Attractors in quasiregular semigroups

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Abstract

Suppose that f generates a $K-$quasimeromorphic semigroup in a domain D of $\overline{\mathbf{R}^n}$, where $n \geq 2$. Suppose that U is a topological ball with $\overline{f(U)} \subset U$ and $\overline{U} \subset D$, and that $f|U$ is a homeomorphism. We prove that then U contains a unique fixed point w of f (so that $f(w) = w$), and there is a topological ball neighbourhood V of w with $\overline{V} \subset U$ and a quasiconformal homeomorphism g of $\overline{\mathbf{R}^n}$ onto itself with $g(w) = 0$ such that $(g \circ f \circ g^{-1})(x) = x/2$ for all $x \in g(V)$. This allows us to classify the attracting and repelling fixed points of elements of uniformly quasimeromorphic semigroups such that the element is locally homeomorphic at the fixed point, by showing that the element is quasiconformally conjugate to a dilation in a neighbourhood of such a point.

1 Introduction

A *rational semigroup* G is a semigroup of rational functions, defined on the extended complex plane, that is, the Riemann sphere $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, with the semigroup operation being the composition of functions, and such that at least one element of G has degree at least 2. A special case would be a cyclic semigroup $G = \langle g \rangle$ generated by a single rational function g of degree

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at least 2. The study of the dynamics of a rational semigroup is therefore a generalisation of the iteration theory of a single rational function.

A systematic study of dynamics of rational semigroups analogous to the Fatou–Julia theory, based on the concept of a normal family, has been started by us in [**4**, **5**, **6**, **7**]. Beardon's book [**1**] is a general reference to the classical theory.

Quasiregular and quasimeromorphic mappings defined in domains in $\mathbf{R}^n$, where $n \geq 2$, are generalisations of analytic and meromorphic functions, with whom they share many properties. Therefore one can ask if an analogous theory of dynamics could be developed for uniformly quasiregular and uniformly quasimeromorphic semigroups, to be defined below. For uniformly quasiconformal groups of homeomorphisms there is already a lot of literature. The point of the present work is to study non-homeomorphic mappings, which can only form a semigroup and not a group.

We next recall the definition of a quasiregular mapping. We identify the compactification $\mathbf{R}^n \cup \{\infty\} = \overline{\mathbf{R}^n}$ with the $n-$sphere $\mathbf{S}^n$ in the usual way. If D is a domain in $\mathbf{R}^n$, then a mapping $F : D \to \mathbf{R}^n$ of Sobolev class $W^{1,n}(D)$ is said to be $K-$*quasiregular*, where $1 \leq K < \infty$, if

$$J_F(x) \geq 0 \quad \text{a.e.} \qquad \text{or} \qquad J_F(x) \leq 0 \quad \text{a.e.}$$

and

$$\max_{|\xi|=1} |DF(x)\xi| \leq K \min_{|\xi|=1} |DF(x)\xi|$$

for almost every $x \in D$, where $\xi \in T_x\mathbf{S}^n$. The smallest number K for which the above inequality holds is called the *maximal dilatation* of F. A non-constant quasiregular mapping can be redefined on a set of measure zero so as to make it continuous, open, and discrete, and we shall always assume that this has been done. It is also differentiable with Jacobian determinant $J_F(x) \neq 0$ almost everywhere.

Let D be a domain in $\mathbf{S}^n$. A function $f : D \to \mathbf{S}^n$ is said to be $K-$quasimeromorphic if each point of D has a neighbourhood contained in D in which f or $f/|f|^2$ is $K-$quasiregular (if $\infty \in D$, then in a neighbourhood of infinity, we consider $f(z/|z|^2)$ or $f(z/|z|^2)/|f(z/|z|^2)|^2$).

A family G of quasimeromorphic mappings $F : \mathbf{S}^n \to \mathbf{S}^n$ which is closed under composition is called a *uniformly quasimeromorphic semigroup* if there is some $K < \infty$ such that each element of G is $K-$quasimeromorphic. An example of a uniformly quasimeromorphic semigroup can be constructed as follows. Let Γ be a measurable conformal structure on $\mathbf{S}^n$. By this we mean that at each point $x \in \mathbf{S}^n$, the function $\Gamma(x)$ is a linear bijection

$$\Gamma(x) : T_x\mathbf{S}^n \to T_x\mathbf{S}^n$$

of the inner product space $T_x\mathbf{S}^n$, such that $\Gamma(x)$ is symmetric, positive definite, has determinant 1, and satisfies the uniform ellipticity condition

$$K^{-1}\,|\xi|^2 \leq \langle \Gamma(x)\xi, \xi\rangle \leq K|\xi|^2,$$

where $K \geq 1$ is independent of x.

