Geometry of Pseudospheres I.

T.H.Marshall

Abstract

The n-dimensional pseudospheres are the surfaces in $\mathbf{R}^{n+1}$ given by the equations ${x_1}^2 + {x_2}^2 + \dots {x_k}^2 - {x_{k+1}}^2 - \dots - {x_{n+1}}^2 = 1$ $(1 \leq k \leq n+1)$.

We consider the pseudospheres as surfaces in $E_{n+1,k}$, where $E_{m,k} = \mathbf{R}^k \times (i\mathbf{R})^{m-k}$, and investigate their geometry in terms of the linear algebra of these spaces. Each of the spaces $E_{m,k}$ has a natural (not generally positive definite) metric, which is inherited by the pseudospheres.

We prove that each matrix with columns in $E_{m,k}$ can be put into a canonical form by premultiplying by an orthogonal matrix (a matrix which effects an isometry of $E_{m,k}$). We term a matrix in this form *bitriangular*. This generalizes upper triangular form for real square matrices.

1 Introduction

There are many obvious similarities between the two geometries of nonzero constant curvature, spherical and hyperbolic, and it is natural to seek methods which deal with both of them at once. One way of doing this is to regard both spaces as "slices" through the complexified n-sphere

$$\tilde{S}^n = \{\mathbf{z} \in \mathbf{C}^{n+1} | z_1^2 + z_2^2 + \dots + z_{n+1}^2 = 1\}$$

where we denote the kth component of $\mathbf{z}$, $(1 \leq k \leq n+1)$, by $z_k = x_k + iy_k$.

We define a the *semi-Euclidean* spaces, $E_{n+1,k}$, by

$$E_{n+1,k} = \mathbf{R}^k \times (i\mathbf{R})^{n+1-k} \qquad (0 \leq k \leq n+1)$$

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Each $E_{n+1,k}$ is thus an $(n+1)$-dimensional *real* vector subspace of $\mathbf{C}^{n+1}$ obtained by taking the real parts only of the first k coordinates and the imaginary parts only of the remaining $n+1-k$ coordinates.

Now if we take the intersection of $\tilde{S}^n$ with $E_{n+1,k}$ for $k=n+1$ and $k=1$, we obtain, respectively, the ordinary n-sphere, and a pair of hyperboloids. In the latter case, we denote by Q_n the component lying in the halfspace $x_1>0$. By changing the sign of the metric on Q_n we obtain the hyperboloid model for hyperbolic n-space. It can be seen then that spherical and hyperbolic space occur essentially as two instances of the more general class of *pseudospheres*, $S_{n,k}$ defined by

$$S_{n,k} = \tilde{S}^n \cap E_{n+1,k} \qquad (1 \leq k \leq n+1)$$

Explicitly, $S_{n,k}$ is given by,

$$\begin{aligned} S_{n,k}=\{(x_1, x_2, \ldots x_k, iy_{k+1}, \ldots, iy_{n+1} \in E_{n+1,k} \mid \\ {x_1}^2 + {x_2}^2 + \ldots {x_k}^2 - {y_{k+1}}^2 - \cdots - {y_{n+1}}^2 = 1\} \end{aligned}$$

It is these spaces that are the subject of this paper.

Our basic approach to these geometries is linear algebraic. We adapt the methods and results of linear algebra from $\mathbf{R}^n$ to the spaces $E_{n+1,k}$ and to the matrices with columns in these spaces.

Clearly each $S_{n,k}$ is a smooth, imbedded and, for $k>1$, connected submanifold of $E_{n+1,k}$. In Section 2 we define a metric on each $S_{n,k}$, this metric being inherited in a natural way from the complexified sphere. One of the features that distinguishes spherical and hyperbolic spaces from the other pseudospheres is that they alone have a positive definite metric. The metrics on pseudospheres in general only satisfy the weaker criterion of non-degeneracy.

In Section 3 we define and characterize, for each $S_{n,k}$, its *hyperplanes*, which are essentially the totally geodesic submanifolds of codimension one. With each hyperplane we associate a *normal vector* in $E_{n+1,k}$. Generally we can then determine an orientation on a hyperplane by a particular choice of normal vector. In section 4 we characterize isometries on hyperplanes in terms of matrix operations on their normal vectors. We identify the group of isometries of each space $E_{m,k}$ and characterize the matrices, which effect these. Following the real case ($k=m$) we term such matrices *orthogonal*.

The main result of this paper is Theorem 4, which states that every matrix with columns in $E_{m,k}$ can be put into a unique canonical form by premultiplying by an orthogonal matrix. We call a matrix in this canonical form *bitriangular*. For real square matrices bitriangular means simply simply upper triangular, with the added stipulation that the first nonzero entry in each row is positive to guarantee uniqueness.

The second part of this paper [M] will appear separately. In it we apply bitriangular form to characterize geometric properties of sequences of hyperplanes, such as their incidence properties, in algebraic terms. We also derive an n-dimensional version of the cosine rule for spherical and hyperbolic spaces.

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2 Metrics

Each point in $\mathbf{C}^{n+1}$ has an $n+1$ complex dimensional tangent space which is naturally isomorphic to $\mathbf{C}^{n+1}$ itself. The usual dot product defined by

$$\mathbf{z}^{(1)} \cdot \mathbf{z}^{(2)} = \sum_{i=0}^{n+1} z_i^{(1)} z_i^{(2)}$$

(Note, this should be distinguished from the Hermitian product) determines a smooth complex valued bilinear form on $\mathbf{C}^{n+1}$. The tangent spaces and bilinear form so defined are inherited in an obvious way by the complex sphere $\tilde{S^n}$.

The subspaces $E_{n+1,k}$ and their submanifolds also inherit, in the obvious way, tangent spaces and bilinear forms from $\mathbf{C}^{n+1}$. As in the Euclidean case, there is an obvious isomorphism between the tangent space at each point of $E_{n+1,k}$ and $E_{n+1,k}$ itself, which we use tacitly. The bilinear form is positive definite on the real sphere $S_{n,n+1}$, negative definite on $S_{n,1}$, and indefinite in other cases. For example, whenever $2 \leq k \leq n$ the two vectors $\mathbf{u} = (0, 1, 0, \ldots, 0)$ and $\mathbf{v} = (0, 0, \ldots, i)$ are tangent to $S_{n,k}$ at the point $(1, 0, \ldots, 0)$ and we have $\mathbf{u} \cdot \mathbf{u} = 1$, $\mathbf{v} \cdot \mathbf{v} = -1$.

In contrast to the real case, $\mathbf{u} \cdot \mathbf{u}$ may be negative, or zero for nonzero $\mathbf{u}$ so that its square root, the "norm" of $\mathbf{u}$, need not be real. The following terminology originates from relativity theory.

