Truncated Tetrahedra and their Reflection Groups

T.H.Marshall

Abstract

We classify, up to isometry, all tetrahedra in hyperbolic space with one or more vertices truncated, for which the dihedral angles along the edges formed by the truncations are all $\pi/2$, and those remaining are all submultiples of π. We show how to find the volumes of these polyhedra, and find presentations and small generating sets for the orientation-preserving subgroups of their reflection groups.

For particular families of these groups, we find low index torsion free subgroups, and construct associated manifolds and manifolds with boundary. In particular, for each $g \geq 2$, we find a sequence of hyperbolic manifolds with totally geodesic boundary of genus g, which we conjecture to be of least volume among such manifolds.

1 Introduction

A *Coxeter polytope* is a polytope whose dihedral angles are all submultiples of π. The main interest of these polytopes is that their reflection groups (i.e. the groups generated by reflections through their faces) are discrete. Coxeter himself [Co] classified these polytopes in spherical and Euclidean space, showing that they must be, respectively, simplices, and Cartesian products of simplices. In hyperbolic space the situation is more complicated, and we consider only the 3-dimensional case. The situation for tetrahedra is well known, there being, up to isomorphism, exactly 32 Coxeter tetrahedra

[0]1991 *Mathematics Subject Classification.* Primary 30F40, 57N10; Secondary 20F55

Key Words and Phrases. Coxeter polytope, Truncated tetrahedron, Co-volume, Torsion-free subgroup, Hyperbolic Manifold

in hyperbolic 3-space, including those with one or more ideal vertices [L]. For other combinatorial types, the Coxeter polyhedra can be identified using Andreev's theorem [An], which we review in Section 2

The subject of this paper are the *truncated tetrahedra* which are also Coxeter polyhedra, where we define a truncated tetrahedron to be a polyhedron which is combinatorially equivalent to a tetrahedron with k $(1 \leq k \leq 4)$ of its vertices "cut off" perpendicularly, in the obvious way. An example, with $k = 2$, is illustrated in Fig. 1. For $k = 1$ a truncated tetrahedron is a triangular prism. These polyhedra and their reflection groups have been discussed by Conder and Martin [CM]. In this paper we extend and develop some of their ideas.

Figure 1. *A truncated tetrahedron*

We begin by classifying, up to isometry, all the Coxeter truncated tetrahedra with between one and four truncations. This is a fairly straightforward application of Andreev's theorem. The truncated tetrahedra can be grouped naturally into a finite number of infinite families, which are listed in Tables 1–7, along with some properties of the associated reflection groups.

It is these groups that we consider in Section 3. Poincaré's polyhedral theorem [Mas; Theorem H 11] not only tells us that they are discrete, but also provides a presentation for them. Given a presentation for such a group Γ we can then derive one for the order two subgroup Γ^0 comprising the orientation-preserving isometries in Γ. The main subject of Section 3 is the question of how many generators are required to generate Γ^0. Usually this is no more than two fewer than the number of faces of the associated polyhedron. This problem naturally leads to the same question for Coxeter groups in general, and in particular for extended triangle groups, and we give a number of results in this area.

In Section 4 we indicate how to calculate the volume of a truncated tetrahedron, and so the co-volume of the associated reflection group.

Sections 5 and 6 deal with another group-theoretic problem— that of finding low index torsion-free subgroups of a given reflection group. The geometrical interest of such groups is that the associated orbit space is a manifold. If a group G, with fundamental domain D has a torsion-free subgroup of index n, then we can construct a manifold by glueing together n copies of D, according to the face pairings determined by G, and conversely. We do just such a construction in Section 6.

In these sections we concentrate entirely on two particular families of groups, $\Gamma^0(q)$, also considered in [CM], [1] and $\tilde{\Gamma}^0(q)$ $(7 \leq q < \infty)$. These are the orientation-preserving groups associated with, respectively, a family of prisms $P(q)$ (see Fig. 2, Section 5), and the family of unbounded polyhedra obtained by omitting one of the faces of $P(q)$ (whence, for each q, $\tilde{\Gamma}^0(q)$ is a subgroup of $\Gamma^0(q)$).

We find low-index torsion-free subgroups for each of these groups, and show that, for many of the $\Gamma^0(q)$, and all of the $\tilde{\Gamma}^0(q)$, the index is the smallest possible (Theorems 16 and 19).

In Section 6 we explicitly construct manifolds, and manifolds with totally geodesic boundary, associated with torsion-free subgroups of $\Gamma^0(q)$ and $\tilde{\Gamma}^0(q)$, respectively, using the group theory of the previous section. In particular, we present a sequence of low volume manifolds with totally geodesic boundary of any prescribed genus $g \geq 2$. We conjecture that these manifolds are of least volume among manifolds with totally geodesic boundary of the same genus (this is known to be true for $g = 2$ [KM]).

This paper is essentially a chapter of my Ph.D. thesis. I gratefully acknowledge the help and advice of my supervisor Prof. G.J. Martin, and also Prof. M.D.E. Conder and Dr B. Everitt for helpful discussions, and advice

[1] where they are denoted by $\Gamma_{16}(q)$

with CAYLEY programming.

1.1 Notation

We denote the four vertices of a tetrahedron by v_i $(1 \le i \le 4)$ and the face opposite v_i by f_i. In the case of a truncated tetrahedron we suppose that the k $(1 \le k \le 4)$ vertices cut off are v_i for $1 \le i \le k$. The new face formed by cutting off v_i will be denoted by c_i. The edge joining v_i (or c_i) to v_j (or c_j) will be denoted by e_{ij} and $\alpha_{ij} = \pi/m_{ij}$ will denote the dihedral angle along this edge. This terminology is illustrated (for the case $k = 2$) in Figure 1.

Unless stated otherwise, *angle*, in this paper will always refer to a *dihedral* angle between two faces of a polyhedron

2 Classification

We now classify the truncated tetrahedra, with k $(1 \le k \le 4)$ truncations, for which the angles α_{ij} are all submultiples of π and the remaining $3k$ dihedral angles are all $\pi/2$. We begin by stating Andreev's theorem [An], which characterizes the finite volume hyperbolic polyhedra of a given combinatorial type (other than tetrahedra), none of whose angles exceed $\pi/2$. We note that at most three faces of such a polyhedron can meet at a finite vertex, and at most four at an ideal vertex, since every spherical polygon with four or more sides, and every Euclidean polygon with five or more sides, has at least one obtuse angle.

Andreev's theorem has been generalized (see e.g. [H]) but the original result does for our purposes. A preliminary definition is required.

Definition. Let $\Gamma_{i_1}, ..., \Gamma_{i_s}$ be a sequence of faces of a three dimensional polyhedron such that each is adjacent only to the following and the last one only to the first and no three of them intersect at a vertex. We shall say that the faces $\Gamma_{i_1}, ...\Gamma_{i_s}$ are an *s-angled prismatic element.*

Theorem 1 (Andreev) *There exists a convex hyperbolic finite-volume polyhedron $\mathcal{M}$ of given combinatorial type (not being a simplex, and with at most four faces of $\mathcal{M}$ meeting at any vertex) with no dihedral angles of $\mathcal{M}$ exceeding $\pi/2$, if and only if the conditions $\mathcal{M}1$–$\mathcal{M}5$ below are satisfied.* Γ_i *and* Γ_j

denote faces of $\mathcal{M}$, Γ_{ij} *the edge (if any) where they meet, and* γ_{ij} *the angle along this edge.*

$\mathcal{M}0$. *For all edges,* Γ_{ij}

$$0 < \gamma_{ij} \leq \pi/2$$

$\mathcal{M}1$. *If* Γ_i,Γ_j,Γ_k *meet at a vertex of* $\mathcal{M}$ *then*

$$\gamma_{ij} + \gamma_{jk} + \gamma_{ki} \geq \pi;$$

If Γ_i,Γ_j,Γ_k, Γ_l *meet at a vertex of* $\mathcal{M}$ *then*

$$\gamma_{ij} + \gamma_{jk} + \gamma_{kl} + \gamma_{li} = 2\pi;$$

$\mathcal{M}2$. *If* Γ_i,Γ_j,Γ_k *generate a 3-angled prismatic element then*

$$\gamma_{ij} + \gamma_{jk} + \gamma_{ki} < \pi;$$

$\mathcal{M}3$. *If* Γ_i,Γ_j,Γ_k,Γ_l *generate a four-angled prismatic element then*

$$\gamma_{ij} + \gamma_{il} + \gamma_{jk} + \gamma_{kl} < 2\pi;$$

*(*i.e. *at least one of these angles is* $< \pi/2$*)*

$\mathcal{M}4$. *If* $\mathcal{M}$ *is a triangular prism with bases* Γ_i *and* Γ_j, *then at least one of the six dihedral angles associated with the edges around these bases is* $< \pi/2$

$\mathcal{M}5$. *If the faces* Γ_i *and* Γ_j *meet at a vertex but are not adjacent, and if both are adjacent to a third face* Γ_k, *then*

$$\gamma_{ik} + \gamma_{jk} < \pi$$

*(*i.e. *at least one of these two angles is* $< \pi/2$*)*

In view of $\mathcal{M}1$, the angle along a given edge, e can be replaced by any smaller angle if and only if the angles along the four edges that meet e at its end vertices are all $\pi/2$. We refer to such angles as *parameters* and to a family of truncated tetrahedra which has m such angles as an *m-parameter family*. Again by $\mathcal{M}1$, we see that all angles that are not parameters must be at least $\pi/6$ (or $\pi/5$ if ideal vertices are excluded). Hence all polytopes of a given combinatorial type can be classified into a finite number of m-parameter families.

As the value of a parameter π/q approaches zero, the edge e, along which it occurs, moves out towards infinity. In the limit ($q = \infty$), e becomes a vertex of rectangular type on the boundary, and the four edges that met e now run out to this cusp.

We now concentrate on the case when $\mathcal{M}$ is a truncated tetrahedron. Since we are assuming that each angle is a submultiple of π, $\mathcal{M}0$ is automatically satisfied. There are no four-angled prismatic elements in truncated tetrahedra (for any k) so $\mathcal{M}3$ holds vacuously. This is also true of $\mathcal{M}5$ and, when $k \geq 2$, of $\mathcal{M}4$. If $k = 1$ then $\mathcal{M}4$ becomes the requirement that at least one of the six angles along the edges bounding f_1 and c_1 is acute. Since, by assumption, the angles along the edges of c_1 are right angles this becomes

$$\alpha_{23} + \alpha_{24} + \alpha_{34} < 3\pi/2 \tag{1}$$

$\mathcal{M}1$ and $\mathcal{M}2$ become the inequalities.

$$\begin{aligned} \sum_{j \neq i} \alpha_{ij} &< \pi \qquad (i \leq k) && (2) \\ \sum_{j \neq i} \alpha_{ij} &\geq \pi \qquad (i > k) && (3) \end{aligned}$$

with equality holding in (3) exactly when the vertex v_i is ideal.

The details of the classification are straightforward, and we give them only in the illustrative case $k = 2$. Details for the other cases may be found in [M].

When $k = 2$ α_{12} is always a parameter. If α_{34} is also then $\alpha_{14} = \alpha_{24} = \alpha_{13} = \alpha_{23} = \pi/2$ which violates (2). If α_{23} and α_{24} are both parameters then $\alpha_{14} = \alpha_{13} = \pi/2$, again contrary to (2), and symmetrically with α_{14} and α_{13}.

Hence there are most three parameters and, when there are exactly three, we may assume by symmetry that these are α_{12}, α_{24} and α_{13}. In this case all the remaining angles must be $\pi/2$. We thus have exactly one three parameter family of polyhedra. The upper bounds on the parameters, given by (2), simplify, in this case, to

$$\begin{aligned}\alpha_{12}+\alpha_{24} &< \pi/2\\ \alpha_{12}+\alpha_{13} &< \pi/2\end{aligned}$$

Now suppose that there are only two parameters. Then α_{12} is one of these and we may assume by symmetry that the other is α_{24}. In this case $\alpha_{34} = \alpha_{14} = \pi/2$ and $\alpha_{23} < \pi/2$ (else α_{13} is also a parameter). By (2) $\alpha_{13} < \pi/2$ also and, by (3) $\alpha_{13} + \alpha_{23} \geq \pi/2$, with equality when there is a cusp at v_3. Thus (m_{13}, m_{23}) is one of the pairs

$$(3,3),(3,4),(3,5),(3,6),(4,3),(4,4),(5,3),(6,3)$$

The upper bounds on the parameters are

$$\begin{aligned}\alpha_{12}+\alpha_{24} &< \pi-\alpha_{23} \qquad (4)\\ \alpha_{12} &< \pi/2-\alpha_{13} \qquad (5)\end{aligned}$$

Finally, we suppose that α_{12} is the sole parameter. We have $2 \leq m_{34} \leq 6$ and we consider each of these possible values of m_{34} in turn.

