\Gamma$, where Γ is a discrete subgroup of $\text{Möb}^+(n)$ acting as hyperbolic isometries. As $\mathbb{H}^n$ is contractable, M is aspherical and finite dimensional so that Γ is isomorphic to the fundamental group of M and is therefore torsion free. More generally, a *hyperbolic n-orbifold* is defined to be a quotient $Q = \mathbb{H}^n/\Gamma$ where Γ is a not necessarily torsion free group.

We recall that if M is a hyperbolic n-manifold and $f : M \to M$ is an isometry of finite order, then each component of $\text{fix}(f)$ is a complete totally geodesic embedded hyperbolic submanifold. To see this note that locally geodesics are unique in M. Thus if two points are close enough so the geodesic between them is unique, and if they are both fixed by an isometry of finite order, then so is the geodesic between them. This proves total geodesy. To see that this surface is also embedded, observe that if it were to intersect itself we could then choose two points on different sheets so close as to be joined by a unique geodesic. This geodesic would then lie in the fixed point set and so the sheets would have to be the same, a contradiction.

3 A First Collaring Theorem

In order to begin our geometric proof we need to establish the existence of some precisely invariant collar about elliptic fixed points. This is the content of this section.

Lemma 1 . *Let f and g be elliptic Möbius transformations generating a nonelementary group Γ. Then*

$$\rho(\text{fix}(f), \text{fix}(g)) > 0. \tag{21}$$

Proof:- Suppose $\rho(\text{fix}(f), \text{fix}(g)) = 0$. As $\text{fix}(f)$ and $\text{fix}(g)$ are totally geodesic subspaces, f and g have a common fixed point in $\overline{\mathbb{H}^n}$, whence $\langle f, g\rangle$ is elementary, contrary to our assumption.

Theorem 2 *Let Γ be a discrete finitely generated subgroup of Möb$^+(n)$, then there is an $\delta > 0$, for which, if $f, g \in \Gamma$ are elliptic elements of Γ and $\langle f, g\rangle$ is nonelementary, then $\rho(\text{fix}(f), \text{fix}(g)) > \delta$.*

Proof:- As Γ is finitely generated Γ has a torsion free subgroup of finite index. Therefore there is N such that the order of every elliptic in Γ is bounded above by N. Suppose the theorem fails. Then by Lemma 1 there are sequences of elements f_i, g_i in Γ such that $\langle f_i, g_i\rangle$ is discrete nonelementary and $\rho(\text{fix}(f_i), \text{fix}(g_i)) \to 0$. Let $x_i \in \text{fix}(f_i)$ and $y_i \in \text{fix}(g_i)$ be such that $\rho(x_i, y_i) = \rho(\text{fix}(f_i), \text{fix}(g_i))$. For each $i = 1, 2, \dots$ choose a Möbius transformation h_i of $\mathbb{H}^n$ with $h_i(x_i) = e_{n+1}$. Then $h_i(y_i) \to e_{n+1}$ as $i \to \infty$. Next define $f_i' = h_i f_i h_i^{-1}$ and $g_i' = h_i g_i h_i^{-1}$. As f_i' fixes e_{n+1} and as g_i' has a fixed point near e_{n+1}, the compactness properties of the Möbius group imply there are subsequences $\{{f_{i_k}}'\}$ and $\{{g_{i_k}}'\}$ and elliptic Möbius transformations f' and g' of $\mathbb{H}^n$ with ${f_{i_k}}' \to f'$ and ${g_{i_k}}' \to g'$ locally uniformly in $\mathbb{H}^n$. As $\rho(\text{fix}(f_i), \text{fix}(g_i)) = \rho(h(x_i), h(y_i)) = \rho(x_i, y_i) \to 0$, f' and g' have a common fixed point, and so $\langle f', g'\rangle$ is elementary by Lemma 1. However, the groups $\langle h_i f_i {h_i}^{-1}, h_i g_i {h_i}^{-1}\rangle = h_i\langle f_i, g_i\rangle {h_i}^{-1}$ and $\langle f_i', g_i'\rangle$ are isomorphic. As Γ has bounded torsion, so does the sequence of discrete nonelementary subgroups $\langle f_{i_k}', g_{i_k}'\rangle$. This sequence of groups converges algebraically to the group $\langle f', g'\rangle$. By Proposition 5.8 of [Ma] this limit group $\langle f', g'\rangle$ is discrete and nonelementary. This is a contradiction. □