Definition. A vector $\mathbf{u} \in E_{m,k}$ is *spacelike* if $\mathbf{u} \cdot \mathbf{u} > 0$ or $\mathbf{u} = \mathbf{0}$, *timelike* if $\mathbf{u} \cdot \mathbf{u} < 0$ and *null* if $\mathbf{u} \cdot \mathbf{u} = 0$, $\mathbf{u} \neq 0$.

Definition. For $\mathbf{u} \in E_{m,k}$, we define $N(\mathbf{u}) = \sqrt{\mathbf{u} \cdot \mathbf{u}}$, where the square root is chosen to be either nonnegative or a nonnegative multiple of i. We define $\mathbf{u}$ to be a *unit vector* if $|N(\mathbf{u})| = 1$.

The spaces $E_{n+1,k}$ and $S_{n,k}$ are all examples of *semi-Riemannian* manifolds, that is manifolds equipped with a bilinear form $<\ ,\ >$ which is nondegenerate (that is $(\forall \mathbf{v} < \mathbf{u}, \mathbf{v} >= 0) \Rightarrow \mathbf{u} = \mathbf{0}$) but not neccesarily positive definite. We refer to this bilinear form as the *metric*. In the spaces we consider the metric is always given by the dot product.

The general theory of semi-Riemannian manifolds is discussed in O'Neill's book [O] to which we will occasionally refer. Many aspects of Riemannian geometry generalize immediately to the semi-Riemannian case, including connections, parallel transport, the curvature tensor and geodesics.

3 Totally Geodesic Surfaces

A *totally geodesic surface* of a semi-Riemannian manifold is defined to be an imbedded submanifold, M with the property that any geodesic arc joining two points in M, itself lies in M. In particular, a one-dimensional totally geodesic surface is just an open geodesic arc. In the spaces $E_{n+1,k}$ and $S_{n,k}$ the totally geodesic surfaces take a particularly simple form.

Theorem 1 *The complete d-dimensional totally geodesic surfaces of $E_{n+1,k}$ are the d-dimensional affine surfaces in $E_{n+1,k}$. The complete d-dimensional totally geodesic surfaces of $S_{n,k}$ are components of the intersections between $S_{n,k}$ and $(d+1)$-dimensional Euclidean vector subspaces of $E_{n+1,k}$.*

Proof:- The geodesics of $E_{n+1,k}$ are the Euclidean straight lines [O, example 25 p69] from which the first assertion readily follows.

By [O pp111-112] the complete geodesics of $S_{n,k}$ are given by the components of

$$\Pi \bigcap S_{n,k}$$

where Π is a two-dimensional Euclidean plane through the origin of $E_{n+1,k}$ and the second assertion of the theorem follows.$\square$

Definition. a *linear subspace* of $S_{n,k}$ is a non-empty intersection of $S_{n,k}$ with a linear subspace Π of $E_{n+1,k}$.

From the above theorem it follows that the components of linear subspaces are totally geodesic surfaces so the two are very closely related. However, using linear subspaces avoids a number of tedious technical problems that arise from using totally geodesic surfaces. One drawback with linear subspaces is their apparent dependence on a particular imbedding of $S_{n,k}$ in $E_{n+1,k}$. However we shall show in the next section that, for $k > 1$, they can be characterized purely in terms of the geometry of $S_{n,k}$. This is clear immediately when the linear subspace is connected, for then it is simply a totally geodesic surface. When $k = 1$ we agree to define as subspaces of $S_{n,k}$, sets which satisfy the above definition for *some* choice of coordinates.

Definition. A *hyperplane* in $S_{n,k}$ is a linear subspace of codimension one. A *hyperplane* in $E_{m,k}$ is an affine subspace of codimension one.

There is a very simple relationship between the normals to a hyperplane $\Pi \in E_{n+1,k}$ and the normals to $\Pi \bigcap S_{n,k}$.

Lemma 2 *If Π is a hyperplane in $E_{n+1,k}$, $\mathbf{x} \in \Pi \bigcap S_{n,k}$, and $\mathbf{n}$ is normal to Π, then the vector $\mathbf{n}$ in the tangent space to $E_{n+1,k}$ at $\mathbf{x}$ is tangent to $S_{n,k}$ (and hence normal to the hyperplane $\Pi \bigcap S_{n,k}$)*

Proof:- If $\mathbf{y} \in E_{n+1,k}$, we denote by $\mathbf{y}^*$ the vector obtained by dividing the last $n + 1 - k$ components of $\mathbf{y}$ by i. That is $\mathbf{y}^*$ is obtained from $\mathbf{y}$ by viewing all the components as real numbers. Since the notion of tangency is independent of the choice of metric, $\mathbf{n} \in E_{n+1,k}$ is tangent to $S_{n,k}$ at $\mathbf{x}$ if and only if $\mathbf{n}^* \in \mathbf{R}^{n+1}$ is tangent at $\mathbf{x}^*$ to the surface

$$\{\mathbf{x} \mid x_1^2 + x_2^2 \ldots + x_k^2 - x_{k+1}^2 - \cdots - x_{n+1}^2 = 1\}$$

in $\mathbf{R}^{n+1}$. This occurs when

$$\sum\nolimits_{i=1}^{k} n_i^* x_i^* - \sum\nolimits_{i=k+1}^{n+1} n_i^* x_i^* = 0 \tag{1}$$

but the left hand side of this equation is the same as $\mathbf{n} \cdot \mathbf{x}$ which is zero by assumption.$\Box$

We have shown that the normal vectors to Π in $E_{n+1,k}$ and to $\Pi \bigcap S_{n,k}$ in $S_{n,k}$ are the same, where these vectors are construed, in the natural way, as belonging to $E_{n+1,k}$. We define an *oriented hyperplane* in either $E_{n+1,k}$ or $S_{n,k}$ to be a hyperplane with a particular choice of unit normal vector which we will designate the *outward* unit normal. If $\Pi \in E_{n+1,k}$ is an oriented hyperplane, the intersection of Π with $S_{n,k}$ inherits the orientation in the obvious way. We define an oriented hyperplane to be *spacelike* (resp. *null, timelike*) if its outward unit normal is spacelike (resp. null, timelike). Given two oriented hyperplanes P_1 and P_2 in $E_{n+1,k}$ or $S_{n,k}$ with outward unit normals $\mathbf{n}_1$ and $\mathbf{n}_2$ respectively, which are either both spacelike or both timelike, the *angle*, $\theta(\mathbf{n}_1, \mathbf{n}_2)$ between them is given by

$$\theta(\mathbf{n}_1, \mathbf{n}_2) = -\arccos\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{N(\mathbf{n}_1)N(\mathbf{n}_2)}\right) \tag{2}$$

wherever this is defined.