Subcase: $m_{34} = 2$

In this case α_{14}, α_{24}, α_{13}, α_{23} are all less than $\pi/2$ otherwise there would be another parameter. By symmetry, we assume that α_{24} is the largest of these four angles. Subject to this, we maximize, in turn, the angles α_{14} and α_{23}. That is, we have the inequalities.

$$\begin{aligned}
& m_{24} \leq \max(m_{13}, m_{14}, m_{23}) && (6)\\
\text{If } & m_{24} = m_{13} \text{ then } m_{14} \leq m_{23} && (7)\\
\text{If } & m_{24} = m_{23} \text{ then } m_{14} \leq m_{13} && (8)\\
\text{If } & m_{24} = m_{14} \text{ then } m_{23} \leq m_{13} && (9)
\end{aligned}$$

By (3) and (6) the possible values of (m_{24}, m_{14}) are

$$(3,3),(3,4),(3,5),(3,6),(4,4)$$

while (m_{23}, m_{13}) may be any of these and also any of the values

$$(4,3),(5,3),(6,3)$$

apart from the following cases, which are excluded by (6) to (9): (m_{24}, m_{14}, m_{23}, m_{13}) $= (3, n, 3, m)$, $(3, n, m, 3)$, $(3, 3, n, m)$ for $3 \leq m < n \leq 6$ and $(4, 4, p, q)$, for all values of (p, q) except (4,4).

Subcase: $3 \leq m_{34} \leq 6$

By (2), at most two other angles can be $\pi/2$. Suppose first that there are exactly two, and that they occur along edges which do not meet at a vertex, then, by symmetry, we assume that they are α_{13} and α_{24}. Again by symmetry, we assume $m_{14} \leq m_{23}$. Now, (3) gives the inequality $3 \leq m_{14} \leq m_{23} \leq 6$ when $m_{34} = 3$, with the upper bound, 6, being replaced by 4 when $m_{34} = 4$ and 3 when $m_{34} \geq 5$. In the remaining case, at least one of the vertices v_3, v_4 is in no edge with an angle of $\pi/2$. We may assume that this vertex is v_3, in which case we must have $m_{34} = m_{13} = m_{23} = 3$. By symmetry we assume $m_{14} \leq m_{24}$. By (3) we now have either $m_{14} = 2$ and $2 \leq m_{24} \leq 6$ or $m_{14} = m_{24} = 3$.

This completes the classification in the case $k = 2$. We have found a total of 42 families with one parameter, 8 with two, and 5 with 3. These are listed, along with the other families of truncated tetrahedra, in Tables 1-8.

In the case $k = 1$ there is at most one parameter, and this must be one of the angles α_{12}, α_{13} or α_{14}. We choose α_{14} as the parameter. In the degenerate case $\alpha_{14} = 0$ ($m_{14} = \infty$), the edge e_{14} becomes a vertex at infinity. If this vertex is moved to $\infty \in \partial\mathbf{H}^3$, then we obtain an infinite rectangular prism, with the angles round the base being, in turn, α_{12}, α_{23}, α_{13} and $\pi/2$. As noted in [CM], this prism can be cut diagonally into two cusped tetrahedra. When $m_{12} = 2$ and $m_{13} = m_{23}$, these tetrahedra are equal and are Coxeter tetrahedra. This occurs for the limiting prisms of families 1 and 18 of Tables 1 and 2.

Different families may have the same limiting prism. This occurs when $m_{12} = 2$ for both families the other values of m_{ij} are the same for both families, except that m_{13} and m_{23} are interchanged. The three pairs of families 2 and 10, 3 and 12, and 14 and 20 of Table 1 each have equal limiting prisms.

3 Presentations and Generators

For each of the truncated tetrahedra described in the previous section there is a corresponding group generated by the reflections in its faces. We denote this group by $\Gamma(m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34})$, where the π/m_{ij} are the angles of the truncated tetrahedron in question. We will usually abbreviate this to $\Gamma(\mathbf{m})$ or simply to Γ. In this section we consider presentations for these groups and their orientation preserving subgroups, and attempt to find small sets of generators for them. Obviously, when there are k truncations, Γ can be generated by $k+4$ generators– the reflections themselves– and it is easy to show that the orientation preserving subgroup Γ^0 can always be generated by $k+3$ generators. In fact, as Conder and Martin [CM] have found, in the case $k=1$, $k+2$ generators usually suffice. We show that this remains true in most of the cases $k>1$. On the way we find a number of interesting results about generators for general Coxeter groups and, in particular, extended triangle groups.

A presentation for each $\Gamma(\mathbf{m})$ is given by Poincaré's polyhedral theorem [Mas; Theorem H 11] (from which it also follows that each of these groups is discrete). This presentation takes the form, $\Gamma(\mathbf{m}) = (X; R)$, where $X = \{f_1, f_2, f_3, f_4, c_1, \ldots, c_k\}$ (here we have used the same symbol to denote reflection through a face as for the face itself), and R contains the following relators:

$$f_i^2 \qquad (1 \le i \le 4)$$

$$c_i^2 \qquad (1 \le i \le k)$$

$$(f_i f_j)^{m_{pq}} \qquad (1 \le i < j \le 4),$$

$$(f_i c_j)^2 \qquad (1 \le i \le 4) \qquad (1 \le j \le k \qquad i \ne j)$$

where $\{p, q\} = \{1, 2, 3, 4\} \backslash \{i, j\}$

In particular, the reflection groups, $\Gamma(\mathbf{m})$ are all Coxeter groups. Given a presentation $(X; R)$ for any Coxeter group, we term the elements of X, *canonical generators* and the product of any two of these, if of finite order, *edge relators*. In the case of reflection groups, the canonical generators are thus reflections and the edge relators, rotations about edges. In the sequel we will assume X is finite.

Since, every relator is a word of even length, each group, C, has an index 2 subgroup consisting of all the elements of C which can be represented by words of even length. We denote this group by C^0. The definitions of canonical generator, edge relator and C^0 all depend on the particular choice of presentation, not just on the abstract group, so that when we refer to a Coxeter group it is tacitly understood that we have some particular presentation in mind. When C is a reflection group, the choice of presentation is natural, so that there is no danger of ambiguity.

Suppose that the canonical generators for C with presentation $(X; R)$ are $a_1, a_2, \ldots, a_n$. Let $x_i = a_i a_{i+1}$ $(1 \le i \le n-1)$. The set $X' = \{x_i : 1 \le i \le n-1\}$ generates C^0 and $(X'; R')$ is a presentation for C^0 where R' is obtained from R by omitting the relations of the form ${a_i}^2 = 1$ and rewriting the rest in terms of the x_i.

For example, if $\mathbf{m} = (7, 4, 5, 4, 3, 2)$ (in which case $k = 2$), then $\Gamma(\mathbf{m})$ has presentation:

$$\begin{aligned} & \langle f_1, f_2, f_3, f_4, c_1, c_2; {f_1}^2 = {f_2}^2 = {f_3}^2 = {f_4}^2 = {c_1}^2 = {c_2}^2 = (f_1 f_2)^2 \\ = \ & (f_1 f_3)^3 = (f_1 f_4)^4 = (f_2 f_3)^5 = (f_2 f_4)^4 = (f_3 f_4)^7 = (f_2 c_1)^2 = (f_3 c_1)^2 = (f_4 c_1)^2 \\ = \ & (f_1 c_2)^2 = (f_3 c_2)^2 = (f_4 c_2)^2 = 1\rangle. \end{aligned}$$

while the orientation preserving subgroup, $\Gamma^0(\mathbf{m})$, has presentation:

$$\begin{aligned} & \langle x_1, x_2, x_3, x_4, x_5; (x_1)^2 = (x_1 x_2)^3 = (x_1 x_2 x_3)^4 = (x_2)^5 = (x_2 x_3)^4 = (x_3)^7 \\ = \ & (x_2 x_3 x_4)^7 = (x_3 x_4)^2 = (x_4)^2 = (x_1 x_2 x_3 x_4 x_5)^2 = (x_3 x_4 x_5)^2 = (x_4 x_5)^2 = 1\rangle. \end{aligned}$$

where $x_1 = f_1 f_2$, $x_2 = f_2 f_3$, $x_3 = f_3 f_4$, $x_4 = f_4 c_1$, $x_5 = c_1 c_2$. We have thus shown that $\Gamma^0(\mathbf{m})$ is generated by $k + 3$ elements. The following lemma, proved by Conder and Martin [CM], shows that $k + 2$ generators usually suffice.

Lemma 2 *If $\alpha, \beta, \gamma, \delta$ are group elements satisfying the relations*

$$\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = (\alpha\beta)^2 = (\beta\gamma)^2 = (\gamma\delta)^2 = (\delta\alpha)^n = 1$$

where n is odd, then $\langle \alpha\beta, \beta\gamma, \gamma\delta \rangle$ is a 2-generator group, generated for example by $\gamma\alpha$ and $\beta\delta$.

Suppose a polyhedron with p faces has among them four, say a_1, a_2, a_3, a_4, such that, for each $i \le 3$, a_i and a_{i+1} meet at an angle of $\pi/2$, while a_4 and a_1 meet at angle π/n (n odd). In this case the above lemma shows that $< a_1 a_2, a_2 a_3, a_3 a_4 >$ is a 2-generator group, and hence that the orientation preserving group for this polyhedron is $(p-2)$–generator.

In the case of the truncated tetrahedra, there are four such faces if either of the following conditions hold:

1. For some $i \leq k$ there are 2 faces f_j, f_l, adjoining f_i at angles $\pi/2$ and π/n (n odd), respectively; in this case, let $(a_1, a_2, a_3, a_4) = (f_i, f_j, c_i, f_l)$.

2. For some $i \leq k$ one of the three angles between the faces adjoining c_i is $\pi/2$ and another is π/n (n odd); in this case $\{a_1, a_2, a_3, a_4\}$ comprises c_i and its three adjoining faces.

It is also possible that we might, in some cases, take $\{a_1, a_2, a_3, a_4\}$ to be $\{f_1, f_2, f_3, f_4\}$ but this turns out not to be useful. In terms of the m_{ij}, we can state the above two conditions as, respectively,

$\exists i \leq k$ and j, l, m distinct from i and each other such that

$$m_{jl} = 2 \text{ and } m_{lm} \equiv 1 \bmod 2 \tag{10}$$

$\exists i \leq k; j, l$ such that,

$$m_{ij} = 2 \text{ and } m_{il} \equiv 1 \bmod 2 \tag{11}$$

An inspection of Tables 1-2 shows that (10) holds when $k = 1$ except for the seven one-parameter families for which m_{23} is even (10,11,18,19,20,22,23 of Table 1). Of these families, (11) holds for familes 10 and 20 and, when the parameter, m_{14}, is odd, for family 18.

When $k = 2$ or 3 it is routine to verify that one or both of (10) and (11) hold in all cases, with the following exceptions:

$k = 2$, one parameter: Table 5 numbers 37 and 42 and, when m_{12} is even, number 39;

$k = 2$, two parameters: Table 6 number 7, when both parameters are even;

$k = 2$, three parameters all even;

$k = 3$, three parameters: Table 7 number 4 (m_{12}, m_{13} both even), number 5;

$k = 3$: four parameters all even.

To deal with these cases further we prove some more results on triangle groups. In what follows $\Delta = \Delta(p, q, r)$ will denote the extended triangle group, with presentation,

$$\langle a, b, c; a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ac)^r = 1 \rangle$$

The content of the next theorem is implicit in [CM] (Lemma 2.1 and argument preceding Theorem 2.3)

Theorem 3 *The group Δ is a 2-generator group if one of p, q, r is 2 and another is odd, or if two of p, q, r are 3 and the third is not a multiple of 3.*

Proof:- Suppose that the first condition holds. We may assume that $p = 2$, and q is odd. The group $< ac, b >$ contains $[ac, b] = acbcab = acbcba$. Taking the $(q + 1)/2$th power of this element gives $acba$, whence ba and so also a and c are in $< ac, b >$, which is thus the entire group.

In the second case, we may assume that $p = q = 3$ and $r \not\equiv 0 \bmod 3$. The group $< abc, b >$ then contains $(abc)b(cba) = aca$ and $(cba)b(abc) = cac$, whence it also contains $acacac$, a power of which is ac. It then follows that $< abc, b >$ also contains $(abc)(ca) = aba$, $baba = ab$ and hence a and c, so that again $< abc, b >= \Delta$.□

We conjecture that the converse to the above theorem also holds. That is,

Conjecture 4 *The group Δ is a 2-generator group* only *if one of p, q, r is 2 and another is odd, or two of p, q, r are 3 and the third is not a multiple of 3.*

If w is a word in a free group. we say that a letter, a of w occurs in an even (resp. odd) position if $a^{\pm 1}$ is the nth letter of w for some even (resp. odd) n. We have this simple lemma.

