Boundary Control of a Timoshenko Beam with Variable Physical Characteristics

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Abstract

We study the boundary controllability of a vibrating Timoshenko beam with physical characteristics that may vary along the length of the beam. Two cases are considered: A beam which is clamped at one end, the other end being controlled by a torque and transverse force; and a beam which is hinged at one end, where a control torque is applied, and free at the other end, where a control force is applied.

1 Introduction.

In this paper, we study the boundary controllability of a Timoshenko beam, the equations of motion of which are

$$\begin{aligned} \rho\ddot{w} - (K(\psi + w'))' &= 0, \\ I_\rho\ddot{\psi} - (EI\psi')' + K(\psi + w') &= 0. \end{aligned} \tag{1}$$

Here, we use dots to denote time derivatives, and primes to denote derivatives with respect to the space variable, which is the distance of a point on the center line of the beam from one end of the beam. The function w is the transverse displacement of the beam and ψ is the rotation angle of a filament of the beam. We let L denote the length of the beam. The physical parameters appearing here are ρ, the mass density per unit length, E, Young's modulus of elasticity, I, the moment of inertia of a cross section of the beam, I_ρ, the polar moment of inertia of a cross section, and K, the shear modulus. We assume that ρ, EI, I_ρ and K are all positive, C^2 functions of the space variable.

Several authors (see [10], [4], [2], [8], [9], [1]) have considered control problems associated with the uniform Timoshenko beam, for which the physical parameters are constants, but this appears to be the first such analysis for a Timoshenko beam with variable physical parameters.

The mechanical energy of the beam is given by

$$\mathcal{E} = \frac{1}{2}\int_0^L \rho\dot{w}^2 + I_\rho\dot{\psi}^2 + K(\psi + w')^2 + EI(\psi')^2\,dx. \tag{2}$$

We consider two situations. The first is a beam clamped at the origin, and free at its other end. In this case, the control functions are a force f and torque τ, both applied to the free end of the beam. The associated boundary conditions for this case are

$$\begin{aligned} w(0,t) &= 0, & \psi(0,t) &= 0, \\ K(L)(\psi(L,t) + w'(L,t)) &= f(t), & EI(L)\psi'(L,t) &= \tau(t). \end{aligned} \tag{3}$$

In the second situation, we consider the beam to model small motions of a hinged arm, which is hinged at the origin and free at the other end. The control functions are a torque τ applied at the hinged end, and a force f applied at the free end. The associated boundary conditions for this case are

$$\begin{aligned} w(0,t) &= 0, & \psi'(L,t) &= 0, \\ EI(0)\psi'(0,t) &= -\tau(t), & K(L)(\psi(L,t) + w'(L,t)) &= f(t). \end{aligned} \tag{4}$$

In each case, the system is completed by including the initial conditions

$$\begin{aligned} w(x,0) &= w_0(x), & \dot{w}(x,0) &= v_0(x), \\ \psi(x,0) &= \psi_0(x), & \dot{\psi}(x,0) &= \phi_0(x). \end{aligned} \tag{5}$$

There are two wave speeds associated with the system (1),

$$v_1 = \sqrt{K/\rho} \qquad v_2 = \sqrt{EI/I_\rho}. \tag{6}$$

We let T_1 and T_2 denote the times required for the two types of wave to travel along the whole length of the beam. Specifically,

$$T_1 = \int_0^L 1/v_1(x)\,dx, \qquad T_2 = \int_0^L 1/v_2(x)\,dx. \tag{7}$$

We let $T_0 = 2\max(T_1, T_2)$ and suppose that $T > T_0$. For each of the situations described above, we seek control functions f and τ belonging to $L^2(0,T)$ that drive the corresponding system to rest. For the case of the clamped beam, this means that solutions are driven to the state $w(x,T) = \psi(x,T) = \dot{w}(x,T) = \dot{\psi}(x,T) = 0$. For the hinged beam, solutions are driven to one of the states $\dot{w}(x,T) = \dot{\psi}(x,T) = 0$, $w(x,T) = ax$, $\psi(x,T) = -a$, where a is a constant that can be interpreted as being the angle of rotation of the beam about the point $x = 0$. This mathematical model of the hinged beam is valid only for small displacements, and we hope to write a report in the near future which allows for larger rotation angles, and for controllability of the final angle of rotation (a slight modification of the procedure used here will give controllability of the final angle of rotation, but it requires an extra control function).

We show that there is a certain over-determined eigenvalue problem associated with each situation described above, and that controllability is linked to the non-existence of eigenfunctions, and uncontrollability is linked to the existence of such eigenfunctions. For this reason, we call such eigenvalue problems *controllability eigenvalue problems.* Here, each eigenvalue problem consists of the ordinary differential equations

$$\begin{aligned} \mu^2 \rho w + (K(\psi + w'))' &= 0, \\ \mu^2 I_\rho \psi + (EI\psi')' - K(\psi + w') &= 0, \end{aligned} \tag{8}$$

and six homogeneous boundary conditions. The boundary conditions associated with the eigenvalue problem for the clamped beam are

$$\begin{aligned} &w(0) = 0, \quad w(L) = 0, \quad w'(L) = 0, \\ &\psi(0) = 0, \quad \psi(L) = 0, \quad \psi'(L) = 0, \end{aligned} \tag{9}$$

and the boundary conditions associated with the eigenvalue problem for the hinged beam are

$$\begin{aligned} &w(0) = 0, \quad w(L) = 0, \quad w'(L) + \psi(L) = 0, \\ &\psi(0) = 0, \quad \psi'(0) = 0, \quad \psi'(L) = 0. \end{aligned} \tag{10}$$

It is easy to see that the eigenvalue problem (8,9) has no solutions, for even if we dispense with the boundary conditions at the origin, we have an initial value problem for a system of linear ordinary differential equations, the solution of which is unique. Thus, aside from a technical condition (that the wave speeds v_1 and v_2 are different at each point), which could probably be removed, we can conclude that the clamped beam with variable physical characteristics is always controllable. Similarly, we can conclude that the hinged arm problem is controllable provided that the eigenvalue problem (8,10) has no solutions, again with the technical assumption on wave speeds. However, we show in this case that when the coefficients of our differential equations are constant, solutions of the controllability eigenvalue problem exist for certain values of the physical parameters. Thus, this problem is not always controllable.