4 Isometries

Definition. Let M and N be semi-Riemannian manifolds with metrics $< \, , \, >$ and $< \, , \, >'$ respectively. An *isometry* is a diffeomorphism ϕ from M to N such that for any point $p \in \mathrm{M}$ and any two tangent vectors $\mathbf{v}$ and $\mathbf{w}$ at p,

$$< \mathrm{d}\phi(\mathbf{v}), \mathrm{d}\phi(\mathbf{w}) >' = < \mathbf{v}, \mathbf{w} >$$

An isometry *of* M will mean an isometry from M to itself.

In the case of the spaces $E_{m,k}$ and $S_{n,k}$, the isometries are the maps that preserve the dot product. In terms of matrices the isometries of $E_{m,k}$ are given by the following group

Definition. $\mathrm{Q}(m, k)$ is the group of $m \times m$ matrices whose first k columns are in $E_{m,k}$ and whose remaining columns are imaginary multiples of vectors in $E_{m,k}$, for which

$$M^{-1} = M^t$$

Members of $\mathrm{Q}(m, k)$ are referred to as *orthogonal* matrices. In the real case ($k = m$) this, of course, coincides with the familiar definition and, as in that case, we may characterize the orthogonal matrices as those whose column, or row, vectors form an orthonormal set.

It is elementary to show that the linear maps on $E_{m,k}$ given by multiplying by matrices in $\mathrm{Q}(m,k)$, are exactly the isometries of $E_{m,k}$. Moreover, since the orthogonal matrices in $\mathrm{Q}(n+1,k)$ leave $S_{n,k}$ invariant, these matrices also determine isometries on this space. For $k \geq 2$ these are in fact *all* the isometries of $S_{n,k}$ ([O Ch. 9 Proposition 8]), so that in this case the definition of linear subspace is seen to be independent of the choice of coordinates. For the same reason the dot product between the unit normals of two oriented hyperplanes is also invariant for $k \geq 2$.

When $k = 1$ there are isometries of $S_{n,k}$ which are not restrictions of isometries of the ambient space. This is clear if one recalls that $S_{n,1}$ is disconnected so that, isometries of $S_{n,k}$ can be obtained by applying isometries to each component separately.

However the group of isometries of the component Q_n is given by the index two subgroup $\mathrm{Q}^+(n+1,1)$ of $\mathrm{Q}(n+1,1)$ which leaves the components of $S_{n,1}$ invariant ([B] 3.7 or [O] pp237-38). The matrices in this subgroup are exactly those for which the top left entry is positive. This is an immediate consequence of this simple lemma

Lemma 3 *If* $\mathbf{u} = (\alpha; i\mathbf{a})^t$, $\mathbf{v} = (\beta; i\mathbf{b})^t$ *are nonzero spacelike vectors in* $E_{m,1}$ *then* $\mathbf{u}\cdot\mathbf{v}$ *is positive or negative according as the signs of* α *and* β *are the same or different.*

Proof:- Since $N(\mathbf{u})^2 = \alpha^2 - N(\mathbf{a})^2 > 0$, we have $|\alpha| > ||\mathbf{a}||$ and similarly $|\beta| > ||\mathbf{b}||$. If α and β have the same sign then

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= \alpha\beta - \mathbf{a}\cdot\mathbf{b} \\ &\geq \alpha\beta - ||\mathbf{a}||||\mathbf{b}|| > 0 \end{aligned} \tag{3}$$

If α and β have opposite sign then $(-\mathbf{u})\cdot\mathbf{v} > 0$ whence $\mathbf{u}\cdot\mathbf{v} < 0$ $\square$

The group $\mathrm{Q}(n+1,1)$ is thus generated by the subgroup $\mathrm{Q}^+(n+1,1)$ and the matrix $-\mathrm{I}_{n+1}$ which interchanges the components Q_n and $-Q_n$. It is now clear that any isometry of $S_{n,1}$ can be obtained by applying a matrix in $\mathrm{Q}(n+1,1)$ followed by an arbitrary isometry on one of the components. Evidently if the image of two oriented hyperplanes are themselves oriented hyperplanes, the dot product of their respective outward normals will be unchanged.

In spite of the rather awkward structure of the isometry group, we continue, for the sake of uniformity, using $S_{n,1}$ rather than Q_n.

Changing the sign of the metric on Q_n gives a model for hyperbolic n-space. Most of the results found for the pseudosphere $S_{n,1}$ require only trivial modifications to apply to Q_n. The definition of hyperplanes and normal vectors is exactly the same in both cases, and the notion of angle also coincides. Clearly a set of hyperplanes in Q_n has the same incidence properties as the corresponding set in $S_{n,1}$. A feature peculiar to $S_{n,1}$, which is inherited by Q_n is that all hyperplanes are timelike (the spacelike and null hyperplanes in $E_{n,1}$ being all disjoint from $S_{n,1}$)

It is possible to describe the Semi-Euclidean spaces and their linear maps purely in terms of real numbers, using vectors in $\mathbf{R}^m$ and the inner product

$$< \mathbf{u}, \mathbf{v} >= \mathbf{u}_1\mathbf{v}_1 + \ldots + \mathbf{u}_k\mathbf{v}_k - \mathbf{u}_{k+1}\mathbf{v}_{k+1} \ldots - \mathbf{u}_m\mathbf{v}_m$$

in place of vectors in $E_{m,k}$ and the dot product. The slight loss of economy involved in using imaginary numbers is, in our view, more than offset by the gain in naturalness and convenience. It is, in any case, very easy to translate from one notation to the other. In particular, the group of isometries $\mathrm{Q}(m,k)$ is isomorphic to the Lie group $O(m-k,k)$ [O chapter 9]. Explicitly the isomorphism is given by the map

$$\left(\begin{array}{|c|c|}\hline \mathrm{A} & \mathrm{iB} \\ \hline \mathrm{iC} & \mathrm{D} \\ \hline \end{array}\right) \longrightarrow \left(\begin{array}{|c|c|}\hline \mathrm{A} & \text{-B} \\ \hline \mathrm{C} & \mathrm{D} \\ \hline \end{array}\right) \tag{4}$$

where A, B, C, D are real matrices of dimensions $k \times k$, $k \times (m-k)$, $(m-k) \times k$, and $(m-k) \times (m-k)$, respectively.

We conclude this section by defining a special class of orthogonal matrix.

Definition. If $\mathbf{u} \in E_{m,k}$, $N(\mathbf{u}) \neq 0$, we define the *reflection in the plane normal to* $\mathbf{u}$ to be the map effected by the *reflection matrix*

$$R(\mathbf{u}) = I - 2\mathbf{u}\mathbf{u}^t / N(\mathbf{u})^2$$

We refer to a reflection, $R(\mathbf{u})$, as *spacelike* or *timelike* according as the vector $\mathbf{u}$ is.

