On the Discreteness of the Free Product of Finite Cyclic groups

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Abstract

For $p, q \geq 2$, $\max\{p, q\} \geq 3$, $\delta_\infty(p, q)$ is defined to be the smallest number with the following property. If f and g are elliptic Möbius transformations of orders p and q respectively and if the hyperbolic distance $\delta(f, g)$ between their axes is at least $\delta_\infty(p, q)$, then the group $\Gamma = \langle f, g \rangle$ is discrete nonelementary and isomorphic to the free product $\mathbb{Z}_p * \mathbb{Z}_q$. We prove that

$$\cosh(\delta_\infty(p, q)) = \frac{\cos(\pi/p)\cos(\pi/q) + 1}{\sin(\pi/p)\sin(\pi/q)}$$

This valued is obtained in the (p, q, ∞)–triangle group. We give other applications concerning the commutator parameters of the free product of cyclic groups.

1 Introduction

A *Kleinian group* is a discrete nonelementary subgroup of isometries of hyperbolic 3–space $\mathbb{H}^3$. Equivalently such groups are identified with (the Poincaré extensions of) discrete groups of Möbius or conformal transformations of the Riemann sphere $\overline{\mathbb{C}}$. We use [1] and [4] as basic references for the theory of discrete groups. We denote the hyperbolic metric of constant curvature -1

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1991 *Mathematics Subject Classification.* Primary 30F40, 20H10 on $\mathbb{H}^3$ by $\rho(\cdot,\cdot)$. The elements of a Kleinian group, other than the identity, are either *loxodromic*, *elliptic* or *parabolic*. Each elliptic or loxodromic element f fixes two points of $\overline{\mathbb{C}} = \partial\mathbb{H}^3$ and the hyperbolic line joining these two points is called the *axis* of f, denoted $\mathrm{ax}(f)$. If f and g are elliptic Möbius transformations, then we set

$$\delta(f,g) = \rho(\mathrm{ax}(f), \mathrm{ax}(g)) \tag{1}$$

and call $\delta(f,g)$ the *axial distance* between f and g.

We associate with each Möbius transformation

$$f = \frac{az+b}{cz+d}, \ ad - bc = 1, \tag{2}$$

the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \tag{3}$$

and set $\mathrm{tr}(f) = \mathrm{tr}(A)$ where $\mathrm{tr}(A)$ denotes the trace of the matrix A. Next for each pair of Möbius transformations f and g we let $[f,g]$ denote the multiplicative commutator $fgf^{-1}g^{-1}$. We call the three complex numbers

$$\beta(f) = \mathrm{tr}^2(f) - 4, \ \beta(g) = \mathrm{tr}^2(g) - 4, \ \gamma(f,g) = \mathrm{tr}([f,g]) - 2 \tag{4}$$

the parameters of the 2–generator group $\langle f,g\rangle$ and write

$$\mathrm{par}(\langle f,g\rangle) = (\gamma(f,g), \beta(f), \beta(g)). \tag{5}$$

These parameters are independent of the choice of matrix representations for f and g in $\mathrm{SL}(2,\mathbb{C})$ and they determine $\langle f,g\rangle$ uniquely up to conjugacy whenever $\gamma(f,g) \neq 0$. Recall that $\gamma(f,g) = 0$ if and only if f,g have a common fixed point in $\overline{\mathbb{C}}$. If f is a primitive elliptic of order p, then

$$\beta(f) = -4\sin^2(\pi/p) \tag{6}$$

Thus if $\Gamma = \langle f,g\rangle$ is a Kleinian group generated by elliptics of orders p and q respectively, then there is a complex number γ such that

$$\mathrm{par}(\Gamma) = (\gamma, -4\sin^2(\pi/p), -4\sin^2(\pi/q)) \tag{7}$$

Thus up to conjugacy the space of all such discrete groups is determined uniquely by one complex parameter.

There is a relationship between the axial distance and the parameters of a discrete group encoded in the following lemma [3].

Lemma 1.1 *Let f and g be Möbius transformations with distinct pairs of fixed points. Then*

$$\begin{aligned} \frac{4\gamma(f,g)}{\beta(f)\beta(g)} &= \sinh^2(\delta \pm i\phi) \\ \cosh(2\delta) &= \left|\frac{4\gamma(f,g)}{\beta(f)\beta(g)}+1\right| + \left|\frac{4\gamma(f,g)}{\beta(f)\beta(g)}\right| \\ \cos(2\phi) &= \left|\frac{4\gamma(f,g)}{\beta(f)\beta(g)}+1\right| - \left|\frac{4\gamma(f,g)}{\beta(f)\beta(g)}\right| \end{aligned} \tag{8}$$

where $\delta = \delta(f,g)$ is the hyperbolic distance between the axes of f and g and $\phi = \phi(f,g)$ is the angle between the spheres or hyperplanes which contain $\mathrm{ax}(f)$ or $\mathrm{ax}(g)$ and the common perpendicular of $\mathrm{ax}(f)$ and $\mathrm{ax}(g)$.

Next, for each p and q with $\max\{p,q\} \geq 3$ we set

$$\delta_\infty(p,q) = \operatorname{arccosh}\left(\frac{\cos(\pi/p)\cos(\pi/q)+1}{\sin(\pi/p)\sin(\pi/q)}\right) \tag{9}$$

Our main result is the following theorem.