The solutions F of the equation

$$D^tF(x)\Gamma(F(x))DF(x) = J_F(x)^{2/n}\Gamma(x)$$

for mappings of Sobolev class $W^{1,n}(\mathbf{S}^n)$ form a semigroup under composition. Each such solution is a $K-$quasimeromorphic mapping of $\mathbf{S}^n$. We call Γ an equivariant measurable conformal structure for G.

In an analogous way, one defines a uniformly quasiregular or quasimeromorphic semigroup of non-constant mappings taking a domain D in $\mathbf{R}^n$ or $\mathbf{S}^n$ into itself.

In [**8**], it is proved that if a uniformly quasimeromorphic semigroup G defined in a domain in the Riemann sphere $\mathbf{S}^2$ satisfies a certain algebraic condition, which is true for all abelian semigroups, then G can be conjugated by a quasiconformal homeomorphism to a semigroup of meromorphic functions (which are rational if the domain is the whole sphere). In [**9**], it is shown that under similar algebraic conditions in $\mathbf{R}^n$, a quasiregular semigroup has an equivariant conformal structure. In [**9**], an example is also given of a cyclic quasiregular semigroup with a nontrivial branch set in $\mathbf{R}^n$, where $n \geq 3$.

The purpose of this paper is to begin the classification of the local dynamics of functions generating uniformly quasiregular semigroups close to their fixed points. In this note we consider maps that are locally homeomorphic at a fixed point and are attracting or repelling at that point.

Theorem 1.1 *Suppose that f generates a $K-$quasimeromorphic semigroup in a domain D of $\overline{\mathbf{R}^n}$, where $n \geq 2$. Suppose that U is a topological ball with $\overline{f(U)} \subset U$ and $\overline{U} \subset D$, and that $f|U$ is a homeomorphism. Then U contains a unique fixed point w of f (so that $f(w) = w$), and there is a topological ball* *neighbourhood V of w with $\overline{V} \subset U$ and a quasiconformal homeomorphism g of $\overline{\mathbf{R}^n}$ onto itself with $g(w) = 0$ such that $(g \circ f \circ g^{-1})(x) = x/2$ for all $x \in g(V)$.*

Of course, we could have $D = U$, in which case the assumption states that f generates a $K-$quasiconformal semigroup and maps U onto a compact subset of U. Naturally, f need not generate a uniformly quasiconformal group as f^{-1} and its iterates need not be defined in all of U.

Even though f^{-1} is usually not globally quasiregular in D (since f need not be one-to-one), so that we cannot obtain a result corresponding to Theorem 1.1 for repelling fixed points just by considering f^{-1} instead of f in D, we may make such a modification locally, and then Theorem 1.1 yields the following result.

Theorem 1.2 *Suppose that f generates a $K-$quasimeromorphic semigroup in a domain D of $\overline{\mathbf{R}^n}$, where $n \geq 2$. Suppose that U is a topological ball with $\overline{U} \subset f(U)$ and $\overline{f(U)} \subset D$, and that $f|U$ is a homeomorphism. Then U contains a unique fixed point w of f (so that $f(w) = w$), and there is a topological ball neighbourhood V of w with $\overline{V} \subset U$ and a quasiconformal homeomorphism g of $\overline{\mathbf{R}^n}$ onto itself with $g(w) = 0$ such that $(g \circ f \circ g^{-1})(x) = 2x$ for all $x \in g(V)$.*

2 Proof of Theorem 1.1

Let the assumptions of Theorem 1.1 be satisfied. The family $\{f^n|U : n \geq 1\}$ is normal [**12**, Corollary IV.3.14] so that there is a subsequence $f^{n_k}|U$ that tends, locally uniformly in U, to a limit function φ, which is either a constant or a non-constant $K-$quasiregular mapping in U [**11**]. The sets $\overline{f^n(U)}$ form a decreasing sequence of compact connected sets whose intersection E is non-empty, compact, and connected.