Since $\mathrm{R}(\mathbf{u}) = \mathrm{R}(\lambda\mathbf{u})$ for any real nonzero λ, there is no loss of generality in assuming that $N(u)$ is 1 or i.

It is routinely verified that $\mathrm{R}(\mathbf{u})$ is an orthogonal symmetric involutive matrix in $\mathrm{Q}(m,k)$ which leaves the hyperplane perpendicular to $\mathbf{u}$ invariant.

5 The Extended Pseudosphere

Just as for hyperbolic space, the notion of points "at infinity" can be introduced for the pseudospheres $S_{n,k}$. We do this as follows. Recall that, for $\mathbf{x} \in E_{n+1,k}$, $\mathbf{x}^*$ denotes the corresponding vector in $\mathbf{R}^{n+1}$, obtained by dividing the imaginary terms of $\mathbf{x}$ by i. Each $S_{n,k}$ can now be mapped onto the *Euclidean* unit sphere, $S = \{\mathbf{x} \mid \|\mathbf{x}^*\| = 1\}$ by the mapping π which divides $\mathbf{x} \in E_{n+1,k}$ by its Euclidean norm, $\|\mathbf{x}^*\|$. The image of π is thus exactly the set of spacelike vectors in S.

If we identify $S_{n,k}$ with its image under π then we may define the *extended pseudosphere* $\overline{S_{n,k}}$ to be the closure of $\pi(S_{n,k})$, that is the set of spacelike and null vectors in S. The *points at infinity* will be the boundary of $\pi(S_{n,k})$.

We say that a collection, $\mathcal{C}$, of subsets of $S_{n,k}$ meets at infinity if

$$\bigcap_{C \in \mathcal{C}} \overline{\pi(C)}$$

contains a point at infinity.

6 Bitriangular Form

By the well known QR decomposition, a real square matrix M may be transformed into an upper triangular matrix by premultiplying by an orthogonal matrix. In geometrical terms, this corresponds to transforming the column vectors of M by a series of reflections so that the first vector lies along the x_1 axis, the second in the x_1x_2 plane, the third in the $x_1x_2x_3$ hyperplane and so on. We now state and prove a corresponding result for matrices with columns in $E_{m,k}$. In the real case, $k = m$, this is essentially the triangularization result just referred to. However, for $k < m$, the situation is considerably complicated by the presence of timelike and, in particular, null vectors. As we will show, this is closely related to the fact that, whereas in spherical space, any two hyperplanes meet, it is possible in $S_{n,k}$ $(k < n)$ for two hyperplanes to meet only at infinity or not at all.

Definition. If M is a matrix with n columns then $r_j = r_j(\mathrm{M})$ is the rank of the matrix comprising the first j columns of M $(1 \le j \le n)$. In addition we set $r_0 = 0$.

We define *bitriangular form* by induction on the dimensions of the matrix.

Definition. An $m \times n$ matrix M with columns in $E_{m,k}$ is in *bitriangular form* (or, for brevity, is *bitriangular*) if it has the following block structure.

$$\mathrm{M} = \left(\begin{array}{c|c} \mathrm{A} & \mathrm{B} \\ \hline \mathbf{O} & \begin{array}{c|c} \mathbf{u} & \mathbf{v} \\ \hline \multicolumn{2}{c}{\begin{array}{c|c} \mathbf{0} & M' \end{array}} \\ \hline i\mathbf{u} & i\mathbf{w} \end{array} \\ \hline i\mathrm{C} & i\mathrm{D} \end{array}\right) \qquad (5)$$

where:

(1) A and C are real matrices with no zero rows, for which $r_{j+1}(\mathrm{A}) \leq r_j(\mathrm{A}) + 1$ and $r_{j+1}(\mathrm{C}) \leq r_j(\mathrm{C}) + 1$ $(j > 0)$, and with all the entries in the jth column of A below the $r_j(\mathrm{A})$th (respectively all entries in the jth column of C above the $r_j(\mathrm{C})$th) equal to zero, and the first nonzero entry in each row positive.

We include the possibility that the blocks A and B, or the blocks iC and iD, or all four of these blocks are absent from the matrix in (5).

(2) $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are real row vectors.

(3) B and D are real matrices. The blocks $\mathbf{O}$ and $\mathbf{0}$ denote, respectively, a matrix, and a column vector, of zeros.

(4) $\forall j$ an entry in $\mathbf{u}$ is equal to one if it occurs in the jth column of M and $r_j > r_{j-1}$,

(5) The first entries of $\mathbf{v}$ and $\mathbf{w}$ are distinct from each other.

(6) If the first entry of $\mathbf{v}$ lies above the jth column of $\mathbf{M}'$ then $r_j(\mathbf{M}') = r_{j-1}(\mathbf{M}')$

(7) $\mathbf{M}'$ is in bitriangular form.

We define $s = s(\mathrm{M})$, $t = t(\mathrm{M})$, $u = u(\mathrm{M})$ as the widths of A, $\mathbf{u}$, and $\mathbf{v}$, respectively. We permit s, t and u to take the value zero. That is A, $\mathbf{u}$ and $\mathbf{v}$ may vanish entirely. We require however that $t = 0$ only when $u = 0$. That is $\mathbf{u}$ may vanish only when $\mathbf{v}$ does. In this case, M is particularly simple, being essentially two triangular matrices put together.

Remarks:-

(1) In the real case, $k = m$, we must have $t = u = 0$ so that all the blocks in (5) vanish except for A and B. If M is also square then B also vanishes and M is upper triangular, with the additional condition that the first nonzero entry in each row is positive ensuring uniqueness.

(2) The bitriangular matrix in (5) has a symmetry which may be roughly expressed by saying that M remains bitriangular when it is turned upside-down. More precisely, if the rows of M are listed in reverse order, the real rows multiplied by i, and the imaginary rows divided by i, then the resulting matrix is also in bitriangular form (with column vectors in $E_{m,m-k}$).

We now state the main result of this paper.

Theorem 4 *For every $m \times n$ matrix M with columns in $E_{m,k}$, there is a unique bitriangular matrix C for which*

$$\mathrm{PM} = \mathrm{C}$$

where P is orthogonal (the matrix P is not generally unique). We will refer to the matrix, C, as the bitriangular form *of M.*

Our approach to the proof of Theorem 4 is similar to the construction of the QR decomposition, using Householder transformations (see e.g. [ND]). We will in fact show that P can be taken to be a product of reflections. This will allow us to prove (Proposition 10) that every orthogonal matrix is a product of reflections. We begin with a few simple lemmas about reflections.