The technique that we use involves demonstrating a smoothing property of auxiliary problems consisting of a semi-infinite beam and an infinite beam for the clamped and hinged problems respectively. This technique was first introduced by W. Littman and S. W. Taylor [13] to investigate the controllability of an Euler-Bernoulli beam that is pinned at several points along its length. The method, which has its origins in an earlier paper [12] by Littman and Taylor, has also been used to investigate the controllability of an Euler-Bernoulli beam and point mass system [14]. A much earlier technique, introduced by W. Littman [6] and used by W. Littman and L. Markus [7] for a uniform Euler-Bernoulli beam and later by Taylor [11] for a non-uniform Euler-Bernoulli beam, could also be used to study the controllability of the clamped beam described above. However, the technique of [6] will not work when there are homogeneous boundary conditions at each end of the beam, which is the case for the hinged beam.

2 Remarks on the existence of solutions to the beam equations.

Here we outline the existence theory of each of the systems (1, 3, 5) and (1, 4, 5). One approach to work with a variational formulation of the equations, as Lagnese, Leugering, and Schmidt do for systems of uniform Timoshenko beams in [5]. However, we use the classical method of characteristics, because consideration of characteristics is an important element in our development of the smoothing properties of the beam equations in the next section. In this section, we assume that ρ, EI, I_ρ and K are all positive, C^1 functions of the space variable.

We begin by transforming the equations to first order systems by introducing the variables

$$u_1 = \frac{1}{2}(K^{1/2}(\psi + w') - \rho^{1/2}\dot{w}), \qquad u_2 = \frac{1}{2}(K^{1/2}(\psi + w') + \rho^{1/2}\dot{w}),$$
$$u_3 = \frac{1}{2}((EI)^{1/2}\psi' - I_\rho^{1/2}\dot{\psi}), \qquad u_4 = \frac{1}{2}((EI)^{1/2}\psi' + I_\rho^{1/2}\dot{\psi}). \tag{11}$$

In the new variables, the beam equations (1) take the form

$$\dot{u} + \Lambda u' = Au - \frac{1}{2}\Lambda' u, \tag{12}$$

where Λ is the 4 by 4 diagonal matrix with diagonal entries $\lambda_{11} = v_1$, $\lambda_{22} = -v_1$, $\lambda_{33} = v_2$, $\lambda_{44} = -v_2$, where the characteristic speeds are given by (6); and A is the skew-symmetric matrix given by

$$2a_{12} = K^{1/2}(\rho^{-1/2})' - (K^{1/2})'\rho^{-1/2},$$
$$a_{13} = -a_{14} = a_{23} = -a_{24} = -\frac{1}{2}K^{1/2}I_\rho^{-1/2},$$
$$2a_{34} = (EI)^{1/2}(I_\rho^{-1/2})' - ((EI)^{1/2})'I_\rho^{-1/2}. \tag{13}$$

The energy (2) now has the simple form

$$\mathcal{E} = \frac{1}{2}\int_0^L u_1^2 + u_2^2 + u_3^2 + u_4^2 \, dx. \tag{14}$$

The clamped beam's boundary conditions (3) now take the form

$$\begin{aligned} u_2(0,t) - u_1(0,t) = 0, &\quad u_1(L,t) + u_2(L,t) = K(L)^{-1/2} f(t), \\ u_4(0,t) - u_3(0,t) = 0, &\quad u_3(L,t) + u_4(L,t) = EI(L)^{-1/2} \tau(t), \end{aligned} \tag{15}$$

and the hinged beam's boundary conditions (4) take the form

$$\begin{aligned} u_2(0,t) - u_1(0,t) = 0, &\quad u_1(L,t) + u_2(L,t) = K(L)^{-1/2} f(t), \\ u_3(0,t) + u_4(0,t) = -EI(0)^{-1/2} \tau(t), &\quad u_3(L,t) + u_4(L,t) = 0. \end{aligned} \tag{16}$$

We complete the description of each system by specifying the initial condition

$$u(x,0) = \phi(x). \tag{17}$$

As usual, we use the term *classical solution* to denote a C^1 solution of either (12, 15, 17) or (12, 16, 17). It is clear that such solutions must satisfy compatibility conditions. There are eight such conditions for each of the systems, four arising from the continuity of $u(x,t)$ at $(0,0)$ and $(L,0)$, and four more arising from the compatibility of the initial and boundary data with the partial differential equations (12) at $(0,0)$ and $(L,0)$. We leave the specific details of these to the reader.

Theorem 2.1 (Classical Solutions). *If the boundary data f, τ and the initial data ϕ are continuously differentiable and satisfy the compatibility conditions, then each of the systems (12, 15, 17) and (12, 16, 17) has a unique classical solution*

The proof involves making use of the characteristic curves of the equations (12) to set up a system of integral equations, which one solves by the contraction mapping principle. This is a very standard, classical method of proof (see, for example, F. John [3], p. 46), so we omit the details.

It is easy to check that classical solutions of our first order systems correspond to classical solutions of the original beam systems (1, 3, 5) and (1, 4, 5), and vice versa. It is useful to note that we can differentiate the energy (14) of a classical solution and that

$$\dot{\mathcal{E}}(t) = \frac{1}{2}\rho(L)^{-1/2}(u_2(L,t) - u_1(L,t))f(t) + \frac{1}{2}I_\rho(L)^{-1/2}(u_4(L,t) - u_3(L,t))\tau(t) \tag{18}$$

for the clamped system, and

$$\dot{\mathcal{E}}(t) = \frac{1}{2}\rho(L)^{-1/2}(u_2(L,t) - u_1(L,t))f(t) + \frac{1}{2}I_\rho(0)^{-1/2}(u_4(0,t) - u_3(0,t))\tau(t) \tag{19}$$

for the hinged system.

Given $T > 0$, let P_c denote the set of functions $p \in C^1([0,L] \times [0,T])$ that satisfy

$$\begin{aligned} p_2(0,t) - p_1(0,t) = 0, &\quad p_1(L,t) + p_2(L,t) = 0, \\ p_4(0,t) - p_3(0,t) = 0, &\quad p_3(L,t) + p_4(L,t) = 0, \\ p(x,T) = 0, & \end{aligned} \tag{20}$$

and let P_h denote the set of functions $p \in C^1([0,L]\times[0,T])$ that satisfy

$$\begin{aligned} p_2(0,t) - p_1(0,t) = 0, &\quad p_1(L,t) + p_2(L,t) = 0, \\ p_3(0,t) + p_4(0,t) = 0, &\quad p_3(L,t) + p_4(L,t) = 0, \\ p(x,T) = 0. & \end{aligned} \tag{21}$$

