CULMS Newsletter

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Community for Undergraduate Learning in the Mathematical Sciences

The CULMS Newsletter

CULMS is the Community for Undergraduate Learning in the Mathematical Sciences.

This newsletter is for mathematical science providers at universities with a focus on teaching and learning.

Each issue will share local and international knowledge and research as well as provide information about resources and conferences.

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Editorial

This is the first issue of a Newsletter published by the Community for Learning in the Mathematical Sciences (CULMS) of The University of Auckland that focuses on undergraduate teaching and learning. Our intention is to publish the Newsletter twice a year and we welcome relevant submissions from all areas of the Mathematical Sciences.

It is apparent on inspection of the literature that, in comparison with research in school mathematics, there is relatively little research into teaching and learning of mathematical sciences at the undergraduate level, and very few academic journals that focus on publishing such research. In recent years our own research has moved to focus more on university mathematics and we have experienced the difficulty of finding a suitable journal to submit articles to. While this Newsletter is not a journal it is our hope that it will help fill the need to provide a voice for short research articles and opinion pieces on aspects of undergraduate teaching and learning.

There are a number of issues that we believe to be important in this growing field of research into the nature of undergraduate teaching and learning in the mathematical sciences. For example, what theoretical frameworks can inform undergraduate teaching practice? Is the lecture and tutorial model the best teaching approach, or can we improve on it? If so, how? What should be the role of technology in the undergraduate mathematical sciences? Should it be used, and if so what kind of technology and to what extent? Is there a role for CAS calculators, for example, in undergraduate courses? How might course content change when technology is integrated? What should be the relationships between the teaching of the various branches of mathematics, and between mathematics and other mathematical sciences? How important is service teaching of mathematics, and should service courses differ from those for mathematics majors? Do bridging courses have a role to play in the learning of mathematical sciences, and how should they relate to issues in the transition from school to university? Is the traditional written examination still an appropriate way to assess mathematical understanding and proficiency? What are other appropriate ways of assessing mathematical knowledge and thinking? What are the differences and similarities between teaching pure mathematics, statistics, applied mathematics, engineering science. What about finer refinements like discrete mathematics versus analysis? If you have a view on questions such as these, or similar related issues, or have been researching in these fields, then as the editors of the CULMS Newsletter we would be pleased to receive a contribution from you.

In this issue Bill Barton sets out his vision for a new approach to undergraduate teaching of mathematics. He lays down a challenge to undertake an analysis of traditional delivery methods and comes up with new possibilities that he claims are within the economies of scale of universities. The second article by Jamie Sneddon describes a team-based approach that incorporates student team learning into undergraduate teaching. While this has been employed in many subjects around the world it is not often used in mathematics. This University of Auckland trial in just part of a small mathematics class enabled deeper concepts to be explored and the students reported that they found it enjoyable. Another new approach to supplementing teaching in the form of guidance tutoring is described by Terhi Hautala. This peer support method was instigated at the University of Helsinki, and it is claimed that students gained an improved learning experience, were better motivated for study, and formed a more cohesive student body. It is hoped that these benefits will assist in stemming the drop out from mathematics courses. The following article presents Derek Holton's personal perspective on what is involved in mathematics research, and how undergraduate teaching should be truly informed by this. He makes a plea for a move away from techniques for solving 'standard' textbook style questions to the use of more open problems. In this way, he argues, students will not only see what mathematics is really about but will be better prepared for many types of employment. The final article in this issue by Joanna Mamona-Downs and Martin Downs considers an approach to the completeness of the real numbers using infinite decimals. The aim of avoiding an axiomatic approach to this concept is to make links between the symbolic and formal worlds of mathematical thinking (Tall, 2008), and thus ease students' transition into the formal mathematical thinking expected at university.

Reference

Tall, D. O. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal*, *20*(2), 5–24.

Bill Barton and Mike Thomas
Editors

The Pleasure Principle in Undergraduate Mathematics[1]

Bill Barton
The University of Auckland

I wish to live in a world where…

- our yachting population will understand the vector explanation of why you do not sail with the wind directly behind you (this should be demonstrated in secondary school);
- I will not wake up to a headline proclaiming that examination results will tell you how your child's school is performing (journalists should know enough statistics to understand that such a statement is nonsense);
- those who frequent art galleries will see the mathematical as well as artistic beauty in Peter James Smith's paintings (mathematics should be widely valued for its beauty as well as its utility).