Lemma 5 *If $\mathbf{u}$, $\mathbf{v}$ are in $E_{m,k}$ and $\mathbf{u}$ is a unit vector then*

(1) $\mathrm{R}(\mathbf{u})\mathbf{v} = \mathbf{v} \pm 2(\mathbf{u} \cdot \mathbf{v})\mathbf{u}$

(2) $\mathrm{R}(\mathbf{u})\mathbf{v} = \mathbf{v} \Leftrightarrow \mathbf{u} \cdot \mathbf{v} = \mathbf{0}$

the sign in (1) *being opposite to that of* $N(\mathbf{u})^2$

Proof:- The first assertion is a straightforward calculation from the definition of $\mathrm{R}(\mathbf{u})$ and the second follows immediately.$\square$

Lemma 6 *For any* $i \leq m$ *the elementary* $m \times m$ *matrix which changes the sign of the* i*th entry of a vector in* $E_{m,k}$ *is a reflection. If* $i, j \leq k$ *or* $i, j > k$ *then the matrix which interchanges the* i*th and* j*th entries of a vector in* $E_{m,k}$ *is a product of reflections.*

Proof:- For $l \leq m$ let $\mathbf{a}_l = \mathbf{e}_l$ when $l \leq k$ and $\mathbf{a}_l = i\mathbf{e}_l$ when $l > k$. An easy calculation shows that $\mathrm{R}(\mathbf{a}_i)$ and $\mathrm{R}(\mathbf{a}_i)\mathrm{R}(\mathbf{a}_j)\mathrm{R}(\frac{1}{\sqrt{2}}(\mathbf{a}_i + \mathbf{a}_j))$ effect the sign change and swapping operations respectively.$\square$

We may define a linear operation on a vector $\mathbf{v}$ by leaving some of the entries unaltered and applying a reflection to the vector left after deleting these entries from $\mathbf{v}$. The next result, which is a simple consequence of lemma 5, shows that this operation is itself a reflection.

Lemma 7 *Let* T *be the linear transformation on* $\mathbf{u} \in E_{m,k}$ *obtained by leaving the* i_1*th,* i_2*th, ...,* i_l*th entries of* $\mathbf{u}$ *unchanged and applying the reflection,* $\mathrm{R}(\mathbf{u})$ *to the vector left after deleting these entries from* $\mathbf{u}$*. The transformation* T *is then the reflection* $\mathrm{R}(\mathbf{u}')$*, where* $\mathbf{u}' \in E_{m,k}$ *and has entries of zero in the* i_1*th,* i_2*th, ...,* i_l*th positions, and the entries of* $\mathbf{u}$*, listed in order, in the other positions.*

The above lemmas will be used, often tacitly, in proving Theorem 4.

If M, with m rows, is bitriangular, then we may subdivide the rows of M into four classes as follows. If $\mathbf{r}_k$ denotes the kth row of M, then M determines disjoint subsets Z, R, R', and F of $\{1, 2, \ldots, m\}$ for which

$$\mathbf{r}_k = \mathbf{0} \text{ if } k \in Z, \tag{6}$$

the rows $\{\mathbf{r}_k | k \in R\}$ are all real and nonzero, and there is a bijection, $\sigma : R \mapsto R'$ such that

$$\mathbf{r}_{\sigma(k)} = i\mathbf{r}_k \quad (k \in R) \tag{7}$$

(For brevity, we will refer to such a pair of nonzero rows in ratio 1:i as *paired*) and the set $\{\mathbf{r}_k | k \in F\}$ contains no zero rows and no paired rows.

Clearly the rank of M is the cardinality of $F \bigcup R$.

The following lemma follows from a straightforward induction on the number of columns in M.

Lemma 8 *With the above notation,* $\mathbf{w} \in \mathbf{C}^m$ *is orthogonal to every column of* M *if and only if* $\mathbf{w}_k = 0$ *for* $k \in F$ *and* $\mathbf{w}_{\sigma(k)} = i\mathbf{w}_k$ *for* $k \in R$.

We will investigate the geometric significance of this result in [M]. It will be useful to identify the following set of bitriangular matrices, which take a particularly simple form.

Definition. A matrix M is *regular* if its bitriangular form has no paired rows.

The main lemma used in the proof of Theorem 4 will be,

Lemma 9 *If* $\mathbf{v} \in E_{m,k}$*,* $\mathbf{v} \neq \mathbf{0}$ *then there is a (possibly empty) product of reflections* R *for which*

(1) $\mathrm{R}\mathbf{v} = (N(\mathbf{v}), 0, \ldots, 0)^t$ *if* $\mathbf{v}$ *is spacelike*

(2) $\mathrm{R}\mathbf{v} = (0, 0, \ldots, \mathbf{N}(\mathbf{v}))^t$ *if* $\mathbf{v}$ *is timelike*

(3) $\mathrm{R}\mathbf{v} = (1, 0, \ldots, 0, i)^t$ *if* $\mathbf{v}$ *is null*

Proof:- (1) Since $N(\mathbf{v}) > 0$, $k \geq 1$. Using Lemma 6, we may assume that the first entry, v_1, of $\mathbf{v}$ is non positive so that

$$\beta = N(\mathbf{v})(N(\mathbf{v}) - v_1) > 0$$

Now let

$$\mathbf{u} = \frac{1}{\sqrt{2\beta}}(v_1 - N(\mathbf{v}), v_2, \ldots, v_m)^t$$

Routine calculation shows that $N(\mathbf{u}) = 1$ whence, using Lemma 5, $\mathrm{R}(\mathbf{u})\mathbf{v} = (N(\mathbf{v}), 0, \ldots, 0)^t$. Observe that we have in fact got $\mathbf{v}$ into the required form using only *spacelike* reflections. We will use this fact below.