Thus, functions in P_c satisfy the homogeneous boundary conditions of a clamped-free beam, and the functions in P_h satisfy the homogeneous boundary conditions of a hinged-free beam. Suppose that u is a classical solution of the system (12, 15, 17). Taking the vector scalar product of (12) with $p \in P_c$ and integrating over $[0,L]\times[0,T]$, we obtain

$$\begin{aligned} \int_0^T \int_0^L u^*(\dot{p} + \Lambda p' + \frac{1}{2}\Lambda' p - Ap)\, dxdt = -\int_0^L \phi^*(x) p(x,0)\, dx + \\ \int_0^T \rho(L)^{-1/2} f^*(t) p_1(L,t) + I_\rho(L)^{-1/2}\tau^*(t) p_3(L,t)\, dt. \end{aligned} \tag{22}$$

As usual, we say that a function u is a weak solution of (12, 15, 17), if (22) holds for all $p \in P_c$. We define weak solutions of (12, 16, 17) similarly. We note that *weak solutions are unique.* To see this, suppose that we have a weak solution u of the clamped system with zero initial and boundary data. Given $F \in P_c$, find $p \in P_c$ such that $\dot{p} + \Lambda p' + \frac{1}{2}\Lambda' p - Ap = F$. The fact that we can find a classical solution of this problem follows from Theorem 2.1 and Duhamel's principle. Hence (22) implies that

$$\int_0^T \int_0^L u^* F\, dxdt = 0$$

for all such F, and thus $u = 0$. The same argument works for weak solutions of the hinged problem.

The physically meaningful solutions of the beam equations are those with finite energy. Thus, we define the *finite energy space* $\mathcal{H} = (L^2(0,L))^4$, the norm of which is given by

$$||u|| = (\int_0^L |u_1|^2 + |u_2|^2 + |u_3|^2 + |u_4|^2\, dx)^{1/2},$$

and say that a weak solution u is a *finite energy solution* if $u \in L^\infty(0,T;\mathcal{H})$.

Theorem 2.2 (Finite Energy Solutions). *If the boundary data f, τ are in $L^2(0,T)$ and the initial data $\phi \in \mathcal{H}$, then each of the systems (12, 15, 17) and (12, 16, 17) has a unique finite energy solution u. In fact, $u \in C(0,T;\mathcal{H})$.*

Proof. We prove the theorem for the case of the clamped beam system. The proof for the hinged beam is similar. We note that uniqueness has already been established.

Suppose first that f and τ are in $C_0^1(0,T)$, and $\phi \in C_0^1(0,L)$. Let u be the classical solution, the existence of which is guaranteed by Theorem 2.1. Let T_1 and T_2 be given by (7) and let $t_0 < \min(T_1, T_2)$. Let γ be the characteristic curve with speed v_1 that ends at (L, t_0), i.e. γ is parameterized by $x = X(t)$, where

$$X(t_0) = L, \qquad \dot{X} = v_1(X).$$

Let $x_0 = X(0)$. We have

$$\dot{u}_1 + v_1 u_1' + \frac{1}{2} v_1' u_1 = \sum_{k=1}^{4} a_{1k} u_k$$

We multiply this by u_1 and integrate the equation over the region Ω bounded by γ, the x-axis, and the t-axis. An application of Green's Theorem then gives

$$\frac{1}{2}\int_0^{t_0} v_1(L) u_1(L,t)^2\, dt = \frac{1}{2}\int_{x_0}^{L} u_1(x,0)^2\, dx + \sum_{k=1}^{4} \iint_\Omega a_{1k} u_k\, dx dt. \tag{23}$$

A similar equation holds for u_3. Thus, we see that

$$\frac{1}{2}\int_0^{t_0} v_1(L) u_1(L,t)^2 + v_2(L) u_3(L,t)^2\, dt \le \mathcal{E}(0) + C\int_0^{t_0} \mathcal{E}(t)\, dt. \tag{24}$$

But integration of (18) and taking into account the boundary conditions (15)gives

$$\begin{aligned}
\mathcal{E}(t) - \mathcal{E}(0) &= \frac{1}{2}(K(L)\rho(L))^{-1/2}\int_0^{t_0} f(t)^2\, dt \\
&+ \frac{1}{2}(EI(L)I_\rho(L))^{-1/2}\int_0^{t_0} \tau(t)^2\, dt \\
&- \rho(L)^{-1/2}\int_0^{t_0} u_1(t) f(t)\, dt - I_\rho(L)^{-1/2}\int_0^{t_0} u_3(t)\tau(t)\, dt \\
&\le (K(L)\rho(L))^{-1/2}\int_0^{t_0} f(t)^2\, dt + (EI(L)I_\rho(L))^{-1/2}\int_0^{t_0} \tau(t)^2\, dt \\
&+ \frac{1}{2}\int_0^{t_0} v_1(L) u_1(L,t)^2 + v_2(L) u_3(L,t)^2\, dt
\end{aligned} \tag{25}$$

Estimates (24) and (25) imply that there is a constant $c1$, independent of the initial and boundary data, such that

$$\mathcal{E}(t_0) \le c1(\mathcal{E}(0) + \int_0^{t_0} f(t)^2 + \tau(t)^2\, dt) \tag{26}$$

for all $0 \le t_0 \le t_1 = \min(T_1, T_2)$. However, we can repeat the analysis over the interval $[t_1, 2t_1]$, then over $[2t_1, 3t_1]$, and so on. We conclude that (26) holds for all $0 \leq t_0 \leq T$. Moreover, the proof reveals (see 24) that the components of $u(L,t)$ are all in $L^2(0,T)$ and have L^2 norms bounded by a constant times the sum of the L^2 norms of the initial and boundary data.

We now see the existence of the finite energy solutions, as follows. Given initial data $\phi \in \mathcal{H}$ and boundary data f and τ in $L^2(0,T)$, pick a sequence ϕ_n in $C_0^\infty(0,L)$ converging to ϕ in $\mathcal{H}$, and sequences f_n and τ_n in $C_0^\infty(0,T)$ converging to f and τ in $L^2(0,T)$ respectively. Let u_n be the sequence of classical solutions with initial data ϕ_n and boundary data f_n and τ_n. The estimate (26) shows that u_n is a Cauchy sequence in $L^\infty((0,T);\mathcal{H})$, and it is clear that the limit u satisfies (22).