For Fuchsian and Kleinian groups somewhat more is true. We have $\rho(\text{fix}(f), \text{fix}(g)) \geq \delta(p)$ where $p = \max\{\text{ord}(f), \text{ord}(g)\}$ and $\delta(p) \to \infty$ as $p \to \infty$ [GM 2]. However, the next example shows that this is not in general the case in higher dimensions.

Theorem 3 *Let $n \geq 4$. Then for each $N \geq 3$ there is a discrete nonelementary subgroup $\langle f, g\rangle$ of* Möb*(n) with* $\text{ord}(f) = \text{ord}(g) \geq N$ *and*

$$\rho(\text{fix}(f), \text{fix}(g)) \leq 0.57 \tag{22}$$

Proof:- We use the ball model $\mathbf{B}^n$ for hyperbolic geometry. There are two elliptic elements f' and g' both of order 3 in the (2,3,7) triangle group for which $\rho(\text{fix}(f'), \text{fix}(g')) = 0.5662\ldots$. Extend the (2,3,7) triangle group via the Poincaré extension [Be] to a group of isometries of hyperbolic n-space $\mathbf{B}^n$. This extension still setwise fixes the disk $\mathbf{B}^2$. As $n-2 \geq 2$, there is a rotation h of order N which pointwise fixes the unit disk $\mathbf{B}^2$ and commutes with both f and g (and therefore with their Poincaré extensions). Set $f = hf'$ and $g = hg'$ (here f' and g' denote the Poincaré extension also). Then $\langle f, g\rangle$ is discrete and nonelementary as it acts discontinuously on the invariant 2-disk $\mathbf{B}^2$ with limit set the boundary circle. However $\rho(\text{fix}(f), \text{fix}(g)) \leq \rho(\text{fix}(f'), \text{fix}(g')) \leq 0.57$, while $\text{ord}(f)$, $\text{ord}(g) \geq N$. $\square$

It seems clear from this example that what goes wrong with a possible general collaring theorem along the lines we are seeking is that there might be a lower dimensional invariant subspace on which the elliptics act with small order. Thus we expect that there are general collaring theorems for prime orders, or more generally, as the reduced order $\text{r-ord}(f) \to \infty$.

4 Groups of Rotations

Let $M \in O(n)$ have (not necessarily distinct) rotation angles $\theta_1, \theta_2, \ldots, \theta_k$. In the group of $n \times n$ unitary matrices $U(n)$, M can be diagonalized, the eigenvalues of M being $\mathrm{e}^{\pm i\theta_j}$ $(1 \leq j \leq k)$. Recall that each rotation angle θ is associated with a Jordan block $B(\theta)$ and an invariant subspace $\Pi(\theta)$, which is one-dimensional if $\theta = 0, \pi$ and two-dimensional otherwise. These spaces are mutually perpendicular and that their sum is $\mathbb{R}^n$.

We show that a similar decomposition of $\mathbb{R}^n$ into invariant subspaces can be associated with each abelian group in $O(n)$. Presumably this result is well known. We begin with two simple lemmas. The first is straightforward to prove, and the second easily follows.

Lemma 4 *A 2×2 matrix commutes with some $B(\theta)$ ($\theta \neq 0, \pi$) if and only if it is a scalar multiple of a rotation matrix.*

A $2n \times 2n$ matrix is in *rotation block form* if it subdivides into n^2 lots of 2×2 blocks, each of which is a scalar multiple of a rotation matrix.