(2) We let B be the $m \times m$ matrix with entries of i on the subsidiary diagonal and zeros elsewhwere. Clearly B is symmetric, $\mathrm{BB}^t = \mathrm{BB} = -I_m$ and, if $\mathbf{u}$ is a spacelike unit vector in $E_{m,k}$ then $\mathrm{B}\mathbf{u}$ is a timelike unit vector in $E_{m,m-k}$. Also,

$$\begin{aligned} -\mathrm{BR}(\mathbf{u})\mathrm{B} &= -\mathrm{B}(I - 2\mathbf{u}\mathbf{u}^t)\mathrm{B} \\ &= I - 2(\mathrm{B}\mathbf{u})(\mathrm{B}\mathbf{u})^t / N(\mathrm{B}\mathbf{u})^2 \\ &= \mathrm{R}(\mathrm{B}\mathbf{u}) \end{aligned} \tag{8}$$

Now let $\mathbf{v} \in E_{m,k}$ be timelike. By (1) there is a sequence of $k \geq 0$ spacelike unit vectors $\mathbf{u}_1, \mathbf{u}_2 \ldots \mathbf{u}_k$ in $E_{m,m-k}$ for which

$$\mathrm{R}(\mathbf{u}_1)\mathrm{R}(\mathbf{u}_2)\ldots\mathrm{R}(\mathbf{u}_k)\mathrm{B}\mathbf{v} = (-iN(\mathbf{v}), 0, 0, \ldots, 0)^t$$

whence

$$\begin{aligned} -\mathrm{R}(\mathrm{B}\mathbf{u}_1)\mathrm{R}(\mathrm{B}\mathbf{u}_2)\ldots\mathrm{R}(\mathrm{B}\mathbf{u}_k)\mathbf{v} =& \mathrm{BR}(\mathbf{u}_1)\mathrm{R}(\mathbf{u}_2)\ldots\mathrm{R}(\mathbf{u}_k)\mathrm{B}\mathbf{v} \\ =& \mathrm{B}(-iN(\mathbf{v}), 0, 0, \ldots, 0)^t \\ =& (0, 0, \ldots, N(\mathbf{v}))^t \end{aligned} \tag{9}$$

Which proves (2).

(3) Since $\mathrm{N}(\mathbf{v}) = 0$, $\mathbf{v} \neq \mathbf{0}$, we have $1 \leq k \leq m-1$. Using (2) and Lemma 7 we can find a product of reflections, R , for which $\mathrm{R}\mathbf{v}$ takes the form

$$(b, 0, 0, \ldots, 0, ai)$$

Using Lemma 6 if neccesary we may assume $a, b > 0$. Since $\mathrm{N}(\mathrm{R}(\mathbf{v})) = \mathrm{N}(\mathbf{v}) = 0$, we have $b = a$. The proof is completed by finding a unit vector $\mathbf{u}$ in $E_{m,k}$ for which

$$\mathrm{R}(\mathbf{u})(a, 0, \ldots, 0, ia) = (-1, 0, \ldots, 0, i)$$

and applying Lemma 6. It is routine to show that the vector $\mathbf{u} \in E_{m,k}$ with first entry $(a^{1/2} + a^{-1/2})/2$, last entry, $i(a^{1/2} - a^{-1/2})/2$ and zeros elsewhere, has this property.$\square$

Proof of Theorem 4:- We use induction on the dimension of M. The theorem is trivial if M has a single row, and a consequence of Lemma 9 if M has a single column. For the induction step we suppose that $\mathrm{M} = (\mathbf{v_1}; \mathrm{M}')$, $\mathbf{v_1}$ being the first column of M. We observe that, whenever M is bitriangular and $\mathbf{v_1}$ is spacelike or timelike, the matrix obtained by deleting the first row of M' is also bitriangular.

If $\mathbf{v_1}$ is spacelike then, by Lemma 9(1), there is a product of reflections, R, for which

$$\mathrm{R}\mathbf{v_1} = (N(\mathbf{v_1}), 0, \ldots, 0)^t$$

Let M'' be the matrix obtained by deleting the top row from RM'. By induction hypothesis, there is an $(m-1) \times (m-1)$ product of reflections R''

for which R″ M″ = C″ is bitriangular. It follows, by lemma 7, that there is an $m \times m$ product of reflections R′ with the property that deleting the top row of R′RM′ gives C″. It follows that R′RM is bitriangular.

To prove the uniqueness part of the theorem in this case, we suppose that P′ and P″ are both orthogonal and that P′M and P″M are both bitriangular. Letting $\mathrm{P} = \mathrm{P}''(\mathrm{P}')^{-1}$ and B = P′M, we then have B and PB both bitriangular and it suffices to show that these two matrices are in fact equal. Since the first columns of these matrices are equal and P is orthogonal, it follows that P must have an entry of 1 in the top left corner, and entries of 0 elsewhere in the first row and column. The submatrix Q of P obtained by deleting the first row and column is thus also orthogonal. The matrices B_1 and B_2 obtained by deleting the first row and column of B and PB respectively are both bitriangular and we have $\mathrm{QB}_1 = \mathrm{B}_2$. By the induction hypothesis, we then have $\mathrm{B}_1 = \mathrm{B}_2$ and it readily follows that B = PB as required.

When $\mathbf{v_1}$ is timelike, the induction step is similar, this time using (2) of Lemma 9 and defining M″ to be the matrix obtained by deleting the *bottom* row of RM′. The induction step when $\mathbf{v_1} = \mathbf{0}$ is easy since, in this case, for any matrix P, PM is bitriangular if and only if PM′ is.

The remaining case–$\mathbf{v_1}$ is null– is more involved and we prove the existence and uniqueness parts separately. In this case, M must take the form

$$M = \left(\begin{array}{c|c|c} \multicolumn{2}{c|}{\mathbf{u}} & \mathbf{v} \\ \hline \mathbf{0} & \multicolumn{2}{c}{M'} \\ \hline \multicolumn{2}{c|}{i\mathbf{u}} & i\mathbf{w} \end{array}\right) \tag{10}$$

meeting the same conditions as the block form (5), of which it is a special instance.

To prove existence, let M_1 be the matrix obtained by deleting the last column of M. By the induction hypothesis there is a product of reflections, P, for which PM_1 is bitriangular. Let N′ be the matrix obtained by deleting the top and bottom rows of PM_1. Since the initial column of PM_1 is null, N′ is also bitriangular. Again using induction hypothesis, we multiply PM by another product of reflections, P′, which leaves the top and bottom rows of PM unchanged and for which the matrix, N″, obtained by deleting the top and bottom rows of P′PM is bitriangular. We set

$$\mathrm{S} = \mathrm{P}'\mathrm{PM} = \left(\begin{array}{|c|c|} \hline \mathbf{a} & p \\ \hline \mathrm{N} & \mathbf{y} \\ \hline i\mathbf{b} & iq \\ \hline \end{array}\right)$$

where the row vectors, $\mathbf{a}$, $\mathbf{b}$ and the scalars p and q are all real and $y \in E_{m-2,k-1}$. We also denote the last column of S by $\mathbf{z}$ and the matrix obtained by deleting $\mathbf{z}$ from S by S_1.

Since N is bitriangular and is an orthogonal multiple of N′, the induction hypothesis (uniqueness part) gives $\mathrm{N} = \mathrm{N}'$, so that S differs from PM only in, possibly, the last column, $\mathbf{z}$. In particular, $\mathrm{S}_1 = \mathrm{PM}_1$ is bitriangular.