Mathematics should be a gate through which we are welcomed to a greater understanding of our world, not a gate that keeps out a significant proportion of the population from participation. This is a problem for all levels of education, but how can we change undergraduate education in particular to make this transformation?

Mathematicians speak like addicts about their subject, deep mathematical knowledge is a source of intense and intimate pleasure. Vaughan Jones (NZ's Fields Medallist):

> I was on a high for months as the result revealed itself. All I wanted to do was think about the maths and any worldly thing was an intrusion.

Marcus du Sautoy (UK's Mathematics communicator):

> I love the buzz of discovering some new eternal truth about the mathematical world. The adrenaline rush of creating a strange symmetrical object never seen before, with interesting new properties, is addictive.

Hauke Groot (a practising roading engineer):

> Finally I got it. It was such a buzz. Partly because I had spent so much time but finally did it. And then I thought. Oh, there is another way to do this, so I worked on, and came to the same solution but using another method, and then a third one. It was so satisfying. But I could not charge my client for the last three hours work, because I already had the solution, I had just got carried away with the

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[1] This article is based on a Public Lecture given at The University of Auckland in April, 2010. It is based on a large body of research, especially on undergraduate mathematics. This is not referenced in the text, but the major sources are listed in the References section.

mathematics.

What would the undergraduate experience look like if it was directed at getting students addicted? Rather than relying on tradition to guide our practice, I propose a research-based design for tempting undergraduates into deep mathematical engagement.

How do you get people addicted? Classical behaviourist theory tells us how: give intermittent, irregular reinforcement. This may be what research in mathematics provides for its practitioners—expanses of frustration, lots of work, and mazes of dead-ends followed by the rare thrill of a breakthrough. But it is not how most mathematics courses are designed. Students either get little or no reinforcement at all, or get it frequently and regularly for not much effort. We need, rather, to design a curriculum so that students experience the struggle and rewards of mathematical endeavour. Such a principle (I call it the Pleasure Principle) should override the usual curricular principle of getting through a stairway of topics.

There are some problems here. For example, how can we start reinforcing behaviour if the behaviour we want to reinforce is struggle? At undergraduate level, this should not be a problem. The students come to us (for better or worse) with a history of struggle. They expect it. What we need to arrange is the appropriate rewards for the appropriate struggles. Also I acknowledge that such a principle may not work for all students. However I believe that is will work for a larger number than currently get stimulated by the undergraduate experience.

The Pedagogical Task

Let us start by thinking carefully about what we would like undergraduate students to learn. We must pay attention both to those who are hoping to become mathematicians of one sort or another, and those who are studying mathematics for other purposes. My list of learning outcomes (for the end of three years undergraduate study) is as follows:

- a broad grasp of the field;
- an appreciation of its power and its applicability;
- yes, some technical skills and familiarity with content;
- significant experience of working in mathematics (including with technology);
- rational thinking (of a special kind, let us call it mathematico-rational thinking);
- perseverance (of a special kind—driven by confidence that rational thinking will work, that I can learn any mathematics that I need, and that “I can do it”); and
- a desire to bring mathematics to bear and to critique its use.

And the work habit outcomes? It is necessary that we develop in students the skills of individual learning and taking responsibility for their own behaviour. They must become self-motivated, they will need to use multiple sources and modes, and, this last is a big change from school learning, they must become self-monitoring.

Now we have the pedagogical task of thinking about how these things may come about. How can we implement the “Pleasure Principle” that encapsulates what has happened to those who love mathematics the most? Tradition, which has been the strongest factor in guiding current practice at universities, has not served us particularly well. Education research tells us quite clearly that lectures are a very poor means of communicating content, or guiding students how to do mathematical problems, or even explaining the intricacies of mathematical concepts. Lecture notes are an inefficient means of recording mathematical knowledge that students need—each lecturer reproducing a new set of notes when multiple versions of mathematical texts exist already in printed and electronic form is a huge waste of valuable time. Tutorials, as places where mathematical communication can take place, and where students can engage in authentic mathematical tasks, are potentially successful learning environments. However, research again tells us that tutorials can get modified (by poor tutoring, by assessments, by task design) to be places where students work on repetitive exercises (rather than mathematical tasks), places where lecture material is repeated, or places where students are rarely questioned in ways that will lead to conceptual learning.