Lemma 5 *A $2n \times 2n$ matrix commutes with a matrix comprising n diagonal blocks of $B(\theta)$ $(\theta \neq 0, \pi)$ if and only if it is in rotation block form.*

Each $2m \times 2m$ matrix in rotation block form can be mapped to a $m \times m$ complex matrix by taking the block $kB(\theta)$ to the complex number $ke^{i\theta}$. Call this map φ. It is easy to verify that φ determines an isomorphism between the group of rotation block matrices in $O(2m)$ and the group of unitary matrices $U(m)$.

Theorem 6 *Let G be an abelian subgroup of $O(n)$. Then there is a decomposition of $\mathbb{R}^n$ into mutually perpendicular subspaces $X_1, X_2, \ldots, X_k$, each of dimension 1 or 2, and invariant under G, with the action of G restricted to each 2-dimensional X_i being a rotation.*

Proof:- We use induction on n. The theorem is trivial for $n = 1$. We assume that it holds for all $m < n$.

Let $M \subseteq G$, have eigenvalue $\lambda = e^{i\theta}$ (θ is a rotation angle of M) and corresponding eigenspace $X \subseteq \mathbb{C}^n$. For any $N \in G$, $N(X)$ is the eigenspace corresponding to λ in NMN^{-1}. Since G is abelian, $NMN^{-1} = M$ and so $N(X) = X$, so each eigenspace is invariant under G. The intersection of $\mathbb{R}^n$ with the direct sum of the eigenspaces corresponding to λ and $\overline{\lambda}$, is thus a G-invariant subspace of $\mathbb{R}^n$ (in fact the direct sum of the $\Pi(\theta_i)$ for which $\theta_i = \theta$). If any of these is a proper subspace of $\mathbb{R}^n$ (i.e. if M has at least two distinct rotation angles) then we may apply the induction hypothesis.

In the remaining case, each $M \in G$ has only one rotation angle θ. We may assume that, for some M, $\theta \neq 0, \pi$ (the theorem being trivial otherwise), and that M is in Jordan block form. It follows that n is even, say $n = 2m$. By Lemma 5 each matrix in G is in rotation block form. The map φ, defined above, maps G to an abelian subgroup of $U(m)$. By Schur's lemma this group can be conjugated by a unitary matrix U to a group of diagonal matrices. It follows that conjugating G by the orthogonal matrix $\varphi^{-1}(U)$ gives a group of matrices, each in Jordan block form, with m mutually perpendicular 2-dimensional invariant subspaces, and that G restricted to each of these subspaces is a rotation, as required. $\square$

5 Spherical Packings

We let β_n denote the volume of the unit n-ball $\mathbf{B}^n$, and ω_n the volume of its boundary, the unit $(n-1)$–sphere. Recall the formula [A §V (12)]

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{23}$$

whence,

$$\beta_n = \frac{\pi^{n/2}}{\Gamma(1+(n/2))} \tag{24}$$

A *lattice packing* of balls in $\mathbb{R}^n$ is a packing for which the centres of the balls form a lattice in $\mathbb{R}^n$. We let Δ_n denote the maximum possible lattice packing density of balls in $\mathbb{R}^n$. Slightly more is known about lattice packings than about arbitrary packings, but the problem of estimating Δ_n is still very difficult. Asymptotically $n^{-1}\log_2\Delta_n$ lies between -1 and -0.5990 [CS (46)]. We consider the closely related problem of packing balls into a torus.

Let $T = T(r_1, r_2, \ldots r_k)$ be the torus,

$$C_1 \times C_2 \times \ldots \times C_k, \tag{25}$$

where C_i is a circle of radius $r_i > 0$, with the metric induced from the usual embedding in $\mathbb{R}^{2k}$. This metric is the same as that induced by the standard universal covering projection $\mathbb{R}^k \to T$. Let R be the group of all isometries of T obtained by rotating each factor C_i. A subset of T is a *lattice* if is the orbit of a point of T under a discrete (and hence finite) subgroup of R. We define $p(\theta, T)$ to be the largest possible number of disjoint closed balls of radius θ (in the metric of T), contained in T, whose centres form a lattice. Equivalently, $p(\theta, T)$ is the largest possible number of points in a lattice of T, with the distance between any two distinct points being greater than 2θ.