Lemma 5 *Let F be free on the generators $a_1, a_2, \ldots, a_n$, and let π be the canonical projection from F to the n-fold free product of Z_2. If $w \in F$ and $\pi(w) = 1$ then each letter a_i has an equal number of occurrences in odd and even positions.*

Proof:- We use induction on the length of w. If $w = 1$ then the result is trivial. Otherwise, if $\pi(w) = 1$ then w must contain two adjacent occurences of the same letter. By the induction hypothesis, the word w' obtained by deleting these two occcurences satisfies the conclusion of the lemma, whence it readily follows that w does also.□

Lemma 6 *If C is a Coxeter group and $w \in C \backslash C^0$, then w^2 is in the commutator, $(C^0)'$.*

Proof:- Let C have canonical generators, $a_1, a_2, \ldots, a_n$. Setting $b = a_n$, it follows that the elements $y_1 = ba_1, y_2 = ba_2, \ldots, y_{n-1} = ba_{n-1}$ generate C^0, and $by_i^{\pm 1} = y_i^{\mp 1}b$ $(1 \leq i \leq n-1)$. We may now write $w = bv$ where v is a word in the y_i, since $v \in C^0$. Now $w^2 = bvbv$. We advance the leftmost b to the right until it meets and cancels the second b. Since the letters of the first v are replaced by their inverses, w^2 is a word in the y_i in which, for each i, the sum of the exponents of y_i is zero. Thus $w^2 \in (C^0)'$.□

Lemma 7 *Let C be a Coxeter group with n canonical generators, for which the orders of the edge relators have a common factor greater than one. Then the abelianized group, $C^\circ/(C^0)'$ requires at least $n-1$ generators.*

Proof:- Let $h > 1$ be a common factor of the orders of the edge relators. The presentation for C^0 has $n-1$ generators, and the relators in this presentation are all words raised to a multiple of h. Clearly then, we can obtain the direct product of $n-1$ copies of Z_h as a quotient of the abelianized group, $C^0/(C^0)'$, and the lemma follows.□

Corollary 8 *Let $\Delta = \Delta(p, q, r)$ where p, q, and r have a common factor greater than one. Then the abelianized triangle group, $\Delta^0/(\Delta^0)'$, is not cyclic.*

Proposition 9 *If P represents the k-fold free product of Z_2 with itself, and C is a Coxeter group with n canonical generators, and for which the powers of the edge relators have a common factor greater than one then the group*

$$(P \times C)^0$$

requires at least $n + k - 1$ generators.

Proof:- Suppose, for a contradiction, that $(P \times C)^0$ can be generated by $n + k - 2$ generators. By the Grushko-Neumann theorem ([MKS] 4.1 p191), we may assume these generators are of the form,

$$(\alpha_1, u_1),\ (\alpha_2, u_2), \ldots (\alpha_k, u_k), (1, v_1), (1, v_2) \ldots (1, v_{n-2})$$

where the α_i are the canonical generators of P , $u_i \in C \backslash C^0$ $(1 \leq i \leq k)$, and $v_j \in C^0$ $(1 \leq j \leq n-2)$.

It follows that for any (x, y) in the group, there is a pair of words $(\overline{W}, W)$, which represents (x, y), where W is a word in the u_i, and v_j, and $\overline{W}$ is obtained by substituting α_i for u_i, in W, and omitting the v_j.

Since, in particular, these generators generate any word of the form $(1, x)$, any $x \in C^0$ can be represented by a word, W, for which $\overline{W}$ represents the identity of P. Since, by Lemma 6, $u_i^2 \in (C^0)'$, this x may be written, modulo $(C^0)'$, as a word, W' containing the u_i to positive powers only. The corresponding word $\overline{W'}$ again represents the identity of P. As in the proof of lemma 6, we let the canonical generators of C be, $a_1, a_2, \ldots, a_n$ and define $b = a_n$ and $y_i = ba_i$ $(1 \leq i \leq n-1)$. We may now write each u_i as a word bw_i, where w_i is a word in the y_j, and each v_i as a word in the y_j only. By substituting these words for the u_i and v_i, we also express W' as a word in b and the y_j.

Again, as in the proof of lemma 6, we now advance each occurence of b to the right of the word, starting from the rightmost occurence of b and working left. These occurences of b then accumulate at the right of the word and cancel. The exponent of each y_i, changes sign as many times as there are occurences of b to the left of it. That is an exponent of a y_j that has been substituted for the nth occurrence from the left in W', of a letter in $\cup u_i$, changes sign n times. Since $\overline{W'}$ represents the identity of P, it follows, using Lemma 5, that W' reduces, modulo $(C^0)'$, to a product of the v_j. Consequently, the abelianized group $C^0/(C^0)'$ can be generated by $n-2$ generators. But this is contrary to lemma 7.□

Remark:- Since easily, $(Z_2 \times C)^0$ is isomorphic to C, it follows, when C satisfies the hypotheses of this proposition, that C requires n generators.

We now apply these results to the groups $\Gamma^0(\mathbf{m})$

Corollary 10 *If $k = 1$ and m_{23}, m_{24} and m_{34} are even then $\Gamma^0(\mathbf{m})$ requires four generators when m_{12}, m_{13}, m_{14} have a common factor greater than one.*

Proof:- Since the m_{23}, m_{24}, m_{34} are even, $(Z_2 * Z_2) \times \Delta(m_{12}, m_{13}, m_{14})$ is a quotient group of Γ, and the corollary follows immediately from the above result.□

On the other hand,

Lemma 11 *If, for some $i \leq k$, $\Delta(m_{ij}, m_{ik}, m_{il})$ is two-generator, where $\{j, k, l\} = \{1, 2, 3, 4\} \backslash \{i\}$, then $\Gamma^0(\mathbf{m})$ is $(k+2)$-generator.*

Proof:- For convenience we assume $i = 1$, then Γ^0 is generated by $x = c_1 f_2$, $y = c_1 f_3$, $z = c_1 f_4$, $w = c_1 f_1$ and $v_i = c_1 c_i$ $(1 < i \leq k)$. Rewriting the relators from the presentation for Γ in terms of the new generators, it is clear that the subgroup $< x, y, z >$ is the extended triangle group $\Delta(m_{ij}, m_{ik}, m_{il})$, which is two-generator by hypothesis, whence Γ^0 is $(k + 2)$-generator.$\Box$

Theorem 3 and the above two results allow the minimum number of generators to be found for most of the outstanding groups Γ^0, in the case $k = 1$, and these are given in Table 1. Families 19, with m_{14} odd, and 22 remain unresolved but, by Corollary 10, the groups from these families will require four generators if the extended triangle groups $\Delta(3, 4, q)$ and $\Delta(4, 4, q)$ respectively, require three generators, which will be true if Conjecture 4 holds.

When each of the m_{ij} is even, replacing each of the m_{ij} by 2 and abelianizing gives the $(k + 4)$-fold product of Z_2. Since this group requires $k + 4$ generators, the same is true of Γ, whence Γ^0 requires $k + 3$ generators. For $k = 2$ or $k = 3$ this argument deals with most of the remaining cases. The only exceptions are,

1. $\sharp$ 37 from Table 5, when the parameter is odd.
2. $\sharp$ 42 from Table 5
3. $\sharp$ 4 from Table 7 when m_{23} is odd and the other two parameters are even.
4. $\sharp$ 5 from Table 7

In the first of these, Lemma 11 shows that Γ^0 requires 5 generators, if, as we have conjectured, the extended triangle group $\Delta(p, 4, 4)$ requires three generators. In the second, proposition 3 and corollary 10, show that Γ^0 is four generator at least when m_{12} is not a multiple of 3. Otherwise, we conjecture that five generators are needed. In the third case, six generators are required if $\Delta(4, q, r)$ requires three generators, for odd q and even $r > 2$, as we have conjectured.

In the last case, Γ^0 is 5-generator in at least some cases, for example when one parameter is 3 and another is not a multiple of 3. In general though it seems likely that 6 generators are required.

Similar arguments show that, when $k = 4$, Γ^0 requires only 6 generators in some cases and 7 in others, according to the values of the parameters. We omit details.

4 Volumes

We now show how to calculate the volume of a truncated tetrahedron, and hence the co-volumes of the groups $\Gamma(\mathbf{m})$ and $\Gamma^0(\mathbf{m})$.

By Schläfli's equation (see e.g. [V, Chapter 7, 2.2]), if P is a hyperbolic polyhedron of given combinatorial type with dihedral angles $\theta_1,\theta_2,\ldots,\theta_m$, then its volume, $V = V(\theta_1,\theta_2,\ldots,\theta_m)$ has derivatives:

$$\partial V/\partial\theta_i = -l_i/2 \qquad (1 \leq i \leq m) \tag{12}$$

where l_i is the length of the edge along which the angle θ_i occurs.

Given this result, we may find V as a function of $\boldsymbol{\theta} = (\theta_1, \theta_2,\ldots,\theta_m)$ provided we know the edge lengths as a function of $\boldsymbol{\theta}$ along with the volume at some particular initial value $\boldsymbol{\theta}_0$ of $\boldsymbol{\theta}$. The volume at $\boldsymbol{\theta} = \boldsymbol{\theta}_1$ is then given by the line integral:

$$V(\boldsymbol{\theta}_1) = V(\boldsymbol{\theta}_0) + \int_{\boldsymbol{\theta}_0}^{\boldsymbol{\theta}_1} \nabla V \cdot \mathbf{d}\boldsymbol{\theta} \tag{13}$$

We now apply these results to truncated tetrahedra. Throughout, θ_{ij} will be the variable of integration corresponding to the angle α_{ij}, and $\boldsymbol{\alpha}$ will denote the vector $(\alpha_{12},\ldots\alpha_{34})$

When k=1 we vary only the three angles θ_{23}, θ_{24} and θ_{34}. We set $\boldsymbol{\theta} = (\theta_{23},\theta_{24},\theta_{34})$ The θ_{1j} $(2 \leq j \leq 4)$ will be fixed at value α_{1j} while the remaining three angles are all $\pi/2$. For the initial value we set:

$$\boldsymbol{\theta} = \boldsymbol{\alpha}_0 = (\pi/2, \pi/2, \pi/2) \tag{14}$$

This choice of angles determines a degenerate prism of zero volume. Faces f_1 and c_1 coincide, both being triangles with angles θ_{12},θ_{13} and θ_{14} (note that when $\boldsymbol{\theta} = \boldsymbol{\alpha}_0$ (1) does not hold which is why we get a degenerate rather than a proper polyhedron). The volume of the prism is now given by:

$$V = V(\alpha_{12},\alpha_{13},\alpha_{14},\alpha_{23},\alpha_{24},\alpha_{34}) = \int_{\boldsymbol{\alpha}_0}^{\boldsymbol{\alpha}_1} \nabla V \cdot \mathbf{d}\boldsymbol{\theta} \tag{15}$$

where $\boldsymbol{\alpha}_1 = (\alpha_{23},\alpha_{24},\alpha_{34})$.

The length l_{ij} of the edge e_{ij} is given by:

$$l_{ij} = \operatorname{arcsinh}\sqrt{\frac{|G||G^{ij}_{ij}|}{|G^i_i||G^j_j|}} \tag{16}$$

(see e.g. equation 1.40 of [M]) where G is the Gram matrix corresponding to faces f_1, f_2, f_3, f_4. In view of (12) and (16), evaluating the line integral in (15) is a straightforward numerical calculation. For simplicity, the path of integration may be taken as a line segment joining the endpoints.

In the cases where the angle α_{14} is a parameter, the angles α_{24} and α_{34} are both $\pi/2$. Hence, in this case, we may calculate the volume by varying α_{23} only. This method also works in the degenerate case $\alpha_{14} = 0$

The volume of the prism is the co-volume of the corresponding reflection group $\Gamma(\mathbf{m})$ and these volumes are doubled to get the co-volumes of the index two subgroup $\Gamma^0(\mathbf{m})$. Some of these volumes are given in Tables 2–5.

When $2 \leq k \leq 3$ we can calculate volumes using induction on k. In these cases we vary only the angle $\phi = \theta_{12}$ the other angles remaining fixed.

Let $\phi_1 = \alpha_{12}$ and ϕ_0 be the least value of ϕ for which at least one of $\phi + \alpha_{13} + \alpha_{14}$, $\phi + \alpha_{23} + \alpha_{24}$ is π. Geometrically, ϕ_0 is obtained by continuously increasing ϕ until one of the truncations at v_1 and v_2 becomes instead an ideal vertex (which may be viewed as a truncation "at infinity"). If we let α' be the vector got by replacing θ_{12} in α by ϕ_0 then (12) and (15) give

$$\begin{aligned} V(\alpha) &= V(\alpha') + \int_{\phi_0}^{\phi_1} \frac{\partial V}{\partial \phi} \mathrm{d}\phi & (17) \\ &= V(\alpha') + \frac{1}{2} \int_{\phi_1}^{\phi_0} l_{12} \mathrm{d}\phi & (18) \end{aligned}$$

In the degenerate case, $\phi_1 = \alpha_{12} = 0$, the integral (18) is improper.