We now establish continuity of the solution as an $\mathcal{H}$-valued function. We know that the components of $u(L,t)$ are all in $L^2(0,T)$ and have L^2 norms bounded by a constant times the sum of the L^2 norms of the initial and boundary data. Classical solutions satisfy (18), which integrates to give

$$\begin{aligned}\mathcal{E}(s_2) - \mathcal{E}(s_1) &= \frac{1}{2}\int_{s_1}^{s_2} \rho(L)^{-1/2}(u_2(L,t) - u_1(L,t))f(t) \\ &+ I_\rho(L)^{-1/2}(u_4(L,t) - u_3(L,t))\tau(t)\,dt.\end{aligned}$$

But a limit argument shows that this holds for finite energy solutions as well. Thus, we see that $t \to ||u(t)||$ is continuous (here $u(t)$ is taken to mean $u(\cdot,t)$. Further, we have

$$||u(s2) - u(s_1)||^2 = ||u(s_2)||^2 + ||u(s_1)||^2 - 2(u(s_2), u(s_1)),$$

so the left side of this equation will tend to zero as $s_2 \to s_1$, provided that we can show that $(u(s_2), u(s_1)) \to ||u(s_1)||^2$. This will follow from the weak continuity of u. But the weak continuity is easily established by taking the scalar product of (12) with $p \in C_0^\infty(0,L)$, and integrating by parts to give

$$(u(s_2) - u(s_1), p) = \int_{s_1}^{s_2} (u(t), \Lambda p' + \frac{1}{2}\Lambda' p - Ap)\,dt$$

This is equation is, of course, derived for classical solutions, but it holds for finite energy solutions by the usual limit argument. Since any $v \in \mathcal{H}$ can be approximated by such a p, we see that $(u(s_2) - u(s_1), v) \to 0$ as $s_2 \to s_1$. Thus the continuity is established. This completes the proof.

We should say a little about what this theorem says about the existence of finite energy solutions of the original beam systems (1, 3, 5), and (1, 4, 5). Weak solutions of each system are defined in the usual way. We give details for the clamped system, the hinged system being similar. Let $\mathcal{C}$ be the set of all q and χ in $C^2([0,L] \times [0,T])$ that vanish at $t = T$ and satisfy the homogeneous boundary conditions of the clamped-free system, i.e.

$$\begin{aligned} q(0,t) = 0, &\qquad \chi(0,t) = 0, \\ K(L)(\chi(L,t) + q'(L,t)) = 0, &\quad EI(L)\chi'(L,t) = 0,\end{aligned}$$

for $0 \le t \le T$ and

$$q(x,T) = 0, \quad \chi(x,T) = 0,$$

for $0 \le x \le L$. We say that (w, ψ) is a weak solution of (1, 3, 5) if the following holds for all $(q, \chi) \in \mathcal{C}$:

$$\begin{aligned}
0 &= \int_0^T \int_0^L w(\rho \ddot{q} - (K(q' + \chi)') + \psi(I_\rho \ddot{\chi} - (EI\chi')' + K(\chi + q'))\, dxdt \\
&+ \int_0^L \rho(x)(\dot{q}(x,0)w_0(x) - q(x,0)v_0(x))\, dx \\
&+ \int_0^L I_\rho(x)(\dot{\chi}(x,0)\psi_0(x) - \chi(x,0)\phi_0(x))\, dx \\
&- \int_0^T q(L,t)f(t) + \chi(L,t)\tau(t)\, dt.
\end{aligned}$$

A similar criterion holds for weak solutions of (1, 4, 5) We set $\mathcal{H}_0 = (L^2(0,L))^2$, $\mathcal{V}_c = \{(w,\psi) \in H^1(0,L)^2 : w(0) = \psi(0) = 0$, and $\mathcal{V}_h = \{(w,\psi) \in H^1(0,L)^2 : w(0) = 0$.

Theorem 2.3 (Finite Energy Solutions). *If the boundary data f, τ are in $L^2(0,T)$ and $(w_0, \psi_0) \in \mathcal{V}_c$ and $(v_0, \phi_0) \in \mathcal{H}_0$, then the system (1, 3, 5) has a unique weak solution (w, ψ) such that $(w, \psi) \in C(0, T, \mathcal{V}_c)$, $(\dot{w}, \dot{\psi}) \in C(0, T, \mathcal{H}_0)$.*

Note that we can state a similar theorem for (1, 4, 5). We again call such solutions *finite energy solutions.*

Proof. It is easy to see that classical solutions of (1, 3, 5) correspond to classical solutions of (12, 15, 17) under the transformation (11). The proof of Theorem 2.2 exhibited finite energy solutions of (12, 15, 17) as limits of classical solutions. It is a simple task to verify that the images of these sequences under the transformation (11) converge to finite energy solutions of (1, 3, 5).

3 Smoothing Properties of the Beam Equations.

Here we consider two auxiliary problems concerning the beam equations. In this section, we establish smoothing properties of these auxiliary problems. In the next section, we show that the smoothing properties are associated with the controllability problems posed in the introduction.

The first system is associated with the finite clamped system already considered. This system consists of a semi-infinite beam, the end of which is clamped at the origin. Equation (1) must be satisfied for $0 < x < \infty$, and the clamped end conditions,

$$w(0,t) = 0, \quad \psi(0,t) = 0,$$

must hold at $x = 0$.

The second system is associated with the finite hinged system. This system is most easily thought of as consisting of two semi-infinite beams, the first satisfying (1) for $x < L$, the second satisfying the equations for $x > L$. The beams are connected by a hinge at their ends at $x = L$, and the first beam is connected to the origin by a hinge. The conditions at $x = 0$ and $x = L$ are

$$w(0,t) = 0, \qquad \psi'(L,t) = 0.$$

In this case, $\psi'(0^-,t) = \psi'(0^+,t)$, since there is no external applied torque at the origin, and $w(L^-,t) = w(L^+,t)$, since the displacement of each beam is the same at $x = L$.

In order to prove existence of solutions (Theorems 3.1 and 3.2), we assume in this section that ρ, EI, I_ρ and K are all positive, C^1 functions of the space variable, and that they are all constant in the exterior of a bounded interval, although the latter assumption is not essential. For the smoothing result, Theorem 3.4, in addition to these assumptions, we assume that the functions are C^2 functions of the space variable and that the wave speeds (6) are different at each point.