Lemma 7 *$p(\theta, T)$ is an increasing function in each r_i.*

Proof:- Let $r_i' \geq r_i > 0$ $(1 \leq i \leq k)$. Let Λ be a lattice on $T = T(r_1, \ldots, r_k)$ with the distance between any two distinct points greater than 2θ. We construct a lattice on $T' = T(r_1', \ldots, r_k')$ with the same property by lifting Λ to $\mathbb{R}^k$, applying the linear map which takes $\mathbf{e}_i$ to $(r_i'/r_i)\mathbf{e}_i$ $(1 \leq i \leq k)$ and projecting down again to T'. □

Theorem 8 *If $r_i \geq \theta/\pi$ for each i, then*

$$\begin{aligned} p(\theta, T) &\leq \Delta_k \beta_k^{-1} \theta^{-k} k\text{-vol(T)} \\ &= \Delta_k \beta_k^{-1} \theta^{-k} \prod_{i=1}^{k} (2\pi r_i) \end{aligned} \tag{26}$$

Proof:- In this case all balls of radius θ in T are simply connected. Given a lattice packing of T by K such balls, the union of their lifts gives a lattice packing of balls of radius θ in $\mathbb{R}^k$ with density

$$K \beta_k \theta^k / \ k\text{-vol}(T). \tag{27}$$

The theorem follows from the definition of Δ_k. $\square$.

For our applications we need to consider tori imbedded in the unit n-sphere $\mathbf{S}^n$ having the form

$$C_1 \times C_2 \times \ldots \times C_m \times \{u\} \tag{28}$$

where $m = [n + 1/2]$ ($[x]$ denoting the greatest integer not exceeding x), C_i is the circle of radiusr_i and $u \in \mathbb{R}$. The last factor $\{u\}$ is absent when n is even. We thus have

$$r_1^2 + r_2^2 + \ldots + r_m^2 + u^2 = 1, \tag{29}$$

Theorem 9 *Let the torus T be given by (28). We define $\eta_n(\theta, T)$ to be the largest number k, for which there is a lattice L of T and a set of k disjoint balls of spherical radius (that is* solid angle*) θ in $\mathbb{R}^{n+1}$, each exterior to the open unit ball and tangent to $\mathbf{S}^n$ at a point of the lattice L in T. We have,*

$$\eta_n(\theta, T) \leq p(\theta, T) \tag{30}$$

Proof:- Let B be a ball of spherical radius θ, tangent to $\mathbf{S}^n$ at a point in T. The radial projection of B onto $\mathbf{S}^n$ is a ball (in the metric of the sphere) of radius θ. Since the metric of T is inherited from that of the sphere, each of these projections meets T in a region which contains a ball of radius θ in T, and the required inequality follows.$\square$

Corollary 10 *If T is as in (28), and*

$$\eta_n(\theta, T) = K \geq \mathrm{e}^{m/2}\Delta_m\beta_m^{-1}2^m \tag{31}$$

then,

$$\theta \leq \frac{2\pi}{\sqrt{m}}\left(\frac{\Delta_m}{K\beta_m}\right)^{1/m} \tag{32}$$

Proof:- Suppose p of the r_i are greater then θ/π. We may suppose that these are $r_1, r_2, \ldots r_p$. By Lemmas 7, 8 and 9,

$$\begin{aligned}
\eta_n(\theta, T) &\leq p(\theta, T) \\
&\leq p(\theta, T(r_1, r_2, \ldots, r_p, \theta/\pi, \theta/\pi, \ldots \theta/\pi) \\
&\leq \Delta_m\beta_m^{-1}\theta^{-m}(2\pi)^m(\theta/\pi)^{m-p}\textstyle\prod_{i=1}^p(r_i) \\
&\leq \Delta_m\beta_m^{-1}\theta^{-m}(2\pi)^m(\theta/\pi)^{m-p}p^{-p/2} \qquad (33)
\end{aligned}$$