Since α' determines a tetrahedron with $(k-1)$ or fewer truncations, its volume can be calculated by the induction hypothesis.

To complete the calculation we need only find l_{12} as a function of ϕ. Since $2 \leq k \leq 3$, f_3 is a pentagon with sides a_1, l_{12}, a_2, l_{24} and l_{14}, where a_i $(i = 1, 2)$ is the edge between f_3 and c_i. If we let ψ be the face angle between edges l_{14} and l_{24}, a formula of Beardon [B Theorem 7.18.1(i)] gives

$$\cosh(l_{12}) = \frac{\cosh(a_1)\cosh(a_2) + \cos(\psi)}{\sinh(a_1)\sinh(a_2)} \tag{19}$$

Spherical cosine rule applied to a small sphere centred at v_4 gives

$$\cos(\psi) = \frac{\cos(\theta_{14})\cos(\theta_{24}) + \cos(\theta_{34})}{\sin(\theta_{14})\sin(\theta_{24})} \tag{20}$$

while hyperbolic cosine rule applied to c_1 (whose face angles are just the dihedral angles ϕ, θ_{13},θ_{14}) gives

$$\cosh(a_1) = \frac{\cos(\phi)\cos(\theta_{14}) + \cos(\theta_{13})}{\sin(\phi)\sin(\theta_{14})} \tag{21}$$

and symmetrically

$$\cosh(a_2) = \frac{\cos(\phi)\cos(\theta_{24}) + \cos(\theta_{23})}{\sin(\phi)\sin(\theta_{24})} \tag{22}$$

whence $\sinh(a_1)$, $\sinh(a_2)$ can also be calculated. Using (19)–(22), we may now calculate numerically l_{12} as a function of ϕ and so the integral (18).

When $k = 4$, a similar approach could be used to reduce the calculation to the case $k = 3$. However it is more convenient to proceed as follows. In (15) set $\boldsymbol{\theta}_0 = (\pi/3, \pi/3, \pi/3, \pi/3, \pi/3, \pi/3)$ and $\boldsymbol{\theta}_1 = \boldsymbol{\alpha}$. Thus the initial polyhedron is ideal and so its volume is easily calculated by the well known formula

$$\mathrm{volume}(\Delta) = \eta(\alpha) + \eta(\beta) + \eta(\gamma) \tag{23}$$

where Δ is an ideal simplex, α, β, γ, the dihedral angles along the three edges meeting at a given vertex of Δ (the choice of vertex being immaterial) and $\eta(\theta)$ is the Lobachevsky function defined by

$$\eta(\theta) = -\int_0^\theta \log \mid 2\sin(u) \mid \mathrm{d}u \tag{24}$$

(see e.g. [Mi, p18 lemma 2]) Combining (15) and (23) gives

$$V = V(\alpha) = 3\eta(\pi/3) + \int_{\boldsymbol{\alpha}_0}^{\boldsymbol{\alpha}_1} \nabla V \cdot \mathbf{d}\boldsymbol{\theta} \tag{25}$$

Again we may take the integration path to be a line segment. It is easy to verify that θ at any interior point of this path, determines a tetrahedron with four truncations. Hence each f_i is a hexagon (if $k < 4$ this is not generally true, which is why this approach does not work in these cases) To calculate the integral in (25) is again simply a question of calculating the lengths l_{ij} as a function of θ.

As an example, we show how to calculate l_{12}, the other edge lengths all being found similarly. The required edge length l_{12} is the length of one of the sides of f_4, the others being in turn, b_2, l_{23}, b_3, l_{13} and b_1, where b_i $(1 \leq i \leq 3)$ is the length of the edge between f_4 and c_i. All the angles in this hexagon are right angles which allows us to apply another formula of Beardon's [B,Theorem 7.19.2] which gives

$$\cosh(l_{12}) = \frac{\cosh(b_1)\cosh(b_2) + \cosh(b_3)}{\sinh(b_1)\sinh(b_2)} \tag{26}$$

and, as in (21), (22), the $\cosh(b_i)$, and hence also the $\sinh(b_i)$, can be expressed in terms of θ by using cosine rule on the triangular face c_i.

5 Torsion-free Subgroups

It is a consequence of Selberg's lemma [Se] that all finitely generated Fuchsian and Kleinian groups have torsion-free subgroups of finite index.

It is naturally of interest to know what the smallest possible index of a torsion-free subgroup of any given group is. For Fuchsian groups this question has been settled by Edmonds, Ewing and Kulkarni [EEK], but for Kleinian groups in general, it remains open (We discuss this in more detail shortly).

Figure 2. *The Prism $P(q)$*

In this section we find low index torsion-free subgroups of two related families of reflection groups, $\Gamma^0(q)$ and $\tilde{\Gamma}^0(q)$, where, $\Gamma^0(q) = \Gamma^0(2,3,q,3,2,2)$ has presentation

$$< a, p, c, e \mid a^2 = p^2 = c^3 = e^2 = (ac)^3 = (pc)^2 = (ap)^q = (ace)^2 = (pce)^2 >$$

and $\tilde{\Gamma}^0(q)$ is the subgroup $< a, p, c >$ of $\Gamma^0(q)$. Geometrically, $\Gamma^0(q)$ is the orientation preserving subgroup of the reflection group of the prism $P(q)$ pictured in Figure 2.

The dihedral angles of this prism are all $\pi/2$, except where indicated otherwise. If α, β, γ, δ, and ϵ denote respectively reflection through the bottom, left, back, right, and top faces of $P(q)$, then we have $a = \alpha\beta$, $c = \alpha\delta$, $p = \alpha\gamma$, and $e = \delta\epsilon$.

We define $\tilde{\Gamma}^0(q)$, to be the orientation preserving subgroup of the reflection group of the "open" prism obtained by omitting the truncating face c_1 from $P(q)$. We obtain a presentation for $\tilde{\Gamma}^0(q)$ by omitting, from the presentation given for $\Gamma^0(q)$, the generator e, and the relations involving it.

For the groups $\tilde{\Gamma}(q)$ it is useful to include the cases $q \leq 6$. For $q = 1$, $\tilde{\Gamma}(q)$ is trivial. For $q = 6$ it is the reflection group of a cusped tetrahedron. For $2 \leq q \leq 5$ the $\tilde{\Gamma}(q)$ are reflection groups of spherical tetrahedra, with orders 48, 120, 384 and 14400 for $q = 2, 3, 4, 5$, respectively (see [Hu] for details).

A possible method of finding torsion-free subgroups of a group G is to construct a manifold by glueing together copies of its fundamental domain. This, for example, is the approach taken by Edmonds *et al* with Fuchsian groups. We take this geometric viewpoint with the groups $\Gamma^0(q)$ and $\tilde{\Gamma}^0(q)$ in the next section. Here we concentrate on finding the subgroups algebraically.

A practical means of doing this is given by Theorem 13, which is essentially a result of Singerman [Si] although the idea was also used by Fox [F] to prove the existence of torsion-free subgroups of triangle groups. First we need a simple lemma.

Lemma 12 *If G is a discrete group generated by reflection through the faces of a hyperbolic polyhedron, $g \in G^0$ is of finite order if and only if it is conjugate to a power of an edge relator*

Proof:- [CM, Lemma 4.1]

Definition:- If G is a Coxeter group and G^0 acts on a set X, then this group action is said to be *semiregular*, and X a *semiregular* G^0-set, if the permutation effected by the action of each edge relator is semiregular (having all cycles of equal length) and this cycle length is equal to the order of the relator.

Theorem 13 *Let C be a Coxeter group with the property that any $g \in C^0$ is of finite order if and only if it is conjugate to a power of an edge relator. The group C^0 has a torsion-free subgroup of index n if and only if there is a transitive semiregular action of C^0 on a set S of n elements.*

Proof:- Suppose that such an action exists. Since C acts transitively on S the stabilizer of any $x \in S$ is clearly a subgroup of index n.

If a is a power of an edge relator and has order m, then, by assumption, it effects a permutation consisting of m-cycles. Consequently, every element of order m, being conjugate to a power of an edge relator, effects such a permutation. The stabilizer of any x is thus torsion-free.

Conversely, if H is torsion-free subgroup of C^0 of index n, then the action of C^0 on the cosets of H has the required properties.$\Box$

Lemma 12 shows that the above theorem can be applied to discrete hyperbolic reflection groups, and, in particular, to the $\tilde{\Gamma}^0(q)$ for $q \geq 6$. For $2 \leq q < 6$, $\tilde{\Gamma}^0(q)$ is finite so every element is torsion and some of these are *not* conjugate to a power of any edge relator, so that the above theorem is not applicable (of course, since these groups are finite, the only torsion-free subgroups are the trivial ones). Nevertheless, it will be useful to find transitive semiregular $\tilde{\Gamma}^0(q)$-sets for these values of q too, because they can be used to construct such sets for larger values of q (Theorem 16)

We use CAYLEY to find semiregular $\tilde{\Gamma}^0(q)$-sets for small values $q_1, q_2, \ldots q_n$ of q and then show how these sets can be "spliced" together in such a way as to construct a transitive $\tilde{\Gamma}^0(q_1 + q_2 + \ldots + q_n)$-set, also semiregular. By Theorem 13 this gives torsion-free subgroups for these values of $(q_1 + q_2 + \ldots + q_n)$. In this way we eventually find torsion-free subgroups of $\tilde{\Gamma}^0(q)$ of the smallest possible index for every $q \geq 6$ (Theorem 16). Adapting the arguments also gives torsion-free subgroups of $\Gamma^0(q)$ of the smallest possible index in most cases when q is a multiple of 4 (Theorem 19).

It remains to find how small this index can be. A useful lower bound can be found by considering finite subgroups.

Lemma 14 *The index of any finite-index torsion-free subgroup of a group G is divisible by the lowest common multiple of the orders of finite subgroups of G.*

Proof:- Let H be a torsion-free, and A a finite, subgroup of G. We define an equivalence relation $\sim$ on the left cosets of H by $aH \sim bH$ if $b = gah$ for some $g \in A$, $h \in H$. Since H is torsion-free, it readily follows that each equivalence class contains $|A|$ cosets, and the lemma follows.$\square$

The result of Edmonds *et al* [EEK] is that, for Fuchsian groups, the lower bound given by the above lemma is either attained or exceeded by a factor of 2. By contrast, the smallest index of a torsion-free subgroup of a Kleinian group may exceed the LCM of the orders of finite subgroups by an arbitrarily large factor [JR]. However, in the examples we consider, the smallest index of a torsion-free subgroup will be at most double the lower bound of the lemma.

The finite subgroups of $\Gamma^0(q)$ are all conjugate to subgroups of stabilizers of vertices of $P(q)$. There are six vertices and their stabilizers are the triangle groups $\Delta^0(2,3,3)$, $\Delta^0(2,2,2)$, $\Delta^0(2,2,3)$, and $\Delta^0(2,2,q)$, the last two of these groups occuring at two vertices each. These stabilizers have orders 12,4,6, and $2q$ respectively, so that, by Lemma 14, torsion-free subgroups of $\Gamma^0(q)$ have index at least $\mathrm{LCM}(12, 2q)$. The same bound also applies to the groups $\tilde{\Gamma}^0(q)$.

We introduce a set of Coxeter groups which slightly generalize the $\tilde{\Gamma}^0(q)$.

Definition:-

$$G(q) = \{a, p, c \mid a^2 = p^2 = c^m = (pc)^2 = (ac)^n = (ap)^q\}$$

We have $G(q) = \tilde{\Gamma}^0(q)$ when $m = n = 3$. By a slight abuse of notation, we will use the same letters a, p and c to refer to the generators of $G(q)$ for different values of q.

The following results are concerned with actions of $G(q)$ on finite sets. In general, if a group G acts on a set X we refer to a set of the form $\{g^n x | n \in Z\}$ ($x \in X$, $g \in G$) as a g-cycle in X. If g is of finite order m then the sequence $g^{m_1}x, g^{m_2}x, \ldots, g^{m_k}x$, with $0 \le m_i < m$ for each i, is said to be in *cyclic order* if, for some $j \le k$,

$$0 \le m_j < m_{j+1} < \ldots m_k < m_1 < m_2 < \ldots < m_{j-1} < m$$

Clearly a sequence is in cyclic order if and only if any cyclic permutation of it is also.

Definition:- Let X_1 be a $G(q_1)$-set and X_2 a $G(q_2)$-set, for which the group action is semiregular. Let $\Sigma_1 \subseteq X_1$, $\Sigma_2 \subseteq X_2$ both be unions of c-cycles which are disjoint from their images under p. Define $C_1 = \Sigma_1 \cup p(\Sigma_1)$, $C_2 = \Sigma_2 \cup p(\Sigma_2)$. We define an equivalence relation as follows. The pairs (X_1, Σ_1) and (X_2, Σ_2) are *equivalent* if there is a bijection, $\psi : C_1 \to C_2$ for which,

(1) $\psi(\Sigma_1) = \Sigma_2$

(2) $\psi(cx) = c\psi(x)$ $\psi(px) = p\psi(x)$

(3) Two points in C_1 are in the same ap-cycle if and only if their images under ψ are.