It is convenient to work with the first order equations (12). We use the terms *auxiliary problem 1* and *auxiliary problem 2* to refer to the problems for the semi-infinite clamped beam, and the pair of semi-infinite hinged beams respectively. As first order systems, the problems take the following forms:

Auxiliary Problem 1

$$\begin{aligned} \dot{u} + \Lambda u' = Au - \frac{1}{2}\Lambda' u, &\qquad (x,t) \in (0,\infty) \times R, \\ u(x,0) = \phi(x), &\qquad x \in (0,\infty), \\ u_2(0,t) - u_1(0,t) = u_4(0,t) - u_3(0,t) = 0, &\qquad t \in R. \end{aligned} \tag{27}$$

Auxiliary Problem 2

$$\begin{aligned} \dot{u}_{\mathrm{I}} + \Lambda_{\mathrm{I}} u'_{\mathrm{I}} = A_{\mathrm{I}} u - \frac{1}{2}\Lambda'_{\mathrm{I}} u, &\qquad (x,t) \in ((-\infty,0) \cup (0,\infty)) \times R, \\ \dot{u}_{\mathrm{II}} + \Lambda_{\mathrm{II}} u'_{\mathrm{II}} = A_{\mathrm{II}} u - \frac{1}{2}\Lambda'_{\mathrm{II}} u, &\qquad (x,t) \in ((-\infty,L) \cup (L,\infty)) \times R, \\ u(x,0) = \phi(x), &\qquad x \in (-\infty,\infty), \\ u_2(0,t) - u_1(0,t) = 0, &\qquad t \in R, \\ u_4(L,t) - u_3(L,t) = 0, &\qquad t \in R, \end{aligned} \tag{28}$$

where $u_{\mathrm{I}} = (u_1, u_2)^T$, $u_{\mathrm{II}} = (u_3, u_4)^T$, A_{I} is the 2 by 4 matrix obtained by deleting the last two rows of A, A_{II} is the 2 by 4 matrix obtained by deleting the first two rows of A, Λ_{I} is the 2 by 2 matrix obtained by deleting the last two rows and columns of Λ, and Λ_{II} is the 2 by 2 matrix obtained by deleting the first two rows and columns of Λ.

Classical solutions for auxiliary problem 1 are simply functions that are continuously differentiable in the closed right half plane, and satisfy the equations (27). A classical solution of auxiliary problem 2 is a function u for which

1. u is C^1 in the strip $0 \le x \le L$ and the restrictions of u to the sets $x < 0$ and $x > L$ may be extended to be C^1 functions in the closures of these sets.
2. u_1 and u_2 are continuous on the line $x = L$, and u_3 and u_4 are continuous on the line $x = 0$.
3. $u_1 - u_2$ and $u_3 - u_4$ are continuous.
4. The equations (28) are satisfied.

Theorem 3.1 (Classical Solutions).

1. *If $\phi \in C^1[0, \infty)$ and*

$$\phi_1(0) - \phi_2(0) = \phi_3(0) - \phi_4(0) = 0,$$

then (27) has a unique classical solution.

2. *Suppose that the following conditions are satisfied.*

   (a) *ϕ is C^1 in the interval $0 \le x \le L$ and the restrictions of ϕ to the intervals $x < 0$ and $x > L$ may be extended to be C^1 functions in the closures of these intervals.*

   (b) *ϕ_1 and ϕ_2 are continuous at the point $x = L$, and ϕ_3 and ϕ_4 are continuous at the point $x = 0$.*

   (c) *$\phi_1 - \phi_2$ and $\phi_3 - \phi_4$ are continuous, and $\phi_1(0) - \phi_2(0) = 0$, $\phi_3(L) - \phi_4(L) = 0$.*

   Then (28) has a unique classical solution.

The simple proof involves making use of the characteristic curves of the equations (see the comments following the statement of Theorem 2.1).

Let $I_1 = (-\infty, \infty)$ and $I_2 = (0, \infty)$. For $k = 1, 2$, we define the finite energy space of auxiliary problem k to be $\mathcal{H}_k = (L^2(I_k))^4$, with norm given by

$$||u|| = (\int_{I_k} |u_1|^2 + |u_2|^2 + |u_3|^2 + |u_4|^2 \, dx)^{1/2}.$$

(We use the same symbol for each norm, but this will not cause confusion since the two problems are separate, and it will be clear from the context which norm we are referring to). $\frac{1}{2}||u||^2$ represents the mechanical energy of each system, and it is easy to see that for classical solutions this is constant. We use semigroup theory to investigate the existence of finite energy solutions, although an alternative procedure would be to proceed as in the proof of Theorem 2.1.

To this end, let $D_1 = \{u \in (H^1(I_1))^4 : u_1(0) - u_2(0) = u_3(0) - u_4(0) = 0\}$ and consider the operator $\mathcal{A}_1$ on $\mathcal{H}_1$ with domain D_1, given by

$$\mathcal{A}_1 u = -\Lambda u' - \frac{1}{2}\Lambda' u + Au.$$

Similarly, let D_2 be the set of functions $u \in \mathcal{H}_2$ such that

1. u_1 and u_2 are in $H^1(-\infty, 0) \cap H^1(0, \infty)$,
2. u_3 and u_4 are in $H^1(-\infty, L) \cap H^1(L, \infty)$,
3. $u_1 - u_2$ and $u_3 - u_4$ are almost everywhere equal to continuous functions, and in this sense $u_1(0) - u_2(0) = 0$, $u_3(L) - u_4(L) = 0$,

and consider the operator $\mathcal{A}_2$ on $\mathcal{H}_2$ with domain D_2, given by

$$\mathcal{A}_2 u = -\Lambda u' - \frac{1}{2}\Lambda' u + Au.$$

Theorem 3.2 (Finite Energy Solutions). *$\mathcal{A}_1$ and $\mathcal{A}_2$ are the infinitesimal generators of strongly continuous unitary groups $U_1(t)$ and $U_2(t)$ on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively.*

Proof. It is easy to check that both $i\mathcal{A}_1$ and $i\mathcal{A}_2$ are closed, densely defined and symmetric. In the special case $A = 0$, it is easy to check that the ranges of $\mathcal{A}_1 \pm I$ and $\mathcal{A}_2 \pm I$ are $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, since this reduces to solving four first order ordinary differential equations, coupled only by their boundary conditions (one can write down the solution of these explicitly). Thus, if $A = 0$, then $i\mathcal{A}_1$ and $i\mathcal{A}_2$ are self-adjoint. But iA is itself a bounded, self-adjoint operator and perturbations of unbounded self-adjoint operators by bounded self-adjoint operators are self-adjoint. Thus $i\mathcal{A}_1$ and $i\mathcal{A}_2$ are self-adjoint in the general case. Thus, by Stone's Theorem, $\mathcal{A}_1$ and $\mathcal{A}_2$ are the infinitesimal generators of strongly continuous unitary groups. This completes the proof.

We refer to $U_1(t)\phi$, $U_2(t)\phi$, for ϕ in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, as being *finite energy solutions.*

It is convenient to define

$$r_1(x) = \int_0^x \frac{ds}{v_1(s)}, \qquad r_2(x) = \int_0^x \frac{ds}{v_2(s)}.$$

Lemma 3.3 (Trace Property) *The restrictions of components of finite energy solutions to lines parallel to the t-axis are locally L^2 functions. Moreover, if u*

is such a solution, then the mapping $x \to u_k(x,\cdot)$ into $L^2_{loc}(R)$, is continuous everywhere except possibly at $x = 0$ for (28) and $k = 1, 2$, and at $x = L$ for (28) and $k = 3, 4$. At these discontinuities, the left and right limits of the mapping exist.