Consequently,

$$K \leq \max_{0\leq p\leq m} \Delta_m\beta_m^{-1}2^m\pi^p p^{-p/2}\theta^{-p} \tag{34}$$

whence

$$\theta \leq \max_{0\leq p\leq m} \frac{\pi}{\sqrt{p}}\left[\frac{\Delta_m\beta_m^{-1}2^m}{K}\right]^{1/p} \tag{35}$$

Since we have assumed (31) it is an easy exercise in calculus (take logarithmic derivatives with respect to p) to see that the maximum above occurs when $p = m$, and the theorem follows. $\Box$.

For the next result we recall Jordan's theorem, which states that each finite subgroup of $O(n)$ has an abelian subgroup of index not exceeding some bound which depends only on n. We denote this bound by $k(n)$.

Theorem 11 *Let G be a subgroup of Möb(n) with fixed point $a \in \mathbb{H}^n$. Let* $\mathbf{B_0}$ *be the hyperbolic ball with centre a and radius r, and suppose that* $\mathbf{B}$ *is a hyperbolic ball of radius r, tangent to* $\mathbf{B_0}$ *with the property that the images of* $\mathbf{B}$ *under elements of G are pairwise disjoint. Let $m = [n/2]$.*

1. *If G is abelian and $|G| \geq \mathrm{e}^{m/2}\Delta_m\beta_m^{-1}2^m$ then*

$$\cosh r \geq \frac{\sqrt{m}}{4\pi}\left(\frac{|G|\beta_m}{\Delta_m}\right)^{1/m} \tag{36}$$

2. In any case, if $|G| \geq k(n)\mathrm{e}^{m/2}\Delta_m\beta_m^{-1}2^m$ *then*

$$\cosh r \geq \frac{\sqrt{m}}{4\pi}\left(\frac{|G|\beta_m}{k(n)\Delta_m}\right)^{1/m} \tag{37}$$

Proof:- Without loss of generality we may use the ball model of hyperbolic geometry and assume that $\mathbf{B_0}$ is centered at the origin, so that G is a group of rotations. First we suppose that G is abelian.

Since $g \in G$ is orientation preserving, the multiplicity of the eigenvalue -1 will be even, and that of 1 will have the same parity as n. Using Theorem 6, we may thus assume that the m two-dimensional subspaces $L_i = \langle \mathbf{e}_{2i-1}, \mathbf{e}_{2i} \rangle$ $(1 \leq i \leq m)$ are each invariant under G, and that, when n is odd, the space $\langle \mathbf{e}_n \rangle$ is fixed by G.

Let $\mathbf{x} = (x_1, y_1, x_2, y_2, \ldots, x_m, y_m, u) \neq \mathbf{0}$ be in $\mathbb{R}^n$, with the last coordinate absent when n is even. It follows from the above that the projection into L_i of the orbit $G(\mathbf{x})$ lies in a lattice of C_i of radius $r_i = \sqrt{x_i^2 + y_i^2}$, so that the orbit of $\mathbf{x}$ lies in a lattice of the m-torus as in (28) (except for a scale factor of $|\mathbf{x}|$). Now suppose that $\mathbf{x}$ is the point of tangency of $\mathbf{B}$ to $\mathbf{B_0}$. By Corollary 10 the spherical radius θ of $\mathbf{B_0}$ is bounded above by (32) with $K = |G|$. By elementary hyperbolic trigonometry

$$\cosh r = \frac{1}{2\sin\theta} \tag{38}$$

Since $\sin\theta \leq \theta$, (36) follows.