(4) If H_1 is an ap cycle in X_1 and $x_1, x_2, \ldots, x_m$ are the points of $H_1 \cap C_1$ in cyclic order, then the points $\psi(x_1), \psi(x_2), \ldots \psi(x_m)$ are also in cyclic order.

Lemma 15 (splicing) *For $1 \leq i \leq n$, let q_i be a positive integer, X_i a semiregular $G(q_i)$-set, and $\Sigma_i \subseteq X_i$ a union of c-cycles for which Σ_i and $p(\Sigma_i)$ are mutually disjoint. Let $C_i = \Sigma_i \cup p(\Sigma_i)$.*

Suppose also that the (X_i, Σ_i) are all mutually equivalent, and that, for each ap-cycle $H \subseteq X_1$, the difference d between the cardinalities m_1 of $H \cap \Sigma_1$ and m_2 of $H \cap p(\Sigma_1)$ is relatively prime to n. The disjoint union of the X_i can then be made into a semiregular $G(q)$-set, where $q = \sum q_i$.

If X_i is a transitive $G(q_i)$-set for some i, then X is a transitive $G(q)$-set.

Proof:- We may assume that the sets X_i are pairwise disjoint. For $1 \leq i \leq n-1$ let (X_i, Σ_i) and (X_{i+1}, Σ_{i+1}) be equivalent via the map ψ_i. Define $\psi_n : \Sigma_n \to \Sigma_1$ by the condition, that $\psi_n \circ \psi_{n-1} \circ \ldots \circ \psi_1$ is the identity map on C_1. Let $X = \cup_{i=1}^n X_i$.

For $x \in X_i$ we denote the image of x under the given action of $g \in G(q_i)$ by $g *_i x$

We claim that the maps $a, p, c : X \to X$ given by

$$ax = a{*_i}x \quad x \in X_i \tag{27}$$

$$cx = c{*_i}x \quad x \in X_i \tag{28}$$

and

$$px = \begin{cases} p{*_{i+1}}\psi_i(x) & x \in \Sigma_i \\ p{*_{i-1}}\psi_{i-1}^{-1}(x) & x \in p(\Sigma_i) \\ p{*_i}x & x \in X_i \backslash C_i \end{cases} \tag{29}$$

determine a semiregular action of $G(q)$ on X.

Figure 3. *Action of p on X*

The action of p on X is shown schematically in Figure 3, where dotted lines represent the original action of p on the individual X_i, and solid lines represent the action of p on X. The n sets are joined this way in a loop. It is easy to see that the permutations on X given by a, c , p, ac and cp have the required cycle structure. It remains only to show that each ap-cycle in X has length q.

Refer to two points $x, y \in \cup C_i$ as equivalent if, for some i, j, $y = \psi_{j-1} \circ \psi_{j-2} \circ \ldots \psi_i(x)$ (this clearly *is* an equivalence relation)

Now consider the ap-cycle of some $x \in X_i$. Since every such cycle meets C_i, we may suppose that $x \in C_i$. Let H be the ap-cycle of x in X_i, and suppose that $H \cap C_i = \{x_1, x_2, \ldots, x_m\}$ where $x = x_1$ and the elements are listed in cyclic order. We know that m_1 of these points lie in Σ_i and m_2 in $p(\Sigma_i)$ where $m_1 + m_2 = m$ and $m_1 - m_2$ is relatively prime to n.

Let $\{x'_1, x'_2, \ldots, x'_m\}$ be the intersection of $\cup C_j$ with the ap-cycle, in X, of x listed in cyclic order with $x'_1 = x_1$.

Suppose $x_1 \in \Sigma_i$. We have $apx = a{*_{i+1}}p{*_{i+1}}\psi_i(x_1)$. By assumption, $\psi_i(x_1) \in C_{i+1}$ and the next member of the ap-cycle of $\psi_i(x_1)$ to meet C_{i+1} is $\psi_i(x_2)$. a similar argument applies when $x_1 \in p(\Sigma_i)$. Each time the cycle passes through some C_j it "jumps" in this way, either "forwards" to X_{j+1} or " backwards" to X_{j-1} according as the cycle has passed through Σ_j or $p(\Sigma_j)$. A simple induction shows that $\forall j\ x'_j$ is equivalent to x_j.

By the time the cycle again reaches a point y equivalent to x_1, a total of m_1 forward and m_2 backward jumps will have occurred, so that $y \in X_j$ where $j = i + m_1 - m_2$. Since, by assumption, $m_1 - m_2$ is relatively prime to n, it is now elementary number theory to see that this pattern repeats itself $n-1$ more times before the cycle eventually returns to x, and that this cycle is a union of one ap-cycle from each X_i.

Finally we suppose that, for some i, X_i is a transitive $G(q_i)$ -set. We may suppose that $i = 1$. Since we have shown that every ap-cycle meets X_1, we can prove that X is a transitive $G(q)$-set if we can show that the $G(q)$-orbit of any $x \in X_1$ contains all of X_1. Since we have assumed that X_1 is transitive under the original action of $G(q_1)$, it suffices to show that, for each $x \in X_1$, the $G(q)$-orbit of x contains $a{*_1}x$, $c{*_1}x$, and $p{*_1}x$. By (27) and (28), the first two of these inclusions certainly hold, and, as we have shown, the ap-cycle of x in X contains the the ap-cycle of x under the original $G(q_1)$ action on X_1. It follows that the $G(q)$-orbit of x contains $aa{*_1}p{*_1}x = a{*_1}a{*_1}p{*_1}x = p{*_1}x$, as required.$\Box$

Using CAYLEY we have found transitive semiregular $\tilde{\Gamma}^0(q)$ actions on sets of n elements for some particular values of q and n. Necessarily $2q|n$ and we normalize by assuming that the set is $\{1, 2, \ldots, n\}$, and that the actions of p and a are given by,

$$px = \begin{cases} x+1 & \text{if } x \text{ is odd} \\ x-1 & \text{if } x \text{ is even} \end{cases} \tag{30}$$

$$ax = \begin{cases} x-1 & \text{if } x \text{ is odd } x \not\equiv 1 \bmod 2q \\ x+1 & \text{if } x \text{ is even } x \not\equiv 0 \bmod 2q \\ x-2q+1 & \text{if } x \equiv 0 \bmod 2q \\ x+2q-1 & \text{if } x \equiv 1 \bmod 2q \end{cases} \tag{31}$$

It is easy to verify that a, p, and ap have the required cycle lengths. It remains only to specify the action of c. These are given in Tables 8–19

Theorem 16 *For each $q \geq 7$, the group $\tilde{\Gamma}^0(q)$ has a torsion-free subgroup of index* $\mathrm{LCM}(12, 2q)$*, and $\tilde{\Gamma}^0(6)$ has a torsion-free subgroup of index* 24*. These indices are the smallest, possible in each case.*

Proof:- For $q = 6$ the theorem is proved by direct computation with CAYLEY (there is no torsion-free subgroup of index 12 and precisely two of index 24. One of these is given in Table 11)

For the remaining cases, Lemma 14 shows that the index $\mathrm{LCM}(12, 2q)$ is the smallest possible. We prove the existence of torsion-free subgroups, of this index, by exhibiting transitive semiregular group actions. For this we use Lemma 15 and refer to Tables 8–19. We consider four cases, depending on the residue of q mod 6. In each case we specify j numbers $q_1 \ldots q_j$ $(1 \leq j \leq 3)$, and find, for each i $(1 \leq i \leq j)$, a semiregular $\tilde{\Gamma}^0(q_i)$-set X_i, of cardinality $q_i\mathrm{LCM}(12, 2q)/q$, with subset Σ_i such that the (X_i, Σ_i) and are all equivalent. We then verify that Lemma 15 can be applied if each of the n pairs referred to is a copy of one of the (X_i, Σ_i). In all cases the $m_1 - m_2$ referred to in the statement of Lemma 15 will either be ± 1, in which case n may be chosen arbitrarily, or ± 2, in which case n may be any odd number. We can then conclude the existence of a semiregular $\tilde{\Gamma}^0(q)$-set of the required index, whenever $q = n_1 q_1 + \ldots + n_j q_j$, for nonnegative integers n_1 and n_2, not all zero and where sometimes we require $n = n_1 + \ldots + n_j$ to be odd.

In all the cases we consider, the ap-cycles in X_1 will either all have a single point in C_1, or all have a three points in C_1. In the former case the requirements of Lemma 15 are satisfied trivially. In the latter, they are straightforward to verify.

The details for individual cases are given below:

$\underline{q \equiv 0 \mod 6}$
$j = 2$, $q_1 = 12$, $q_2 = 18$
$X_1 = \{1, 2, \ldots, 24\}$, $X_2 = \{1, 2, \ldots, 36\}$
(Action of c given by Tables 16 and 18, respectively)
$\Sigma_1 = \{1, 23, 18\}$ $\Sigma_2 = \{5, 33, 28\}$ whence, by (30),
$p(\Sigma_1) = \{2, 24, 17\}$ $p(\Sigma_2) = \{6, 34, 27\}$
$\psi_1(1) = 5$, $\psi_1(23) = 33$, $\psi_1(18) = 28$ (We specify the values of ψ on Σ_1

only, as this determines the values of ψ on all of C_1.)

In both X_1 and X_2 the two *ap*-cycles are the sets of odd and of even numbers. In X_1, each of these cycles intersects C_1 in three points, the odd cycle having two points in Σ_1, and one in Σ_2. Hence, for this cycle, $m_1 = 2$, $m_2 = 1$, so that $m_1 - m_2 = 1$. Similarly for the even *ap*-cycle in X_1, $m_1 - m_2 = -1$. It is easy to check that ψ maps the intersection of *ap*-cycles with C_1 to the intersection of *ap*-cycles with C_2 and preserves cyclic order.

Every multiple of 6 greater than 6 itself can be expressed as $12n_1 + 18n_2$ $(n_1, n_2 \geq 0)$ so this case is completed

$\underline{q \equiv 3 \mod 6:}$

$j = 1$, $q_1 = 3$

$X_1 = \{1, 2, \ldots, 12\}$

(Action of c given by Table 8)

$\Sigma_1 = \{1, 5, 7\}$ (whence $p(\Sigma_1) = \{2, 6, 8\}$) ψ is the identity map on C_1.

There are 4 *ap*-cycles in X_1, each of which intersects exactly one of Σ_1 and $p(\Sigma_1)$. This intersection may either one or two points. Thus, for each *ap*-cycle, we have $m_1 - m_2 = \pm 1, \pm 2$ so that we may apply Lemma 15 to join an odd number of copies of X_1 together.

$\underline{q \equiv \pm 1 \mod 6:}$

$j = 3$, $q_1 = 4$, $q_2 = 6$, $q_3 = 7$

$X_1 = \{1, 2, \ldots, 48\}$, $X_2 = \{1, 2, \ldots, 72\}$, $X_2 = \{1, 2, \ldots, 84\}$

(Action of c given by Tables 10, 12 and 13, respectively)

$\Sigma_1 = \{1, 9, 17\} \cup \{30, 43, 37\}$

$\Sigma_2 = \{12, 13, 25\} \cup \{44, 70, 54\}$

$\Sigma_3 = \{12, 15, 42\} \cup \{49, 62, 80\}$

Here we have written each Σ_i as a union of c-cycles. The map ψ_1 is determined by mapping the elements of Σ_1, in the order given, to those of Σ_2 in the order given and $\psi_2 : C_2 \to C_3$ is defined similarly. Each of the 12 *ap*-cycles in X_i has a single point in C_i $(1 \leq i \leq 3)$. It is thus easy to verify that (X_1, Σ_1), (X_2, Σ_2), and (X_3, Σ_3) are equivalent and that the requirements of Lemma 15 are met. Every $q \geq 7$ which is $\equiv \pm 1 \mod 6$ is a nonegative integer combination of 4, 6 and 7. Thus this case is complete.

$\underline{q \equiv \pm 2 \mod 6:}$

$q_1 = 4$, $q_2 = 10$

$X_1 = \{1, 2, \ldots, 24\}$, $X_2 = \{1, 2, \ldots, 60\}$
(Action of c given by Tables 9 and 15, respectively)
$\Sigma_1 = \{1, 9, 17\}$
$\Sigma_2 = \{19, 27, 42\}$
and ψ is determined by mapping the elements of Σ_1 in the order given to those of Σ_2 in the order given.

Each of the 6 *ap*-cycles of X_i meets C_i in one point$(1 \leq i \leq 2)$, so (X_1, Σ_1) and (X_2, Σ_2) are clearly equivalent, and we may apply Lemma 15.