Recall that a ring domain W in $\mathbf{R}^n$, where $n \geq 2$, is a domain of the form $W = \mathbf{R}^n \setminus (E_1 \cup E_2)$ where E_1 is a compact subset of $\mathbf{R}^n$, E_2 is an unbounded closed subset of $\mathbf{R}^n$, and $E_1 \cap E_2 = \emptyset$. The module $M(W)$ of W is the module of the family P of locally rectifiable paths joining E_1 and E_2 in W. More precisely,

$$M(W) = \inf \left\{ \int_{\mathbf{R}^n} p(z)^n \, dm(z) : p \in \mathcal{B}, \int_\gamma p(z) \, |dz| \geq 1 \text{ for all } \gamma \in P \right\},$$

where $\mathcal{B}$ denotes the set of all non-negative Borel functions on $\mathbf{R}^n$, and $dm(z)$ indicates integration with respect to the Lebesgue measure on $\mathbf{R}^n$. For example, if $W = \{x \in \mathbf{R}^n : a < |x| < b\}$ then $M(W) = \omega_{n-1}(\log(b/a))^{1-n}$, where ω_{n-1} is the $(n-1)$–dimensional volume of $\mathbf{S}^{n-1}$.

Suppose that E contains at least 2 points. Write $W_1 = U \setminus \overline{f(U)}$. The ring domains $f^k(W_1)$ are disjoint, and $f^k(W_1) \subset f^k(U)$. Since each f^k is $K-$quasiconformal in U, we have $1/K_0 \leq M(f^k(W_1))/M(W_1) \leq K_0$ for all $k \geq 1$, where K_0 depends only on K and n. Hence $M(f^k(W_1)) \leq K_0 M(W_1)$. But if $\varepsilon > 0$, we have $f^k(W_1) \subset N_\varepsilon(E) \setminus E$ for all sufficiently large k, where $N_\varepsilon(E)$ consists of points whose distance from E is less than ε. This implies that $M(f^k(W_1)) \to \infty$ as $k \to \infty$, which is a contradiction. It follows that E consists of exactly one point, say x_0.

If φ is non-constant then $\varphi(U)$ is a subset of the interior E^0 of E, which must therefore be non-empty. Thus φ is constant. Since $E = \{x_0\}$, we have $\varphi \equiv x_0$ for all limit functions φ. It follows that $f^n|U \to x_0$ as $n \to \infty$. In particular, $x_0 = \lim_{n\to\infty} f^n(x_0)$ and so $f(x_0) = \lim_{n\to\infty} f^{n+1}(x_0) = x_0$. If f has the fixed point $x_1 \in U$ then $x_1 = f^n(x_1) \to x_0$ so that the fixed point x_0 is unique. We may and will assume from now on that $x_0 = 0$.

Let B be a closed ball centred at x_0 and of radius R with $B \subset U$. Since each f^n is $K-$quasiconformal in U, it follows from the quasiconformal Schönflies theorem that each $f^n(\partial B)$ is a K_1-quasisphere where K_1 depends on K, U, and B. Recall that a set A in $\mathbf{R}^n$ is called a K_1-quasisphere if there exists a K_1-quasiconformal map of $\mathbf{R}^n$ onto itself that takes the unit sphere onto A. Choose a minimal positive integer m with the property that $f^n(B)$ is a subset of the interior of B for all $n \geq m$. We write $g = f^m$ and $S = \partial B$ so that $S \cap g(S) = \emptyset$. We first show that g can be conjugated to a dilation.

We start by extending g from B to a quasiconformal self-map of $\mathbf{R}^n$. Since g is quasiconformal in a neighbourhood of B and $g(S)$ is a quasisphere, it follows from a combination of results in [**2**, p. 182], [**13**], [**10**], [**14**] that there exists a quasiconformal mapping h of the ring domain $A = \{z \in \mathbf{R}^n : \rho R < |z| < R\} \subset B$ onto the interior of $B \setminus g(B)$ such that h extends to a homeomorphism of $\overline{A}$ onto $\overline{B \setminus g(B)}$, such that $h(z) = z$ for all $z \in S$ and $h(z) = g(z/\rho)$ whenever $|z| = \rho R$. (Note that each of $\mathbf{R}^n \setminus \{0\}/\langle g\rangle$ and $\mathbf{R}^n \setminus \{0\}/\langle x \mapsto \rho x\rangle$ is homeomorphic to $\mathbf{S}^{n-1} \times \mathbf{S}^1$.) Here ρ can be any number in $(0, 1)$ but the maximal dilatation of h, which depends on g and on the set $\overline{B \setminus g(B)}$, will also depend on ρ. (The optimal choice for ρ is probably obtained by taking ρ so that $M(A) = M(B \setminus \overline{g(B)})$.)