Theorem 1.1 *Suppose that f and g are elliptics of order p and q respectively with*

$$\delta(f,g) \geq \delta_\infty(p,q). \tag{10}$$

Then $\Gamma = \langle f,g\rangle$ is discrete and isomorphic to the free product group $\langle f\rangle * \langle g\rangle$. The lower bound is sharp in the sense that it is attained in the (p,q,∞)–triangle group and for every $\epsilon > 0$ there are infinitely many Kleinian groups $\langle f,g\rangle$ generated elliptics of order p and q with

$$\delta_\infty(p,q) - \epsilon \leq \delta(f,g) < \delta_\infty(p,q) \tag{11}$$

which are not isomorphic to the free product of cyclic groups.

Next let

$$\lambda_{p,q} = 4(\cos(\pi/p) + \cos(\pi/q))^2 + 4(\cos(\pi/p)\cos(\pi/q) + 1)^2. \tag{12}$$

Then a little algebraic manipulation combined with the identities in Lemma 1.1 yields the following corollary

Corollary 1.1 *Let Γ be a Möbius group with*

$$\mathrm{par}(\Gamma) = (\gamma, -4\sin^2(\pi/p), -4\sin^2(\pi/q)) \tag{13}$$

If γ lies outside of the open ellipse defined by the equation

$$|z| + |z + 4\sin^2(\pi/p)\sin^2(\pi/q)| < \lambda_{p,q}, \tag{14}$$

then Γ is discrete and isomorphic to the free product of cyclics $\langle f\rangle * \langle g\rangle$.

Again the result is sharp. Closely related results can be found in [2, 3]. Further important applications of the estimates given here can be found in [5].

2 Proofs

Let $p \geq q$. Then $\max\{p,q\} = p \geq 3$. Let f and g be elliptics of order p and q respectively and set

$$\delta = \delta(f,g) \geq \delta_\infty(p,q) \tag{15}$$

and

$$\omega^2 = e^{\delta + i\phi} \tag{16}$$

where ϕ is the angle between the axes of f and g. Then

$$|\omega|^2 = e^\delta \geq e^{\delta_\infty(p,q)} = (\cot(\pi/p) + \csc(\pi/p))(\cot(\pi/q) + \csc(\pi/q)). \tag{17}$$

Next, define matrices A and B as follows.

$$A = \begin{pmatrix} \cos(\pi/p) & i\omega\sin(\pi/p) \\ i\sin(\pi/p)/\omega & \cos(\pi/p) \end{pmatrix} \tag{18}$$

$$B = \begin{pmatrix} \cos(\pi/q) & i\sin(\pi/q)/\omega \\ i\omega\sin(\pi/q) & \cos(\pi/q) \end{pmatrix}. \tag{19}$$

Then $A, B \in SL(2, \mathbb{C})$ correspond to Möbius transformations f and g respectively. Clearly

$$\begin{aligned} \beta(f) &= -4\sin^2(\pi/p) && (20) \\ \beta(g) &= -4\sin^2(\pi/q) && (21) \end{aligned}$$

So f has order p and g has order q. Moreover if $\phi = 0$ and $\delta = \delta_\infty(p, q)$ it is not difficult now to verify that

$$\gamma(f, g) = 4(\cos(\pi/p) + \cos(\pi/q))^2 \tag{22}$$

and in fact in this case $\Gamma = \langle f, g\rangle$ is the (p, q, ∞)–triangle group.

The isometric circles of f are easily calculated from the matrix representative A. They are the two circles

$$|z \pm i\omega\cot(\pi/p)| = |\omega|/\sin(\pi/p) \tag{23}$$

The fixed points of f are the intersection of these two circles and the axis of f is simply the hyperbolic line connecting these two points. A fundamental domain for the action of f on the complex plane is the exterior of these two circles together with the region bounded by their intersection. Similarly the isometric circles of g are the two circles

$$|z \pm i\cot(\pi/q)/\omega| = 1/|\omega\sin(\pi/q)| \tag{24}$$

Again the fixed points of g are the intersection of these two circles. (With obvious modifications if $q = 2$ so that g has order 2.) Next, the isometric circles of g lie in the disk $D(0, r)$, where

$$r = \frac{1 + \cos(\pi/q)}{|\omega|\sin(\pi/q)} \tag{25}$$

Additionally the isometric circles of f contain the disk $D(0, s)$, where

$$s = |\omega|\frac{1 - \cos(\pi/p)}{\sin(\pi/p)} \tag{26}$$

A little manipulation shows that

$$|\omega|^2 = e^\delta \geq e^{\delta_\infty(p,q)} = \frac{\sin(\pi/p)(1 + \cos(\pi/q))}{\sin(\pi/q)(1 - \cos(\pi/p))} \tag{27}$$

and hence $r \leq s$. We have therefore seen that the exterior of a fundamental domain for $\langle g \rangle$ lies inside a fundamental domain for $\langle f \rangle$. It follows from the simplest of the Klein–Maskit combination theorems [4] that the group $\Gamma = \langle f, g \rangle$ is discrete and isomorphic to the free product of the cyclic groups,

$$\Gamma \cong \langle f \rangle * \langle g \rangle \cong \mathbb{Z}_p * \mathbb{Z}_q \tag{28}$$

Finally, in discussing the sharpness of the result we need only point out that if f_t and g_t together generate the (p, q, t)–triangle group, then

$$\delta(f_t, g_t) = \operatorname{arccosh}\left(\frac{\cos(\pi/p)\cos(\pi/q) + \cos(\pi/t)}{\sin(\pi/p)\sin(\pi/q)}\right) \to \delta_\infty(p, q) \tag{29}$$

as $t \to \infty$.

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