Every $q \equiv \pm 2 \mod 6$, can be written as a nonegative integer combination of 4 and 10, so the proof of this case, and of the theorem, is complete.□

We will need the following simple result later

Corollary 17 *In the case $q \equiv 0 \mod 6$ of the above theorem, the subgroup $< ap, pc >$ acts transitively on the $\tilde{\Gamma}^0(q)$-set X constructed in the theorem.*

Proof:- Both of the *ap*-cycles of the $\tilde{\Gamma}^0(q)$-set X constructed in the theorem, consist of a union of one *ap*-cycle from each of the constituent sets, these sets being copies either of the $\tilde{\Gamma}^0(12)$-set, X_1 with action of c given by Table 16 or of the $\tilde{\Gamma}^0(18)$-set, X_2 with action of c given by Table 18. In view of this, and the fact that the action of c on X is the disjoint union of the action of c on these sets, it suffices to show that $< ap, pc >$ acts transitively on either X_1 or X_2. But it is easy to show that $< ap, pc >$ acts transitively on both of these sets. In each case the set of odd numbers is one of the *ap*-cycles, and the set of even numbers the other, so it suffices to show that for some x in X_1 or X_2, the parity of x is changed by the action of *pc*, or, equivalently, unchanged by the action of c. Since the c-cycles are of length 3, this clearly happens for at least one member of each c-cycle. □

Remark:- The action of $< ap, pc >$ is not transitive on all the $\tilde{\Gamma}^0(q)$-sets that we have found. For example, an inspection of Table 10 shows that $< ap, pc >$ has two orbits in the associated $\tilde{\Gamma}^0(4)$-set.

We now consider the groups $\Gamma^0(q)$. To obtain a torsion free subgroup of index n of $\Gamma^0(q)$, it suffices to find a semiregular $\tilde{\Gamma}^0(q)$-set X of n elements and then define an action of e on this set in such a way that *ace*, *pce* and e

are each semiregular and of order two. In practice it is easier to replace the generator e by its conjugate $\pi = (ca)^{-1}e(ca)$. We have,

$$\pi c = ac^2ecac = ac^2eac^2a = (aca)(ace)(aca)^{-1}$$

and

$$\pi apa = ac^2ecaapa = ac^2ecpa = ac^2epc^2a = (acp)(pce)(acp)^{-1}$$

so that the permutations effected by π, πc, and πapa are all semiregular of order two. Conversely, given π with these properties we may define $e = (ca)\pi(ca)^{-1}$. The generator π is slightly more convenient to use than e and, as we will show in the next section, it arises naturally in a geometric context.

Given a semiregular $\tilde{\Gamma}^0(q)$-set X with n elements we can construct a transitive semiregular $\Gamma^0(q)$-set Y with $2n$ elements as follows. Let Y comprise two disjoint copies of X and let the actions of a p and c on Y be such that their restrictions to each copy of X are the same as the original actions except that the action of c on the second copy of X is now defined to be the original action of c^{-1}. It is easy to check that this makes Y into a semiregular $\Gamma^0(q)$-set. If we now let π act on Y by taking a point of one copy of X to the corresponding point of the other copy, then it is also easy to see that π now has required properties.

In view of Theorem 16 this means that, for $q \geq 7$, $\Gamma^0(q)$ has a torsion free subgroup of index $2\mathrm{LCM}(12, 2q)$ (i.e. double the lower bound given by Lemma 14). Conder and Martin [CM] have shown that, in at least one case ($q = 9$) this index is the smallest possible, but that in several other cases (viz. $q = 7, 8,$ and 12) the index can attain the lower bound of $\mathrm{LCM}(12, 2q)$. Using a slight elaboration of Lemma 15, we will show that this is usually true at least whenever q is a multiple of 4.

Corollary 18 *Suppose the X_i in Lemma 15 are all $\tilde{\Gamma}^0(q_i)$-sets that can be made into semiregular $\Gamma^0(q_i)$-sets by specifying an action of π. If further, for each i, Σ_i and $p(\Sigma_i)$ are both invariant under the action of $a\pi a$, and the action of $a\pi a$ commmutes with each equivalence map $\psi_i : C_i \to C_{i+1}$, then X can be made into a semiregular $\Gamma^0(q)$-set.*

Proof (Sketch):- We define the action of π on X by taking the disjoint union of its actions on the X_i. The only non-obvious thing to prove is that the permutation on X given by the action of πapa is order two and semiregular. In fact, we will show that the action of πapa on X coincides with the disjoint union of the original actions of πapa on each X_i. If $ax \in X_i \backslash C_i$ then this is also true of pax and hence $a\pi apax$, so, in this case, the required result follows. If $ax \in C_i$ then p takes ax to C_{i+1} or C_{i-1}, but the hypothesis ensures that the action of p then takes $a\pi apax$ back to C_i and it is straightforward to check that $(\pi apa)x = \pi ap*_i ax$.□

We now apply this result to prove,

Theorem 19 *If q is a multiple of* 4 *other than* 4,16,20,36, *or* 48 *then the group $\Gamma^0(q)$ has a torsion-free subgroup of index* $\mathrm{LCM}(12, 2q)$.

Proof:- q a multiple of 12, $q \neq 36, 48$: Let $X_1 = \{1, 2, \ldots, 24\}$, $X_2 = \{1, 2, \ldots, 48\}$ be, respectively, $\tilde{\Gamma}^0(12)$ and $\tilde{\Gamma}^0(24)$ sets with action determined by (30), (31) and the action of c given by Tables 16 and 19, respectively. From these tables one can verify that, on X_1, $c(ap)^6$ is semiregular of order 2, and, on X_2, that $c(ap)^{12}$ has the same property. Hence we can make X_1 and X_2 into semiregular $\Gamma^0(q)$-sets by taking π as, respectively, $(ap)^6$ and $(ap)^{12}$. Define Σ_1 as the union of c-cycles $\{3, 19, 9\} \cup \{21, 7, 15\}$, whence $p(\Sigma_1) = \{10, 20, 4, 22, 8, 16\}$. Each ap-cycle has 6 points in one of these sets and none in the other, so that $m_1 - m_2 = \pm 6$ for each cycle.

Since, in X_1, $a\pi a = \pi = (ap)^6$, and $(ap)^6 x = x + 12 \pmod{24}$, and since Σ_1 is the union of a c-cycle and its image under π, it is easy to see that Σ_1 and $p(\Sigma_1)$ are both invariant under the action of $a\pi a$. Now we define $\Sigma_2 \subseteq X_2$ by $\Sigma_2 = \{14, 30, 42\} \cup \{18, 6, 38\}$ and determine $\psi : C_1 \to C_2$ by mapping the points in Σ_1 in the order given to those in Σ_2 in the order given.

It is now easy to check that ψ preserves cyclic order, and that $a\pi a$ commutes with ψ, so that Corollary 18 can be applied to n pairs, (X_i, Σ_i), each of which is a copy of either (X_1, Σ_1) or (X_2, Σ_2), whenever n is coprime to 6. Since every positive multiple of 12, other than 36 and 48, can be written as $12n_1 + 24n_2$ where $n_1 + n_2$ is coprime to 6, the proof of this case is complete.

$q \equiv \pm 4$ **mod** 12, $q \neq 4, 16, 20$: Let $X_1 = \{1, 2, \ldots, 48\}$, $X_2 = \{1, 2, \ldots, 72\}$ with $\Gamma^0(q)$ action determined by (30), (31) and Tables 14, 17, respectively. On X_1 and X_2 we have $c(ap)^4$ and $c(ap)^6$ respectively, semiregular of order 2, so we can take π to be, respectively, $(ap)^4$ and $(ap)^6$. The result for $q = 8$ follows. For the other cases, we may take, for example, $\Sigma_1 = \{3, 30, 39\} \cup \{47, 22, 11\}$, $\Sigma_2 = \{49, 3, 48\} \cup \{36, 15, 61\}$. and determine

$\psi : C_1 \to C_2$ by mapping the points in Σ_1 in the order given to those in Σ_2 in the order given. For each of the 6 cycles we now have $m_1 - m_2 = \pm 2$ and it is easy to check that (X_1, Σ_1) and (X_2, Σ_2) are equivalent, and that Corollary 18 can be applied to an odd number n of pairs, each of which is a copy of either (X_1, Σ_1) or (X_2, Σ_2). Since every number $m \geq 24$ and equivalent to $\pm 4 \bmod 12$ can be written as $m = 8n_1 + 12n_2$ where $n_1 + n_2$ is odd, the proof is complete.□

6 Construction of Manifolds

We now describe a direct construction of manifolds by glueing together copies of the prism $P(q)$. We begin by glueing 24 prisms together into a regular truncated tetrahedron, and then glue two or more of these tetrahedra together to construct a manifold with boundary. On the way, we obtain a nice geometric interpretation of semiregular group actions of $\tilde{\Gamma}^0(q)$.

Finally we can obtain a proper manifold either by glueing two of these manifolds with boundary together along their boundaries, or, in some cases, by gluing the boundary of one of these manifolds to *itself*.

The initial step— from prisms to tetrahedra— is indicated in Figure 4. Six copies of $P(q)$ are joined as shown, to give a prism-like figure with a flat top and a ridged base. The three ridges meet at vertex v. The link at this vertex (i.e. the intersection with $P(q)$ of a small sphere centred at this vertex) is a spherical equilateral triangle with three angles of $2\pi/3$. Four such triangles can thus be fitted together to make a sphere, and, correspondingly, four of the polyhedra pictured in Figure 4 can be fitted together to make a regular truncated tetrahedron with common dihedral angle $2\pi/q$.

Figure 4. *Glueing of $P(q)$*

We construct manifolds with boundary by glueing two or more of these truncated tetrahedra together along the faces f_i, the faces c_i (those formed by the truncations) then forming the boundary of the manifold. This construction is done quite explicitly by Fujii [Fu] for two tetrahedra in the case $q = 12$. We can describe the glueing operation algebraically as follows:

Suppose k truncated tetrahedra are glued together. Each tetrahedron has 6 edges that bound two of the faces f_i. We define E to be the set of edge-face pairs (e, f), where e is such an edge and $f = f_i$ one of the faces it bounds. This set has $12k$ elements. We define the *adjacency map* a on E as the map which takes (e, f) to (e, f'), where f and f' are the two faces with e as a boundary edge, and the *cycle map* c by $c((e, f)) = (d, f)$, where d is the next edge bounding f anticlockwise from e (see Figure 5). Clearly a and c are semiregular permutations on E, with cycle lengths 2 and 3 respectively. It is also easy to show that ac is semiregular with cycle length 3.

Now suppose that the faces f_i of the tetrahedra are glued together in pairs in such a way as to construct a manifold with boundary. This pairing of faces induces a third permutation p on E. We define $p((e, f)) = (e_2, f_2)$ where the glueing map takes f to f_2 and e to e_2 (Figure 5). Clearly p is semiregular with cycle length 2. This is also true of cp (because the glueing map is orientation reversing). Finally, because each edges must be identified with $q-1$ others (so that the angle sum is 2π), we require ap to be semiregular with cycle length q. We have thus obtained a semiregular action of $\tilde{\Gamma}^0(q)$ on E. Conversely any such action on a set of $12k$ elements can be interpreted geometrically as a glueing of k truncated tetrahedra to give a manifold with boundary.

Figure 5. *Glueing of two Tetrahedra*

Let M be a manifold constructed out of k truncated tetrahedra in this way. We consider some properties of the boundary ∂M of M. Let E' denote the set of edges of the boundary triangles c_i of the k truncated tetrahedra. There is a natural bijection between E and E', obtained by mapping $(e, f') \in E$ to the corresponding boundary edge in the face opposite f'. For example, in Figure 5, the pair (e, f') corresponds to the edge e'.

This bijection induces an obvious group action of a, p and c on E'. If $e \in E'$ is an edge of the boundary triangle t then ce, is the next edge round *clockwise* in t. The glueing of the tetrahedra induces a corresponding glueing of the boundary triangles in which the edge $e \in E'$ is identified with $apae$. In Figure 5, for example, the pair (e, f'), which corresponds to e', is mapped by a, p and a, applied successively, to (e, f), (e_2, f_2), and finally (e_2, f_2'), which corresponds to the edge e_2' adjoining e.

The edges that meet a given vertex v in ∂M are thus, for some $e \in E'$, $e, ce, apace, capace, (apac)^2e, \ldots (apac)^{q-1}e, c(apac)^{q-1}e$, listed anticlockwise around v.