More precisely, to deduce the existence of h, we may argue as follows (compare [**10**, p. 420]). By the Annulus Theorem, proved by Sullivan ([**13**], compare [**14**, Theorem 3.12, p. 317]) in all dimensions in the quasiconformal category, the set $\overline{B \setminus g(B)}$ is quasiconformally homeomorphic to the closed annulus $\overline{A}$. Following this map by a suitable quasiconformal self-homeomorphism of $\overline{A}$, whose existence follows from [**14**, Theorem 3.14, p. 317], we see that the quotient space of the set $\overline{B \setminus g(B)}$ under the action of g is quasiconformally homeomorphic to the quotient space of the set $\overline{A}$ under the action of $z \mapsto \rho z$ (we remark that both quotient spaces are homeomorphic to the compact space $S^{n-1} \times S^1$). This completes the proof of the existence of h.

Now $B \setminus \{0\} = \cup_{n=0}^{\infty} g^n(B \setminus g(B))$. We set $h(0) = 0$, and if $\rho^{n+1} \leq |z|/R < \rho^n$, we define $h(z) = g^n(h(z/\rho^n))$. Then clearly h defines a quasiconformal homeomorphism of B onto itself such that

$$(h^{-1} \circ g \circ h)(z) = \rho z \tag{1}$$

for all $z \in B$. We extend h to a quasiconformal homeomorphism of $\mathbf{R}^n$ onto itself by setting $h(R^2 z/|z|^2) = R^2 h(z)/|h(z)|^2$ whenever $|z| < R$ (recall that $h|S = Id$). Then we set $\tilde{g}(z) = h(\rho h^{-1}(z))$ for all z. Then $\tilde{g}$ is a quasiconformal homeomorphism of $\mathbf{R}^n$ onto itself. By (1), we have $\tilde{g}(z) = g(z)$ for all $z \in B$. Thus $\tilde{g}$ gives the desired extension of g. It is clear that all the iterates $\tilde{g}^n(z) = h(\rho^n h^{-1}(z))$ are uniformly quasiconformal. We extend g and $\tilde{g}$ to $\mathbf{S}^n$ by setting $g(\infty) = \tilde{g}(\infty) = \infty$.

We have $\lim_{n\to\infty} \tilde{g}^n(z) = 0$ for all $z \in \mathbf{R}^n$. Since $\tilde{g}$ also fixes 0 and ∞, so that $\tilde{g}$ is loxodromic in the terminology of [**3**], it follows [**10**, p. 420] that there is a quasiconformal homeomorphism F of $\mathbf{R}^n$ onto itself such that $(F \circ \tilde{g} \circ F^{-1})(z) = z/2$ for all $z \in \mathbf{R}^n$. In particular, in a neighbourhood of $x_0 = 0$, F conjugates g to the dilation $z \mapsto z/2$. We conjugate f and g by F also but denote the conjugated maps still by $\tilde{g}$, f, and g. So $f^m(z) = z/2$ and $z/2 \equiv \tilde{g}(z) = g(z)$ for all $z \in W$, where W is a suitable topological ball neighbourhood of the origin.

We next extend f to a quasiconformal homeomorphism $\tilde{f}$ of $\mathbf{R}^n$ onto itself. We write $B(r) = \{z : |z| < r\}$. Let $B_1 = B(R_1)$ be such that $\overline{B(2R_1)} \subset W$ and $f(\overline{B(2R_1)}) \subset W$. Note that f commutes with $g = f^m$ in B_1. If $z \in g(B_1) = B(R_1/2)$, we define $\tilde{f}(z) = f(z)$. If $z \notin g(B_1)$, there is a smallest $k \geq$