If $\mathbf{a} \neq \mathbf{b}$, $p = q = 1$, or $\mathbf{y}$ is in the column space of N then S is bitriangular and we are done. We therefore assume that none of these conditions hold. We suppose that the vector $\mathbf{z}$ has j nonzero entries $d_1 \ldots d_j$ lying to the right of zero rows of N and that d_i lies in the l_ith position of $\mathbf{z}$. Since $\mathbf{y}$ is not in the column space of N and N″ is bitriangular, we have $1 \leq j \leq 2$ and, if $j = 2$, then $d_1 = 1$ and $d_2 = i$.

Now suppose $p = q$. If $j = 1$, we define a unit vector $\mathbf{v}$ by,

$$\mathbf{v} = ((p-1)/2d_1, 0, 0, \ldots, 0, \ldots, 1, 0, i(p-1)/2d_1)^t$$

where the entry 1 is in the l_1th position. Let $\mathbf{u} = \mathbf{v}$ when $d_1 \in \mathbf{R}$ and $\mathbf{u} = i\mathbf{v}$ when $d_1 \in i\mathbf{R}$. The vector $\mathbf{u}$ is in $E_{m,k}$ and $\mathrm{N}(\mathbf{u})$ is 1 or i according as d_1 is real or imaginary.

Since $\mathbf{u}$ is perpendicular to each column of S_1, $\mathrm{R}(\mathbf{u})\mathrm{S}_1 = \mathrm{S}_1$ and a straightforward calculation shows that $\mathrm{R}(\mathbf{u})\mathbf{z}$ has entries of 1, $-d_1$, and i in the first, l_1th and last positions, respectively and is otherwise the same as $\mathbf{z}$. It follows that $\mathrm{R}(\mathbf{u})\mathrm{S}$ is bitriangular.

Now suppose that $j = 2$, so that $d_1 = 1$ and $d_2 = i$. In this case we choose positive a arbitrarily and set

$$\alpha = a/(1-p) + (1-p)/2$$

$$\beta = a/(1-p) - (1-p)/2$$

so that

$$\alpha - \beta = 1 - p$$

$$\alpha^2 - \beta^2 = 2a$$

Now let

$$\mathbf{u} = \frac{1}{\sqrt{2a}}(0, 0, \ldots, \alpha, 0, \ldots, 0, i\beta, 0, \ldots, 0)$$

$$\mathbf{v} = -\frac{1}{\sqrt{2a}}(a, 0, \ldots \alpha, 0, \ldots, 0, i\beta, 0, \ldots, ia)$$

where in both these vectors, the entries of α, $i\beta$ occur in the l_1th and l_2th positions, respectively.

The isometry R($\mathbf{v}$)R($\mathbf{u}$) leaves S_1 unchanged. A routine calculation shows that R($\mathbf{u}$)$\mathbf{z}$ differs from $\mathbf{z}$ only in the l_1th and l_2th positions, where it has entries of $(1-p)^2/2a$ and $i(1-p)^2/2a$ respectively, while R($\mathbf{v}$)R($\mathbf{u}$)$\mathbf{z}$ has entries of 1 in the first and l_1th positions and i in the l_2th and last positions and is otherwise the same as $\mathbf{z}$.

Thus

$$\mathrm{R}(\mathbf{v})\mathrm{R}(\mathbf{u})\mathrm{S}$$

is bitriangular. This completes the induction step when $p = q$

Finally suppose $p \neq q$. Let

$$\mathbf{v} = (d_1/(2(p-q)), 0, 0, \ldots, 1, 0, \ldots, 0, 0, id_1/(2(p-q)))^t$$

where the entry of 1 occurs in the l_1th position Let $\mathbf{u} = \mathbf{v}$ when d_1 is real and $\mathbf{u} = i\mathbf{v}$ when d_1 is imaginary. Again $\mathbf{u}$ is in $E_{m,k}$ and N($\mathbf{u}$) is 1 or i according as d_1 is real or imaginary.

The effect of multiplying $\mathbf{z}$ by R($\mathbf{u}$) is to replace the entry d_1 by zero and to change the values of the first and last entries (note however that the ratio between these entries still differs from 1:i). Otherwise $\mathbf{z}$ is unchanged and, as before, R($\mathbf{u}$)$\mathrm{S}_1 = \mathrm{S}_1$. If $j = 1$ the vector obtained by deleting the first and last entries from R($\mathbf{u}$)$\mathbf{z}$ is in the column space of N, so that R($\mathbf{u}$)S is bitriangular. If $j = 2$ then we simply repeat the procedure using R($\mathbf{u}$)S in place of S. For this new matrix, $j = 1$ so the process terminates after at most two steps.

This completes the induction step for the existence part of the theorem.

We will prove the uniqueness part of the theorem by showing that, if P is orthogonal, and both M and PM are in bitriangular form, then PM=M.

Assuming that M and PM are both bitriangular with P orthogonal, we may write M in the block form

$$M = \left(\begin{array}{|c|c|c|}\hline \multicolumn{2}{|c|}{\mathbf{u}} & \mathbf{v} \\ \hline \mathbf{0} & \multicolumn{2}{c|}{M'} \\ \hline \multicolumn{2}{|c|}{i\mathbf{u}} & i\mathbf{w} \\ \hline \end{array}\right) \tag{11}$$

where M satisfies conditions (1)-(7) of Theorem 4. In particular, M′ is bitriangular. We suppose M′ to have k columns. As before we let M_1 denote the matrix obtained by deleting the last column of M. We introduce some further notation. Let M_1' be the matrix obtained by deleting the last column of M′, and write the last column of M as

$$\mathbf{y} = (p; \mathbf{x}; iq)^t$$

where $\mathbf{x}$ is the last column of M′. We will abbreviate $r_j(\mathrm{M}')$ to r_j'.

The matrix PM has the same dimensions as M and is bitriangular by assumption. We denote a subblock of PM by adding a tilde to the symbol used to denote the corresponding block in M. Since, by the induction hypothesis, $\mathrm{PM}_1 = \mathrm{M}_1$, the induction step reduces to proving

$$\mathrm{P}\mathbf{y} = \tilde{\mathbf{y}}.$$

P may be written in the block form

$$\mathrm{P} = \left(\begin{array}{|c|c|c|}\hline \mathrm{a} & \mathbf{f} & \mathrm{ib} \\ \hline \mathbf{e} & \mathrm{Q} & i\mathbf{e}' \\ \hline \mathrm{ic} & i\mathbf{f}' & \mathrm{d} \\ \hline \end{array}\right)$$

$$a - b = 1 \tag{12}$$

$$c + d = 1 \tag{13}$$

$$\mathbf{e}' = \mathbf{e} \tag{14}$$

In view of (14), dot products between the rows of Q are the same as those between the corresponding rows of P. It follows that Q is orthogonal. Thus, considering columns of P, we have also

$$\mathbf{f}' = \mathbf{f} \tag{15}$$

Using the orthogonality properties of the top and bottom rows of P gives the equations

$$a^2 - b^2 + \mathrm{N}(\mathbf{f})^2 = 1 \tag{16}$$

$$d^2 - c^2 - \mathrm{N}(\mathbf{f})^2 = 1 \tag{17}$$

Adding these equations, using (12) and (13) to eliminate b and d, and simplifying then gives,

$$a - c = 1 \tag{18}$$

whence finally,

$$b = c = a - 1 \quad \text{and} \quad d = 2 - a \tag{19}$$

The proof now splits into three subcases according to the length of the vector $\mathbf{v}$.