Since c and apa map an edge in E' to, respectively, the next edge clockwise around in the same triangle, and the edge with which it is identified in the neighbouring triangle, it is clear that there is a bijection between components of ∂M and the orbits in E' under the action of $< apa, c >$. In particular, ∂M is connected if and only if $< apa, c >$ acts transitively on E' (or equivalently on E). This condition can be simplified slightly by using a conjugate group in place of $< apa, c >$. We have $(cpa)apac(cpa)^{-1} = ap$ and $(cpa)c(cpa)^{-1} = pcap$. Hence $< ap, pc >=< ap, pcap >$ is conjugate to $< apac, c >=< apa, c >$ so that we have,

Proposition 20 *The boundary ∂M of the manifold with boundary M is connected if and only if $< ap, pc >$ acts transitively on E*

The boundary ∂M comprises $4k$ boundary triangles each of which is equilateral with angles of $2\pi/q$. The boundary, if connected, thus has Euler characteristic $\chi = 4k - 6k + 12k/q = 12k/q - 2k$. When $q = 6g$ $(g > 1)$, we may, by Theorem 16, take $k = g$, and, in this case, Corollary 17 and the above proposition show that ∂M is connected. In this case ∂M has Euler characteristic $2 - 2g$ and genus g. We have thus shown how to construct a hyperbolic 3-manifold with totally geodesic boundary of any given genus of two or more. The volume V_g of this manifold is $24g$ times the volume of the prism $P(6g)$. As $g \to \infty$ we have $V_g/g \to 24\text{Volume}P(\infty) = 3.6638..$ (see Table 2 $\sharp$ 1, recalling that the co-volume listed there is double that of $P(\infty)$). It is known ([KM]) that, for $g = 2$, this construction gives a manifold with totally geodesic boundary of smallest volume, and we conjecture that for $g > 2$ we obtain a least volume manifold with totally geodesic boundary of genus g.

Finally we consider how to obtain a manifold, from the manifold with boundary M. Clearly we can always glue two copies of M together by identifying corresponding points of ∂M, to give a manifold containing ∂M as a totally geodesic surface. In many cases, however, ∂M double covers a non-orientable surface, and we can use the pairwise identification of points of ∂M to glue M to itself. This gives a manifold which contains a non-orientable totally geodesic surface.

The boundary ∂M isometrically double covers a non-orientable surface if and only if there is an isometry $\tilde{\pi}$ on ∂M which is orientation reversing, of order two and having no fixed points. If we also assume that $\tilde{\pi}$ takes triangles to triangles then $\tilde{\pi}$ induces a permutation of E'. Clearly $\pi^2 = 1$. Also $(\pi c)^2 = 1$ because $\tilde{\pi}$ is orientation reversing, and $(\pi apa)^2 = 1$ because $\tilde{\pi}$ preserves edge identification. Moreover each of the permutations π, πc and πapa must be semiregular. For example, if, for some $x \in E'$, we had $\pi cx = x$, then it would follow that π would transpose x and cx and so $\tilde{\pi}$ would fix the vertex $x \cap cx$, contrary to our assumption that it has no fixed points. The permutation πc is thus semiregular of order two, and similar arguments show that the same is true of π and πapa. We have thus found, geometrically, an action of π which has exactly the properties described in the last section.

Conversely, we can show that a permutation π on E' with the above properties induces an order two, orientation reversing, isometry of ∂M. The edges in the triangulation of ∂M consist of identified pairs of edges in E', $\{e, apae\}$. Since π commutes with apa, it determines a well defined permutation of the edges in ∂M. Further, since $(\pi c)^2 = 1$, if three edges bound a face then so do their π-images, so that π induces a unique isometry of the edge skeleton of ∂M. This can then be extended to an order two, orientation reversing, isometry $\tilde{\pi}$ of ∂M. It is easy to check that $\tilde{\pi}$ leaves no edge invariant (which would require either $\pi x = x$ or $\pi x = apax$ for some $x \in E'$), and that it leaves no face invariant (which, since π is orientation reversing, would require some edge to be invariant also). Therefore, to show that $\tilde{\pi}$ has no fixed points, it suffices to show that it fixes no vertices. Suppose, for a contradiction, that $\tilde{\pi}$ does fix some vertex v, then there is an integer m $(0 \le m \le q)$ and $e \in E'$ such that either $\pi e = (apac)^m e$ or $\pi e = c(apac)^m e$. Both ce and $c(apac)^m e$ have v as an endpoint, while $c^2(apac)^m e$ does not, so that in the former case, $\pi ce = c^2 \pi e = c^2(apac)^m e$, whence v cannot be fixed by $\tilde{\pi}$. In the latter case we have $\pi apace = apac^2 \pi e = c(apac)^{m-1} e$, whence, for $y = apace$ we have $\pi y = c(apac)^{m-2} y$. By repeating this argument we eventually arrive at an $x \in E'$ for which either $\pi x = cx$, or $\pi x = capacx$. If $\pi x = cx$ then πc has a fixed point in E', while if $\pi x = capacx$ then, setting $z = cx$, we have $\pi z = apaz$, whence πapa has a fixed point in E'. Hence in either case we have a contradiction. We have thus proved that the isometry induced by π has no fixed point.

It is not difficult to see that this glueing process can be reversed, i.e. given a manifold M containing a non-orientable totally geodesic surface S, we can cut along S to obtain a manifold with boundary. This boundary is not the surface S but its double cover. Consequently the least volume for a manifold with totally geodesic boundary of genus g does not exceed that of a manifold containing a non-orientable totally geodesic surface of genus $g+1$. If, as we have conjectured, the manifolds that we have constructed with totally geodesic boundary of genus g are the smallest such manifolds, then Theorem 19 gives the reverse inequality, at least for even g other than possibly $g = 4$, because there are manifolds with boundary of these genera, which admit the glueing process described above.

We state finally a result, akin to Corollary 4.5 of [CM]. The proof rests on the fact that orbifold volumes form a well ordered set [T, J], and that the least orbifold volume which is the limit of other orbifold volumes is the volume of $P(\infty)$ [A].

Theorem 21 *For every even $g \neq 4$ there is a manifold F_g, containing a non-orientable totally geodesic surface of genus $g+1$, and volume $V_g = 24g\mathrm{Volume}P(6g)$. For sufficiently large g, the automorphism group of F_g has order $24g$.*

Proof:- The only fact that remains to be established is that the automorphism group of F_g has order $24g$, for sufficiently large g. Suppose for a contradiction that there were infinitely many g for which the automorphism group had order greater than $24g$, and hence at least $48g$. We have

$$\frac{V_g}{24g} \to \mathrm{Volume}P(\infty) = 0.30532..$$

so that given such a sequence of g, we can find a strictly increasing sequence of orbifold volumes, with limit not exceeding half the volume of $P(\infty)$, contrary to the fact that $P(\infty)$ is the smallest limit orbifold volume.□

We observe finally that the manifold constructions done here with truncated tetrahedra can be based on other truncated regular polyhedra. Referring to Fig. 2, if the angle along the lower right edge of $P(q)$ is made to be $\pi/4$ rather than $\pi/3$, then the link at v in Fig. 4 is a spherical equilateral triangle with three angles of $\pi/2$. Thus, in this case, eight of the polyhedra pictured can be fitted together to make a regular truncated cube with common dihedral angle π/q. Similarly, if this angle is made to be $\pi/5$, then twenty of the polyhedron pictured in Fig. 4 can be fitted together to make a regular truncated dodecahedron.

7 Tables

The first six tables in this section list information about the families of truncated tetrahedra, classified according to the number, k, of truncations and the number of parameters. The tables exclude the cases k=2 (3 parameters), k=3 (4 parameters) and k=4 (6 parameters), each of which contain only a single family of polyhedra, but are otherwise complete. The remaining tables give the action of c on a number of specific $\tilde{\Gamma}^0(q)$-sets, with actions of p and a given by (30) and (31) respectively.

Table 1 lists the 23 one-parameter families of singly truncated tetrahedra. In these cases we always have $m_{24} = m_{34} = 2$ while m_{14} is a parameter which, for simplicity, we now denote by q. These families are numbered for reference in the left hand column. The table then lists, from left to right, the remaining values of m_{ij}, the number of cusps polyhedra in the family have, the fewest number of generators required to generate $\Gamma^0(\mathbf{m})$, a "?" indicating that the number given is conjectural only. Table 2 lists, for the same families, the mimimum co-volume this group can have (attained when q is also minimized), the co-volume when $q = \infty$, and the equivalent notation used for $\Gamma^0(\mathbf{m})$ in [CM]. For example, looking at the second entry of Table 2, we see that, for any q ≥ 5, $\Gamma^0(2, 4, q, 3, 2, 2)$ is denoted there by $\Gamma_{2,4}(q)$ and that when q=6 the notation $\Gamma_{1,3}(4)$ is also used. No entry in this column means that the group does not appear in [CM].

Table 3 lists the individual singly-truncated tetrahedra. In these cases $m_{23} = 2$, $m_{24} = 3$. The remaining values of m_{ij} and other data are listed as in Tables 1 and 2. None of the groups $\Gamma^0(\mathbf{m})$ from this table appear in [CM] except for $\Gamma^0(4, 4, 3, 2, 3, 3)$ (denoted there by $\Gamma_{8,2}(3)$).

Tables 4-5 : As in Table 1 we denote the single parameter (m_{12}) by q and list its minimum possible value, the co-volume of $\Gamma^0(\mathbf{m})$ attained at this value of q, and the co-volume at $q = \infty$

Table 6: $m_{34} = 2$. The parameters m_{12} and m_{24} are subject to the inequalities (4) and (5)

Table 7: the parameters (m_{12}, m_{13}, m_{23}) are restricted by the inequalities $\alpha_{12} + \alpha_{13} < \pi - \alpha_{14}$, $\alpha_{12} + \alpha_{32} < \pi - \alpha_{24}$ and $\alpha_{13} + \alpha_{23} < \pi - \alpha_{34}$.

$\sharp$	m_{12}	m_{13}	m_{23}	Min. q	Cusps	$\sharp$ Generators
1	2	3	3	7	0	3
2	2	4	3	5	0	3
3	2	5	3	4	0	3
4	3	3	3	4	0	3
5	3	4	3	3	0	3
6	3	5	3	3	0	3
7	4	4	3	3	0	3
8	4	5	3	2	0	3
9	5	5	3	2	0	3
10	2	3	4	7	0	3
11	3	3	4	4	0	4(($3 \mid$ q), 3($3 \nmid$ q)
12	2	3	5	7	0	3
13	3	3	5	4	0	3
14	2	6	3	4	1	3
15	3	6	3	3	1	3
16	4	6	3	2	1	3
17	5	6	3	2	1	3
18	2	4	4	5	1	3(q odd), 4(q even)
19	3	4	4	3	1	4(?)
20	2	3	6	7	1	3
21	6	6	3	3	2	3
22	4	4	4	3	2	4(q even), 4(?)(q odd)
23	3	3	6	4	2	4($3 \mid$ q), 3($3 \nmid$ q)

Table 1: k=1 one parameter:angles, generators, cusps

$\sharp$	min. Co-vol.	Co-vol.,q=∞	Notation in [CM]
1	0.177123	0.305322	$\Gamma_{1,6}(q)$
2	0.331927	0.501921	$\Gamma_{2,4}(q),\ \Gamma_{13}(4)[q=6]$
3	0.425977	0.664654	$\Gamma'_{2,4}(5)[q=4],\ \Gamma_{1,3}(5)[q=6]$
4	0.428851	0.610644	$\Gamma_{1,2}(3)[q=6],\ \Gamma_{3,3}(q)$
5	0.530596	0.807242	$\Gamma_{2,2}(3)[q=4],\ \Gamma_{1,2}(4)[q=6],\ \Gamma'_{3,3}(4)[q=3]$
6	0.727124	0.969976	$\Gamma'_{3,3}(5)[q=3],\ \Gamma_{1,2}(5)[q=6]$
7	0.784653	1.00384	$\Gamma_{5,2}(q),\ \Gamma_{2,2}(4)[q=4]$
8	0.619699	1.16657	$\Gamma_{5,4}(5)[q=2],\ \Gamma_{2,2}(5)[q=4]$
9	0.851954	1.32931	
10	0.325947	0.501921	$\Gamma_{4,6}(q)$
11	0.751588	1.00384	$\Gamma_{7,3}(q),\ \Gamma_{4,3}(3)[q=6]$
12	0.466516	0.664654	$\Gamma_{6,6}(q)$
13	1.04378	1.32931	$\Gamma_{6,2}(3)[q=6],\ \Gamma_{9,3}(q)$
14	0.634339	0.845785	$\Gamma'_{2,4}(6)[q=4],\ \Gamma_{1,3}(6)[q=6]$
15	0.923164	1.15111	$\Gamma_{1,2}(6)[q=6],\ \Gamma'_{3,3}(6)[q=3]$
16	0.845785	1.34771	$\Gamma_{5,4}(6)[q=2],\ \Gamma_{2,2}(6)[q=4]$
17	1.06284	1.51044	
18	0.681684	0.915966	$\Gamma_{4,3}(4)[q=6]$
19	1.03623	1.41789	$\Gamma_{4,2}(4)[q=6],\ \Gamma'_{7,3}(4)[q=3]$
20	0.635623	0.845785	
21	1.52241	1.69157	
22	1.52661	1.83193	
23	1.38783	1.69157	