1 such that $\tilde{g}^k(z) \in g(B_1)$ (then, in fact, $\tilde{g}^k(z) \in g(B_1) \setminus g^2(B_1)$). Then we set $\tilde{f}(z) = \tilde{g}^{-k}(f(\tilde{g}^k(z))) = 2^k f(2^{-k}z)$. If $z \in \partial g(B_1)$ then $k = 1$, $\tilde{g}(z) = g(z) = f^m(z)$, and $\tilde{f}(z) = \tilde{g}^{-1}(f^{m+1}(z))$. Now $z = g(w)$, where $|w| = R_1$, and by the definition of m, we have $f^{m+1}(w) \in B_1$. Hence $f^{m+1}(z) = g(f^{m+1}(w)) \in g(B_1)$. Thus $\tilde{f}(z) = \tilde{g}^{-1}(f^{m+1}(z)) = g^{-1}(f^{m+1}(z)) = g^{-1}(g(f(z))) = f(z)$. It is now easily verified that $\tilde{f}$ is a quasiconformal homeomorphism of $\mathbf{R}^n$ onto itself with $\tilde{f}|g(B_1) = f|g(B_1)$. We claim that $\tilde{f}^m(z) = \tilde{g}(z)$ for all z, so that $\tilde{f}$ generates a uniformly quasiconformal group in $\mathbf{R}^n$.

To prove that $\tilde{f}^m(z) = \tilde{g}(z)$ for all z, note first that if $z \in g(B_1)$ then $f^n(z) \in B_1$ for all $n \geq 1$. Hence $f^m(z) = g(z)$. Suppose then that $z \notin g(B_1)$, and write $L(a)$ for the map $(L(a))(x) = ax$ when $a > 0$. Then

$$\tilde{f}^m(z) = L(2^{k_m}) \circ f \circ L(2^{k_{m-1}-k_m}) \circ f \circ \cdots \circ f \circ L(2^{k_1-k_2}) \circ f \circ L(2^{-k_1})$$

where the positive integers k_j have the following property. Write $b_0 = z$ and for $1 \leq j \leq m$, define

$$b_j = L(2^{k_j}) \circ f \circ L(2^{k_{j-1}-k_j}) \circ f \circ \cdots \circ f \circ L(2^{k_1-k_2}) \circ f \circ L(2^{-k_1}).$$

By the definition of $\tilde{f}$, we have $2^{-k_j}b_{j-1} \in g(B_1) \setminus g^2(B_1)$ for all j with $1 \leq j \leq m$. Thus we are always applying f to a point ζ in the annular ring domain $g(B_1) \setminus g^2(B_1)$. The resulting value $f(\zeta)$ always belongs to $B(R_1)$. Hence every time when we multiply a point by $2^{k_{j-1}-k_j}$, we are moving a point of $B(R_1)$ to a point in $B(R_1/2) \setminus B(R_1/4)$. Since f commutes with $g = L(1/2)$ in $B(2R_1)$, we deduce that $f(2^{-k_j}b_{j-1}) = 2^{k_{j-1}-k_j}f(2^{-k_{j-1}}b_{j-2})$ for all j. Since $2^{-k_1}b_0 = 2^{-k_1}z \in B(R_1/2) \setminus B(R_1/4)$, induction now shows that $\tilde{f}^m(z) = b_m = 2^{k_m}f(2^{-k_m}b_{m-1}) = 2^{k_1}f^m(2^{-k_1}b_0) = 2^{k_1}g(2^{-k_1}b_0) = 2^{k_1}\tilde{g}(2^{-k_1}b_0) = \tilde{g}(z) = z/2$, as claimed.

Since $\tilde{f}$ generates a uniformly quasiconformal group in $\mathbf{R}^n$, fixes 0 and ∞, and $\lim_{n\to\infty} \tilde{f}^n(z) = 0$ for all $z \in \mathbf{R}^n$, so that $\tilde{f}$ is loxodromic, it follows as above that there is a quasiconformal homeomorphism F of $\mathbf{R}^n$ onto itself such that $(F \circ \tilde{f} \circ F^{-1})(z) = z/2$ for all $z \in \mathbf{R}^n$. In particular, in a neighbourhood of $x_0 = 0$, F conjugates f to the dilation $z \mapsto z/2$. This completes the proof of Theorem 1.1.

3 Proof of Theorem 1.2

Let the assumptions of Theorem 1.2 be satisfied. Then $g = f^{-1}$ is a well defined quasiconformal homeomorphism of $V = f(U)$ onto $U = g(V)$ where

$\overline{U} \subset V$. Thus each g^n is defined in V, and each g^n is $K-$quasiconformal, since each is a branch of the inverse of f^n and by our assumption, each f^n is $K-$quasimeromorphic. Thus g generates a uniformly quasiconformal semigroup in V. Hence the assumptions of Theorem 1.1 are satisfied, and the existence of the unique fixed point of g (and hence of f) and the desired conjugacy now follow from Theorem 1.1 applied to g in V, instead of f in U. This completes the proof of Theorem 1.2.

4 References

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