Proof. It suffices to work with a classical solution and use the usual density argument to get the general result, after appropriate estimates are obtained. Let u be a classical solution of either (27) or (28). Then

$$\frac{\partial}{\partial t}u_2^2 - \frac{\partial}{\partial x}v_1 u_2^2 = 2\sum_{k=1}^{4} a_{2k}u_2u_k. \tag{29}$$

Let $\bar{x} \geq 0$, $\bar{t} > 0$, and let γ_1 be the characteristic curve given by

$$r_1(x) + t = r_1(\bar{x}) + \bar{t}.$$

This curve intersects the x-axis at the point $(x_1, 0)$, where $x_1 = r_1^{-1}(r_1(\bar{x}) + \bar{t})$. Let Ω_1 be the region bounded by γ_1, the line segment from $(\bar{x}, \bar{t})$ to $(\bar{x}, 0)$, and the line segment from $(\bar{x}, 0)$ to $(x_1, 0)$. Integrating (29) over Ω_1 and applying Green's Theorem, we obtain

$$\begin{aligned}\int_0^{\bar{t}} v_1(\bar{x})u_2(\bar{x},t)^2\,dt &= \int_{\bar{x}}^{x_1} u_2(x,0)^2\,dx + 2\sum_{k=1}^{4}\iint_{\Omega_1} a_{2k}u_2u_k\,dxdt \\ &\leq C(1+\bar{t})||u(0)||^2. \end{aligned}\tag{30}$$

Similarly, we let γ_2 denote the curve given by

$$t - r_1(x) = \bar{t} - r_1(\bar{x}),$$

and, if $r_1(\bar{x}) - \bar{t} \geq 0$, we let $x_2 = r_1^{-1}(r_1(\bar{x}) - \bar{t})$. We obtain an estimate for the trace of u_1 on the line $x = \bar{x}$ by integrating the first equation of motion over Ω_2, where, Ω_2 is the region bounded by γ_2, the x-axis, and the line $x = \bar{x}$. However, if $r_1(\bar{x}) - \bar{t} < 0$, we let Ω_2 be the region in the first quadrant bounded by γ_2, the x-axis, the t-axis and the line $x = \bar{x}$. This leads to the estimate

$$\int_0^{\bar{t}} v_1(\bar{x})u_1(\bar{x},t)^2\,dt \leq \int_0^{t_0} v_1(0)u_1(0,t)^2\,dt + C(1+\bar{t})||u(0)||^2,$$

where $t_0 = \bar{t} - r_1(\bar{x})$. But $u_1(0,t) = u_2(0,t)$, so we can use (30) to estimate the integral on the right side of this equation. Estimates for $\bar{t}$ or $\bar{x}$ negative may be obtained similarly. The analysis of u_3 and u_4 is also similar.

To prove the continuity of the mapping $x \to u_2(x,\cdot)$, we integrate (29) over the rectangle bounded by lines $x = x_1$, $x = x_2$, $t = t_1$, $t = t_2$ and obtain

$$\begin{aligned}\int_{t_1}^{t_2} v_1(x_2)u_2(x_2,t)^2\,dt &= \int_{t_1}^{t_2} v_1(x_1)u_2(x_1,t)^2\,dt - 2\sum_{k=1}^{4}\iint_{\Omega_1} a_{2k}u_2u_k\,dxdt \\ &+ \int_{x_1}^{x_2} u_2(x,t_1)^2\,dx - \int_{x_1}^{x_2} u_2(x,t_2)^2\,dx.\end{aligned}$$

This shows that the mapping

$$x \to \int_{t_1}^{t_2} v_1(x) u_2(x,t)^2 \, dt$$

is continuous. Now we may proceed as in the proof of Theorem 2.2 to complete the proof of continuity.

In Theorem 3.4, we assume that ρ, EI, I_ρ and K are all positive, C^2 functions of the space variable, and that they are all constant in the exterior of a bounded interval. We also assume that the wave speeds (6) are different at each point.

Theorem 3.4 (Smoothing Property).

1. *If $u(t) = U_1(t)\phi$, where $\phi \in \mathcal{H}_1$ has support in the interval $[0, L]$, then the following statements are true.*
   - *(a)* $u_1(t) \in H^1(0, r_1^{-1}(t - r_1(L)))$ *if* $t > r_1(L)$.
   - *(b)* $u_3(t) \in H^1(0, r_2^{-1}(t - r_2(L)))$ *if* $t > r_2(L)$.
   - *(c)* $u_2(t) \in H^1(r_1^{-1}(r_1(L) - t), \infty)$ *if* $0 \le t < r_1(L)$,
$u_2(t) \in H^1(0, \infty)$ *if* $t \ge r_1(L)$.
   - *(d)* $u_4(t) \in H^1(r_2^{-1}(r_2(L) - t), \infty)$ *if* $0 \le t < r_2(L)$,
$u_4(t) \in H^1(0, \infty)$ *if* $t \ge r_2(L)$.
2. *If $u(t) = U_2(t)\phi$, where $\phi \in \mathcal{H}_2$ has support in the interval $[0, L]$, then the following statements are true.*
   - *(a)* $u_1(t) \in H^1(0, r_1^{-1}(t - r_1(L)))$ *if* $t > r_1(L)$,
$u_1(t) \in H^1(-\infty, 0)$ *if* $t \ge 0$.
   - *(b)* $u_3(t) \in H^1(-\infty, r_2^{-1}(t))$ *if* $0 \le t < r_2(L)$,
$u_3(t) \in H^1(-\infty, 0)$ *if* $t > r_2(L)$,
$u_3(t) \in H^1(L, \infty)$ *if* $t \ge 0$.
   - *(c)* $u_2(t) \in H^1(r_1^{-1}(r_1(L) - t), \infty)$ *if* $0 \le t < r_1(L)$,
$u_2(t) \in H^1(0, \infty)$ *if* $t \ge r_1(L)$,
$u_2(t) \in H^1(-\infty, 0)$ *if* $t \ge 0$.
   - *(d)* $u_4(t) \in H^1(r_2^{-1}(2r_2(L) - t), L)$ *if* $t > r_2(L)$,
$u_4(t) \in H^1(L, \infty)$ *if* $t \ge 0$.

Proof. It suffices to work with smooth solutions (e.g. with initial data in the domain of the square of the infinitesimal generators), and then use the standard density argument to prove the appropriate estimates.