If G is an arbitrary group then, by Jordan's theorem, G has abelian subgroup of index not exceeding $k(n)$, and (37) follows by applying (36) to this subgroup. $\Box$

We are led to define $r_n(K)$ (resp. $a_n(K)$) to be the infimum of the numbers r for which there exists a subgroup (resp. abelian subgroup) G of Möb(n) of order K, with a ball $\mathbf{B_0}$ of hyperbolic radius r, whose centre is fixed by G, and a ball $\mathbf{B}$ also of radius r, tangent to $\mathbf{B_0}$, whose images under the elements of G are pairwise disjoint.

The previous theorem provides the lower bounds (36) and (37) for $a_n(K)$ and $r_n(K)$ respectively, for sufficiently large K. Observe that, in the case $n = 2$, the images of $\mathbf{B}$ all lie on a unit circle, whence the angular radius θ

of $\mathbf{B}$ is at most π/K, so that

$$\cosh a_2(K) = \cosh r_2(K) \geq \frac{1}{2\sin(\pi/K)} \tag{39}$$

For this inequality to have any content, $K \geq 7$ is required.

In our applications we will have a precisely invariant set X contained in a lower dimensional hyperbolic space. Our collaring theorem will provide a tubular neighbourhood of this set. We then want to compute the volume of this tubular neighbourhood. To do this we need to use hyperbolic cylindical coordinates as follows. Let $\mathbf{T}$ be a totally geodesic m-dimensional subspace of $\mathbb{H}^n$. Suppose $\mathbf{e}_{n+1} \in \mathbf{T}$. Let $\mathbf{S}$ be the orthogonal (totally geodesic) $k = n - m$ dimensional subspace to $\mathbf{T}$ also containing $\mathbf{e}_{n+1}$. Both $\mathbf{S}$ and $\mathbf{T}$ are isometric to the appropriate dimensional hyperbolic spaces. Let $\mathrm{d}\tau^2$ be the usual hyperbolic metric of $\mathbf{T}$ and $\mathrm{d}\sigma^2$ be the hyperbolic metric of $\mathbf{S}$ in hyperbolic spherical coordinates. The easiest way to see this is to use the ball model and think of $\mathbf{T}$ and $\mathbf{S}$ as dimension m and k orthogonal hyperplanes intersecting only at the origin. Then in $\mathbf{T}$ the coordinates are $t = (x_1, x_2, \ldots, x_m)$ and $\sigma = (\rho, \theta)$ where ρ is the hyperbolic distance to 0 and $\theta \in \partial\mathbf{S} = S^{k-1}$. Thus

$$\mathrm{d}\tau^2 = \frac{|dt|^2}{1 - |t|^2} \text{ and } \mathrm{d}\sigma^2 = \mathrm{d}\rho^2 + \sinh(\rho)|\mathrm{d}\theta|^2. \tag{40}$$

These coordinates are orthonormal and we shall now compute the hyperbolic metric $\mathrm{d}\sigma^2$ in these coordinates. Let P be a point in hyperbolic n-space. Then P has coordinates (τ, σ). If σ remains constant and we vary τ, then P moves on a curve at constant distance ρ from $\mathbf{T}$. Thus a change $\mathrm{d}\tau$ causes a displacement $\cosh(\rho)\,\mathrm{d}\tau$. While for fixed τ, the change $\mathrm{d}\sigma$ is the same. Thus in these coordinates the metric is

$$\mathrm{d}s^2 = \cosh^2(\rho)\mathrm{d}\tau^2 + \mathrm{d}\sigma^2 \tag{41}$$

where ρ is the hyperbolic distance of P to $\mathbf{T}$. The volume element in these new coordinates is the determinant of the associated diagonal matrix, as this frame is orthonormal. On the diagonal there are m $\cosh(\rho)$ terms and $k-1$ $\sinh(\rho)$ terms. Hence

$$\mathrm{dVol} = \cosh^m(\rho)\mathrm{d}\tau\mathrm{d}\sigma = \cosh^m(\rho)\sinh^{k-1}(\rho)\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\tau. \tag{42}$$