Table 2: k=1, one parameter: co-volumes, notation

$\sharp$	m_{12}	m_{13}	m_{14}	m_{34}	Cusps	Co-vol.
1	4	5	2	3	0	0.971368
2	5	5	2	3	0	1.2394
3	4	6	2	3	1	1.22084
4	5	6	2	3	1	1.46907
5	5	4	2	4	1	1.499
6	3	4	3	3	1	1.34554
9	4	5	3	3	1	1.91328
10	5	5	3	3	1	2.12568
11	6	6	2	3	2	1.69157
12	6	4	2	4	2	1.76175
13	3	6	3	3	2	1.82054
14	4	6	3	3	2	2.12451
15	5	6	3	3	2	2.33256
16	6	6	3	3	3	2.53735

Table 3: k=1 no parameters

♯	m_{13}	m_{14}	m_{23}	m_{24}	m_{34}	Min. q	Cusps	♯Generators	min. Co-vol.	Co-vol. $q=\infty$
1	3	3	3	3	2	4	0	4	1.50318	2.00768
2	4	3	3	3	2	4	0	4	1.86081	2.31083
3	5	3	3	3	2	4	0	4	2.0872	2.51968
4	4	4	3	3	2	4	0	4	2.16933	2.59109
5	5	4	3	3	2	4	0	4	2.37924	2.79071
6	3	4	4	3	2	3	0	4	1.88694	2.64156
7	3	4	5	3	2	3	0	4	2.15635	2.86313
8	5	5	3	3	2	4	0	4	2.58307	2.98656
9	3	5	5	3	2	3	0	4	2.43411	3.09064
10	2	3	3	2	3	7	0	4	0.978194	1.38261
11	2	3	4	2	3	7	0	4	1.39523	1.72034
12	2	3	5	2	3	7	0	4	1.63438	1.94078
13	2	4	4	2	3	5	0	4	1.58653	2.10198
14	2	4	5	2	3	5	0	4	1.87763	2.34182
15	2	5	5	2	3	4	0	4	1.89671	2.59046
16	2	3	3	2	4	7	0	4	1.31131	1.74096
17	2	3	3	2	5	7	0	4	1.60992	2.05119
18	6	3	3	3	2	4	1	4	2.3008	2.72511
19	4	3	4	3	2	3	1	4	2.07247	2.83577
20	6	4	3	3	2	4	1	4	2.58498	2.99143
21	4	4	4	3	2	3	1	4	2.45412	3.14109

Table 4: k=2 one parameter

♯	m_{13}	m_{14}	m_{23}	m_{24}	m_{34}	Min. q	Cusps	♯Generators	min. Co-vol.	Co-vol. $q=\infty$
22	3	4	6	3	2	3	1	4	2.39003	3.07544
23	6	5	3	3	2	4	1	4	2.78583	3.18535
24	4	5	4	3	2	3	1	4	2.69173	3.35225
25	3	5	6	3	2	3	1	4	2.67227	3.30618
26	2	3	6	2	3	7	1	4	1.85328	2.15165
27	2	4	6	2	3	5	1	4	2.11937	2.56299
28	2	5	6	2	3	4	1	4	2.16543	2.81637
29	2	3	4	2	4	7	1	4	1.94394	2.28991
30	3	2	3	2	3	7	1	4	1.27125	1.69157
31	3	2	3	3	3	7	1	4	1.98307	2.28362
32	3	2	3	4	3	7	1	4	2.28019	2.56422
33	3	2	3	5	3	7	1	4	2.48581	2.76379
34	6	6	3	3	2	4	2	4	2.98711	3.38314
35	4	6	4	3	2	3	2	4	2.91144	3.55921
36	3	6	6	3	2	3	2	4	2.91286	3.5235
37	4	4	4	4	2	3	2	5(?)(q odd), 5(q even)	3.05322	3.66386
38	2	6	6	2	3	4	2	4	2.43735	3.04482
39	2	4	4	2	4	5	2	4(q odd), 5(q even)	2.32775	2.87971
40	2	3	3	2	6	7	2	4	1.95783	2.40539
41	3	2	3	6	3	7	2	4	2.68944	2.96444
42	3	3	3	3	3	4	2	4($3\nmid q$), 5(?)($3\mid q$)	2.77567	3.38314

Table 5: k=2 one parameter (continued)

♯	m_{13}	m_{14}	m_{23}	Cusps	Gen's
1	3	2	3	0	4
2	3	2	4	0	4
3	3	2	5	0	4
4	4	2	3	0	4
5	5	2	3	0	4
6	3	2	6	1	4
7	4	2	4	1	4(q odd), 5 (m_{12}, m_{24}, m_{34} all even), 4 otherwise
8	6	2	3	1	4

Table 6: k=2 two parameters

$\sharp$	m_{14}	m_{24}	m_{34}	Cusps	Gen's
1	2	3	3	0	5
2	2	3	4	0	5
3	2	3	6	1	5
4	2	4	4	1	5, 6(?)
5	3	3	3	1	5, 6(?)

Table 7: k=3 three parameters

x	1	2	3	4	5	6	7	8	9	10	11	12
cx	5	8	12	9	7	2	1	6	11	3	4	10

Table 8: Values of cx: $q = 3$

x	1	2	3	4	5	6	7	8	9	10	11	12
cx	9	18	23	13	16	19	22	12	17	2	7	21
x	13	14	15	16	17	18	19	20	21	22	23	24
cx	24	3	6	20	1	10	15	5	8	11	14	4

Table 9: Values of cx: $q = 4$, index=24

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
cx	9	18	33	28	12	40	19	25	17	2	6	39	35	46	42	21
x	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
cx	1	10	26	8	41	15	31	48	20	7	3	34	38	43	47	24
x	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
cx	27	4	45	14	30	44	5	11	16	22	37	29	13	36	23	32

Table 10: Values of cx: $q = 4$ index=48

x	1	2	3	4	5	6	7	8	9	10	11	12
cx	13	10	7	20	24	15	19	4	1	14	18	21
x	13	14	15	16	17	18	19	20	21	22	23	24
cx	9	2	23	5	12	22	3	8	17	11	6	16

Table 11: Values of cx: $q = 6$

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
cx	19	27	48	16	67	57	52	61	22	37	26	13	25	11	3
x	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
cx	47	59	36	28	2	10	38	50	29	12	14	20	1	49	23
x	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
cx	72	40	45	66	17	60	21	9	31	71	64	56	53	70	65
x	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
cx	34	4	15	24	30	8	62	69	44	41	63	68	5	35	18
x	61	62	63	64	65	66	67	68	69	70	71	72			
cx	51	7	42	55	33	46	58	6	43	54	32	39			

Table 12: Values of cx: $q = 6$ index=72

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14
cx	13	28	26	30	9	56	17	24	55	6	41	15	27	2
x	15	16	17	18	19	20	21	22	23	24	25	26	27	28
cx	42	11	23	8	31	53	44	40	7	18	4	29	1	14
x	29	30	31	32	33	34	35	36	37	38	39	40	41	42
cx	3	25	54	20	57	46	77	74	51	70	21	43	16	12
x	43	44	45	46	47	48	49	50	51	52	53	54	55	56
cx	22	39	33	58	72	65	62	79	69	38	32	19	5	10
x	57	58	59	60	61	62	63	64	65	66	67	68	69	70
cx	45	34	67	75	50	80	84	81	71	47	76	60	37	52
x	71	72	73	74	75	76	77	78	79	80	81	82	83	84
cx	48	66	35	78	68	59	73	36	61	49	83	63	64	82

Table 13: Values of cx: $q = 7$

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
cx	17	14	30	40	25	10	42	45	5	26	47	21	1	18	38	33
x	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
cx	13	2	35	32	48	11	43	28	9	6	23	44	4	39	19	36
x	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
cx	37	15	31	20	16	34	3	29	8	46	27	24	41	7	22	12

Table 14: Values of cx: $q = 8$

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
cx	21	11	56	46	16	32	51	26	36	44	22	1	49	59	6
x	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
cx	31	54	37	27	41	12	2	34	58	7	52	42	20	48	39
x	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
cx	5	15	24	57	10	43	53	17	47	29	28	19	9	35	3
x	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
cx	55	30	40	60	14	25	8	18	38	4	45	23	33	50	13

Table 15: Values of cx: $q = 10$

x	1	2	3	4	5	6	7	8	9	10	11	12
cx	23	17	19	10	14	11	15	22	3	20	13	5
x	13	14	15	16	17	18	19	20	21	22	23	24
cx	6	12	21	8	24	1	9	4	7	16	18	2

Table 16: Values of cx: $q = 12$

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
cx	23	26	48	50	54	51	55	45	40	60	13	38	37	12	61
x	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
cx	35	64	65	34	68	71	27	25	2	1	24	72	21	31	70
x	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
cx	69	30	20	67	62	15	11	14	10	59	57	44	41	58	56
x	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
cx	7	4	49	3	47	53	5	6	52	46	8	43	42	9	39

x	61	62	63	64	65	66	67	68	69	70	71	72
cx	36	16	18	66	63	17	19	33	29	32	28	22

Table 17: Values of cx: $q = 12$ index=72

x	1	2	3	4	5	6	7	8	9	10	11	12
cx	35	7	9	25	33	27	36	1	26	4	19	31
x	13	14	15	16	17	18	19	20	21	22	23	24
cx	15	16	22	21	30	24	32	12	14	13	17	29
x	25	26	27	28	29	30	31	32	33	34	35	36
cx	10	3	34	5	18	23	20	11	28	6	8	2

Table 18: Values of cx: $q = 18$

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
cx	16	46	43	33	17	38	11	24	28	19	23	8	41	30	2	45
x	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
cx	37	6	27	9	40	26	7	12	21	39	10	20	13	42	47	36
x	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
cx	44	3	31	48	5	18	22	25	29	14	34	4	1	15	35	32

Table 19: Values of cx: $q = 24$

Bibliography

[A] C. C. Adams *Limit Volumes of Hyperbolic Three-orbifolds* J. Differential Geom. **34**, (1991) 115-142

[An] E. M. Andreev *On Convex Polyhedra in Lobačevskiĭ Spaces* Math USSR-Sb. **10** No. 3 (1970) 413-440

[B] A. F. Beardon, *The Geometry Of Discrete Groups*, Springer-Verlag 1983

[CM] M. D. E. Conder and G. J. Martin *Cusps, Triangle Groups and Hyperbolic 3-Folds* J. Austral. Math. Soc. Ser. A **55**, (1993) 149-182

[C] H. S. M. Coxeter *Discrete Groups Generated by Reflections* Ann. of Math. **2**, (1934) 588-621

[EEK] A. L. Edmonds, J. H. Ewing, and R. S. Kulkarni *Torsion-free subgroups of Fuchsian groups and Tesselations of Surfaces* Invent. Math. **69**, (1982) 331-346

[F] R. H. Fox *On Fenchel's Conjecture about F-groups* Mat. Tidsskr. B. (1952) 61-65

[Fu] M. Fujii *Hyperbolic 3-Manifolds with Totally Geodesic Boundary which are Decomposed into Hyperbolic Truncated Tetrahedra*, Tokyo J. Math. **13**, (1990) 353-373

[H] C. Hodgson *Deduction of Andreev's Theorem from Rivin's Characterization of Convex Hyperbolic Polyhedra* from Topology 90, Ohio State University Lecture Note Series 185-193, De Gruyter 1990

[Hu] J. E. Humphries *Reflection Groups and Coxeter Groups* Cambridge University Press 1990

[KM] S. Kojima and Y. Miyamoto *The Smallest Hyperbolic 3-Manifolds with Totally Geodesic Boundary* J. Differential Geom. **34**, (1991) 175-192

[JR] K. N. Jones and A. W. Reid *Minimal Index Torsion-free Subgroups of Kleinian Groups* Preprint

[J] T. Jørgensen *On Discrete Groups of Möbius transformations* Amer. J. Math. **98**, (1976) 739-49

[L] F. Lanner *On Complexes with Transitive Groups of Automophisms* Comm. Sém. Math. Univ. Lund. **11**, (1950) 41

[MKS] W. Magnus, A. Karrass, and D. Solitar *Combinatorial Group Theory* Wiley Interscience 1966

[M] T. H. Marshall, Thesis, University of Auckland, 1994

[Ma] B. Maskit, *Kleinian Groups*, Springer-Verlag 1987

[Mi] J. Milnor *Hyperbolic Geometry: The First 150 Years* Bull. Amer. Math. Soc. (N.S.) **6**, (1982) 9-24

[S] A. Selberg *On Discontinuous Groups in Higher-dimensional Spaces*, Tata Institute, Bombay, 1960

[Si] D. Singerman *Subgroups of Fuchsian Groups and Finite Permutation Groups* Bull. London Math. Soc. **2**, (1970) 319-323

[T] W. P. Thurston *The Geometry and Topology of Three-manifolds* (Lecture notes, Princeton University, 1977)

[V] E. B. Vinberg (Ed.) *Geometry II* Encyclopaedia of Mathematical Sciences Vol. 29, Springer-Verlag, 1993