Differentiation of the second of the equations of motion with respect to x gives

$$\dot{u}_2' - v_1 u_2'' - \frac{3}{2} v_1' u_2' = \sum_{j=1}^{4} (a_{2j} u_j' + a_{2j}' u_j).$$

Using the notation

$$D_t = \frac{\partial}{\partial t} - v_1 \frac{\partial}{\partial x},$$

this may be written

$$D_t v_1^{3/2} u_2' = \sum_{j=1}^{4} v_1^{3/2}(a_{2j}u_j' + a_{2j}' u_j).$$

We make use of the fact that the right side of this equation does not involve u_2' because $a_{22} = 0$. The other equations of motion yield

$$\begin{aligned}
2v_1 u_1' &= -D_t u_1 - \frac{1}{2} v_1' u_1 + \sum_{j=1}^{4} a_{1j} u_j, \\
(v_1 + v_2) u_3' &= -D_t u_3 - \frac{1}{2} v_3' u_3 + \sum_{j=1}^{4} a_{3j} u_j, \\
(v_1 - v_2) u_4' &= -D_t u_4 - \frac{1}{2} v_4' u_4 + \sum_{j=1}^{4} a_{4j} u_j. \qquad (31)
\end{aligned}$$

Let γ_2 be the characteristic curve

$$r_1(x) + t = r_1(\bar{x}) + \bar{t}$$

which starts on the x-axis and terminates at the point $(\bar{x}, \bar{t})$. We assume that this curve lies to the right of the curve $r_1(x) + t = r_1(L)$, i.e. $r_1(\bar{x}) + \bar{t} > r_1(L)$. We integrate (3) over γ_2, making use of the identities (31) and the fact that D_t is a directional derivative along γ_2. Assume first that u is a solution of (27). Recall that the initial data vanishes on the x-axis, at points to the right of L. Thus we obtain for (27), after an integration by parts,

$$v_1(\bar{x})^{3/2} u_2'(\bar{x}, \bar{t}) = \sum_{j=1}^{4} (\sigma_{2j}(\bar{x}) u_j(\bar{x}, \bar{t}) + \int_{\gamma_2} \delta_{2j} u_j \, dt), \qquad (32)$$

where the functions σ_{2j} and δ_{2j} are bounded and continuous. Multiplying this by $u_2'(\bar{x}, \bar{t})$, integrating with respect to $\bar{x}$ from $x_0 = \max(r_1^{-1}(r_1(L) - t), 0)$ to ∞, and using the Cauchy-Schwarz inequality yields the estimate

$$(\int_{x_0}^{\infty} |u_2'(\bar{x}, \bar{t})|^2 \, d\bar{x})^{1/2} \leq C(||u(\bar{t})|| + \int_0^{\bar{t}} ||u(t)|| \, dt \leq C(1 + \bar{t})||u(0)||, \qquad (33)$$

where C is a constant independent of u. This proves (1c). The proof of the first two statements of (2c) is similar. The only difference is due to the discontinuity of u_3 and u_4 on the line $x = L$, which leads to an extra term

$$C \int_0^{\bar{t}} |u_3(L^+, t) - u_3(L^-, t)|^2 + |u_4(L^+, t) - u_4(L^-, t)|^2 \, dt)^{1/2}$$

in estimate (33). But this term may be estimated in terms of the initial energy by Lemma 3.3.

Given nonnegative $\bar{x}$ and $\bar{t}$ such that $\bar{t} - r_1(\bar{x}) > r_1(L)$, let γ_1 be the characteristic curve which starts at $(0, \bar{t} - r_1(\bar{x}))$, ends at $(\bar{x}, \bar{t})$ and is given by

$$t - r_1(x) = \bar{t} - r_1(\bar{x}).$$

Let u again be a solution of (27). Proceeding as in the analysis that lead to (32) gives

$$\begin{aligned} v_1(\bar{x})^{3/2} u_1'(\bar{x}, \bar{t}) &= v_1(0)^{3/2} u_1'(0, \bar{t} - r_1(\bar{x})) \\ &+ \sum_{j=1}^{4} (\sigma_{1j}(\bar{x}) u_j(\bar{x}, \bar{t}) - \sigma_{1j}(0) u_j(0, \bar{t} - r_1(\bar{x})) \\ &+ \int_{\gamma_1} \delta_{1j} u_j \, dt). \end{aligned} \tag{34}$$

However, the first two equations of motion and the condition $u_1(0,t) = u_2(0,t)$ give

$$v_1(0) u_1'(0,t) = -v_1(0) u_2'(0,t) - v_1'(0) u_2(0,t) + \sum_{k=1}^{4} (a_{1k}(0) - a_{2k}(0)) u_j(0,t),$$

and we may use (32) to rewrite the u_2' term on the right side of this equation. Substituting the resulting expression for $u_1'(0, \bar{t} - r_1(\bar{x}))$ into (34), multiplying by $u_1'(\bar{x}, \bar{t})$ and integrating with respect to $\bar{x}$ leads to an estimate

$$\left(\int_0^{x_0} |u_1'(\bar{x}, \bar{t})|^2 \, d\bar{x}\right)^{1/2} \leq C(1 + \bar{t}) ||u(0)||, \tag{35}$$

where $x_0 = r_1^{-1}(\bar{t} - r_1(L))$. This proves (1a), and a slight modification of the procedure leads to a proof of the first part of (2a).

At this point, it should be clear that the remainder of the proof is largely a repetition of the arguments already given, so we omit it.

4 Boundary Controllability of the Beams.

The following conditions are relevant to our controllability results:

1. ρ, I_ρ, K, and EI are all positive functions of the space variable and all belong to $C^2([0, L])$.

2. The two wave speeds (6) are different at all points on the beam.

3. $T > 2 \max(T_1, T_2)$, where T_1 and T_2 are given by (7).

Theorem 4.1 (Controllability). *Suppose that conditions (1), (2) and (3) above hold. Then the following statements are true.*

1. *Given finite energy initial data of the clamped beam problem (1, 3, 5), there exist control functions $f \in L^2(0,T)$ and $\tau \in L^2(0,T)$, that drive the system to its rest state at time T:*

$$w(x,T) = \psi(x,T) = \dot{w}(x,T) = \dot{\psi}(x,T) = 0, \qquad 0 < x < L.$$

2. (a) *Suppose that there are no nontrivial solutions of the eigenvalue problem (8, 10). Given finite energy initial data of the hinged beam problem (1,4,5), there exist control functions $f \in L^2(0,T)$ and $\tau \in L^2(0,T)$, that drive the system to one of its rest states at time T:*

$$w(x,T) - ax = \psi(x,T) + a = \dot{w}(x,T) = \dot{\psi}(x,T) = 0, \qquad 0 < x < L. \tag{36}$$

   (b) *If there exist nontrivial solutions of the eigenvalue problem (8, 10), then the hinged beam problem (1,4,5) is not even approximately controllable.*

Remark. The proof will show that in cases (1) and (2a) of Theorem 4.1, there exist bounded linear maps from the space of finite energy initial data to the L^2-normed space of control functions.