Subcase 1: $u(\mathrm{M}) = 0$**.** In this case the vectors $\mathbf{v}$ and $\mathbf{w}$ are absent altogether from (11). By the induction hypothesis, $\mathrm{PM}_1 = \mathrm{M}_1$ whence, by equating the top rows using (12), $\mathbf{f}$ is orthogonal to each column of M_1'. If $r_k' = r_{k-1}'$ then $\mathbf{x}$ is in the column space of M_1' so that $\mathbf{f}$ is orthogonal to $\mathbf{x}$ too. By a similar argument, $\mathbf{f}'$ is also orthogonal to $\mathbf{x}$. Hence,

$$\tilde{p} = ap - bq \quad \text{and} \quad \tilde{q} = cp + dq. \tag{20}$$

and since, in this case $p = q$, it follows that

$$\tilde{p} = p \quad \text{and} \quad \tilde{q} = q. \tag{21}$$

On the other hand, if $r_k' > r_{k-1}'$ then, using (4) of the definition of bitriangular, $\tilde{p} = p = 1$, $\tilde{q} = q = i$ and, in particular, (21) again holds.

Because $u(\mathrm{M}) = 0$, we have $\tilde{\mathrm{M}}' = \mathrm{QM}'$. Since M' and $\tilde{\mathrm{M}}'$ are both in bitriangular form and have fewer rows than M, and Q is orthogonal, the induction hypothesis gives $\mathrm{M}' = \tilde{\mathrm{M}}'$, and, in particular, $\mathbf{x} = \tilde{\mathbf{x}}$. This completes the proof that PM = M in this case.

Subcase 2: $u(\mathrm{M}) = 1$. In this case the vectors $\mathbf{v}$ and $\mathbf{w}$ in (11) are of length one.

By induction hypothesis

$$\tilde{\mathrm{M}}_1 = \mathrm{PM}_1 = \mathrm{M}_1 \tag{22}$$

and, since the top and bottom rows of M_1 are in ratio 1:i, $\mathbf{f}$ is orthogonal to each column of M'_1 and $\mathrm{QM}'_1 = \mathrm{M}'_1$. Since, by (6) of the definition of bitriangular form, $\mathbf{x}$ is in the column space of M'_1, we have

$$\mathbf{f} \cdot \mathbf{x} = 0 \tag{23}$$

and

$$\mathrm{Q}\mathbf{x} = \mathbf{x} \tag{24}$$

By (23), (20) again holds. We must have $\tilde{p} \neq \tilde{q}$ in this case, since $\tilde{p} = \tilde{q}$ would give, by (20),

$$(a - c)p = (b + d)q$$

and since, by (19), $a - c = b + d \neq 0$, we would have $p = q$, contrary to our assumption.

Since $\tilde{p} \neq \tilde{q}$, $\tilde{\mathbf{x}}$ is in the column space of $\tilde{\mathrm{M}}'_1$. Applying the notation defined before Lemma 8 to M'_1, we must have $\tilde{\mathbf{x}}_k = \mathbf{x}_k = 0$ for $k \in Z$ and $\tilde{\mathbf{x}}_k + i\tilde{\mathbf{x}}_{\sigma(k)} = \mathbf{x}_k + \mathbf{x}_{\sigma(k)} = 0$ for $k \in R$. Applying (24) then gives $\mathbf{e}_k = 0$ for $k \in Z$ and $\mathbf{e}_k + i\mathbf{e}_{\sigma(k)} = 0$ for $k \in R$. It follows, using the orthogonality properties of the rows of P, that $\mathbf{f}$ is orthogonal to each vector in the set

$$T = \{\mathbf{q}_k | k \in Z\} \bigcup \{\mathbf{q}_k + i\mathbf{q}_{\sigma(k)} | k \in R\}$$

where $\mathbf{q}_k$ denotes the kth row vector of Q.

The vectors in T are mutually orthogonal and so span a set of dimension equal to the cardinality of $Z \bigcup R$. From (24) we also see that each member of T is orthogonal to the column space of M'_1, which has dimension equal to the cardinality of $F \bigcup R$. Consequently, the set T' comprising T and the columns of M'_1 spans a space of dimension m, that is the whole of $E_{m,k}$. Since $\mathbf{f}$ is orthogonal to every vector in T', we conclude that $\mathbf{f} = \mathbf{0}$.

Now it readily follows that also $\mathbf{e} = \mathbf{0}$, $a = d = 1$ and $b = c = 0$, whence $\tilde{\mathbf{y}} = \mathrm{P}\mathbf{y} = \mathbf{y}$ and the induction step is completed in this case.

Subcase 3: $u(\mathrm{M}) > 1$ Since the previous case applies to the matrix obtained by removing the last $u(\mathrm{M}) - 1$ columns from M, again we have $\mathbf{e} = \mathbf{f} = \mathbf{0}$, $a = d = 1$, $b = c = 0$. It easily follows that $\tilde{\mathrm{M}}' = \mathrm{QM}' = \mathrm{M}'$, the second equation following from the induction hypothesis. Since (20) also clearly holds in this case, the induction step is completed. □

It is convenient to generalize slightly the definition of bitriangular form to include matrices whose columns may be either vectors in $E_{m,k}$ or imaginary multiples of such vectors. We describe such a matrix as bitriangular if the matrix obtained by dividing by i each column vector in $iE_{m,k}$ is bitriangular.

We can now prove,

Proposition 10 *The orthogonal matrices,* $\mathrm{Q}(m,k)$ *are generated by the reflections* $\mathrm{R}(\mathbf{u})$ *for* $\mathbf{u} \in E_{m,k}$.

Proof:- Let M be orthogonal. By the above proof there is a product of reflections R for which RM is bitriangular. As the columns of RM are orthonormal, a simple induction on j shows that the jth column of RM is the jth bitriangular unit vector in $\mathbf{R}^m$. That is RM is the identity matrix, whence $\mathrm{M} = \mathrm{R}^{-1}$, a product of reflections.□

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T. H. Marshall
University of Auckland
Private Bag 92019
Auckland
NEW ZEALAND