Proof. The proof is similar to the corresponding proof in [13], so we sketch it here. We work with the first order systems (12, 15, 17) and (12, 16, 17). To prove (1) and (2a), we show that we can steer the finite energy solutions of the first order systems to zero at time T.

The proofs of (1) and (2a) are essentially the same, so for this proof, we let X, denote either $\mathcal{H}_1$ or $\mathcal{H}_2$, the finite energy spaces of Theorem 3.2. We also denote both $U_1(t)$ and $U_2(t)$ by $\mathcal{U}(t)$ and $\mathcal{A}_1$ and $\mathcal{A}_2$ by $\mathcal{A}$.

Consider the subspace $\mathcal{S}$ of X consisting of functions with support in the interval $[0,L]$. We show that we can extend the initial data of the "finite problems" outside the interval $[0,L]$ in such a way that the projection onto $\mathcal{S}$ of the solution of the problems (27) and (28) vanishes at time T. Since this projection corresponds to the values of the solution for $0 \le x \le L$, we obtain the desired solutions of the control problems by using (15) or (16) to define the control functions. Note that Lemma 3.3 implies that f and τ, if defined this way, will be in $L^2(0,T)$. We now show that such an extension of the initial data exists.

First, let g denote the initial data, extended to be in $\mathcal{S}$. Let P denote the projection onto $\mathcal{S}$ and let $U = \mathcal{U}(T)$. Consider the equation

$$\tilde{h} - PU^{-1}PUP\tilde{h} = g$$

Suppose that this can be solved and set $h = P\tilde{h} - U^{-1}PUP\tilde{h}$. Then $PUh = 0$ and $Ph = g$. Thus h agrees with the initial data g on the interval $[0,L]$ and the solution with initial data h vanishes on the interval $[0, L]$ at time T. Thus, h is the desired extension of g. If we solve for h in terms of g, we obtain $h = Rg$, where

$$R = (P - U^{-1}PUP)(I - PU^{-1}PUP)^{-1}.$$

For this to make sense, it is clearly enough to show that PUP is a contraction. Clearly $||PUP|| \leq 1$ because U is unitary. By the smoothing property, Theorem 3.4, PUP is compact. If we assume that $||PUP|| = 1$, then we can use the compactness to show that the set

$$V = \{z \in \mathcal{S} : Uz \in \mathcal{S}\}$$

is non-empty. V is finite dimensional because it is contained in the kernel of $I - PU^{-1}PUP$ and PUP is compact. Also, if $z \in V$ then $Uz \in \mathcal{D}(\mathcal{A})$, the domain of $\mathcal{A}$, because, by the smoothing property, it is smooth enough to be in $\mathcal{D}(\mathcal{A})$, but since it is in $\mathcal{S}$, it must be in $\mathcal{D}(\mathcal{A})$. $Uz \in \mathcal{D}(\mathcal{A})$ implies that $z \in \mathcal{D}(\mathcal{A})$, so V is a subset of $\mathcal{D}(\mathcal{A})$. Thus, $\mathcal{A}$ is a bounded operator on the finite dimensional space V, and as such, must possess an eigenvalue. It is easy to see that the existence of eigenvectors of $\mathcal{A}$ in $\mathcal{S}$ is equivalent to the existence of nontrivial solutions to either (8, 9) (for the clamped beam problem) or (8,10) (for the hinged beam problem). But (8, 9) has no nontrivial solutions and our assumption is that (8, 10) has no nontrivial solutions. Thus, PUP must be a contraction. This completes the proof of (1) and (2a).

For (2b), let $\langle \cdot, \cdot \rangle$ denote the sesquilinear form associated with the energy functional (2). We note that this is not an inner product on the finite energy space of (1, 4, 5), but it is an inner product on the quotient space of initial data modulo the states (36). Let (W, Ψ) denote a solution of the eigenvalue problem (8, 10). Then $p_\mu(t) = \exp(i\mu t)(W, \Psi)$ is a periodic solution of (1) which satisfies (10). It is easy to check that $\langle p_\mu(t), q(t) \rangle$ is constant for all finite energy solutions q. If $q(T)$ is one of the rest states (36), it follows that $\langle p_\mu(t), q(t) \rangle = 0$. But approximate controllability implies that we can find such a solution q with initial data as close as we please to the initial data of p_μ. But this implies that the energy of the solution p_μ must vanish, which is impossible. This completes the proof of the theorem.

We now investigate the possibility of (2b) occurring for the constant coefficient case.

Theorem 4.2 (Constant Coefficient Case). *Suppose that the coefficients ρ, I_ρ, K, and EI are all constant. Then nontrivial solutions of (8, 10) exist if and only if*

$$\frac{K}{EI} = \frac{\rho}{3I_\rho} = \frac{m^2\pi^2}{2L^2}, \tag{37}$$

where m is an odd integer.

Proof. The calculation is simplified considerably by the fact that the following is a first integral of the equations (8):

$$\begin{aligned}
-\frac{K^2}{EI}w^2 + (K - \frac{3I_\rho K^2}{\rho EI} + \frac{2K^3}{\rho EI\mu^2} - I_\rho\mu^2 + \frac{I_\rho^2 K\mu^2}{\rho EI})\psi^2 \quad & + \\
2\frac{K^2}{\rho EI}(\frac{K}{\mu^2} - I_\rho)\psi w' + 2Kw\psi' + (\frac{I_\rho K}{\rho} - EI - \frac{K^2}{\rho\mu^2})(\psi')^2 \quad & = \quad c
\end{aligned}$$

The boundary conditions (10) at $x = 0$ imply that $c = 0$. Substituting the boundary conditions at $x = L$ into the equation shows that all of the Cauchy data of w and ψ must vanish at $x = L$ (and thus w and ψ vanish on $[0, L]$) unless $\rho EI = I_\rho K$ or $\mu^2 = K/I_\rho$. A straight-forward calculation now reveals that the only nontrivial solutions are

$$w(x) = A\sin(\frac{m\pi x}{L}), \qquad \psi(x) = \frac{LKA}{m\pi EI}(1 - \cos(\frac{m\pi x}{L})),$$

where A is arbitrary, $\mu^2 = K/I_\rho$, m is an odd integer and (37) is satisfied.

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