But lectures and texts should not be abandoned. On the contrary, research also tells us that they are very effective, respectively, at some parts of the necessary learning experience. Lectures are a means whereby large numbers of students can experience a mathematician at work, the thinking they do as they struggle with a problem, the delight when something beautiful happens, the way they bring all their knowledge to bear in a creative way. Lectures can model the desired behaviour (and the Pleasure Principle). Lectures can also be effective communicators of a mathematician’s vision of their discipline—a lecturer can enthuse and educate their audience with the power, beauty, utility, width, history, living experience of mathematics. Our students need such stimulation. They do not need it several times a week.

And self-learning students do need coherent and “correct” accounts of mathematics written in forms that are accessible to students at different stages of their mathematical development. I do not believe that there is one such account, but I also do not believe that there needs to be one account per university course.

Tutorials, or some small group organisation, are well-authenticated in general education research as well as specifically mathematics education research as a place where effective learning can occur. The problem is that the economies of scale of a university do not seem to allow such a form of delivery—or do they? I suggest that, provided we properly use technology, and properly set up student expectations and learning habits, then there IS time for students to have close, small group learning experiences as a central part of their course.

A Possible New Delivery Plan

What follows is a delivery plan—one of many possible. It is based on a course that, traditionally, has lecture streams of 200 students each, has traditionally been delivered by three lectures plus one tutorial per week, with assignments, a mid-semester test and an examination. Such a course is standard at The University of Auckland, and is supported by a fulltime "Assistance Room", and some facility for one-on-one tutorial assistance. Students do four courses at a time, so are expected to spend at least ten hours per week on each course.

The new delivery plan described below assumes that students have been already inducted into a high level of self-learning responsibility and new "habits". A first-year course would need to pay attention to developing these characteristics.

The new plan is centred around "Engagement Sessions". This is a place where students experience the authentic struggle, frustration, and reward of mathematical activity. This is where they will learn, in a supportive environment, what it means to think rationally, the deep and connected nature of mathematical concepts, to be creative in their approach to mathematics, to justify and hypothesise and question and prove mathematical conjectures, to be part of a mathematically literate community. The opportunity to develop mathematical discourse, to be challenged, and to operate with personal support are all strongly indicated by research to be both favoured by students, and to be effective learning conditions.

In addition, there would be a single lecture per week, where a "star" lecturer would model the thinking, attitudes, expertise, habits, language, processes of a working mathematician, and/or would talk widely about the subject, its applications, its contemporary developments, its cross-links, its expanse and beauty.

In practical terms, each student would, with ten others, meet with a mathematics lecturer once a week for two hours, plus one lecture. In addition students would be expected to work for four hours per week developing and practicing mathematical skills in a computer laboratory using existing tutorials of various kinds. The final three hours would be in unsupervised peer-group (or individual) sessions working on authentic problems to present and discuss at the engagement session.

But surely that will not be the same lecturer/tutor time commitment as the traditional delivery method. Traditionally there are four 1-hour lectures, and four 2-hour preparation periods, plus eight 1-hour tutorials (200 students in groups of 25)—a total of 20 hours/week. We replace it with a lecture and 20 2-hour engagement sessions. $20 \neq 41$. The answer lies in the demand for tutor assistance and in the assessment.

Currently such a course requires additional tutor assistance of the order of 15 hours per week (either in tutorials or Assistance Room or one-on-one provision). The saving would be equivalent to about 0.25 of a contract lecturer. So there are more lecturers.

Assessment

With respect to assessment, it is interesting that, in my discussions with mathematicians, mathematics educators, and university educators there is a refreshing willingness to rethink much of university education. There is no difficulty talking about, and imagining, university mathematics courses that have much broader aims than at present. It is not difficult to give away content lists, even skill lists, in favour of developing good attitudes towards mathematics, generic rational thinking skills, and a willingness to struggle with problems.

There is also no difficulty talking about alternative modes of delivery: problem-based courses; project courses; changed and reduced roles for lecturing; peer-mentoring; innovative tutorials; self-planned and independent learning; reverse engineered courses where the student task is to write the textbook or set the examination. But when it comes to assessment, the shutters come down. My attempts to get people to contemplate assessment-free courses have all failed, and ended up in accusations of irrational pushing of boundaries.

Now this is curious. Most people display a willingness to critique current practices and talk about alternative assessment, they will discuss assessment distinctions (such as the difference between formative and summative assessment, or the need to assess both skills and understanding). Many heavily criticise final examinations. Furthermore, most discussions included a heavy critique of school assessments, particularly the idea of high-stakes examinations at the end of every year (or even every three years). But no-one will contemplate letting go university assessments that occur three times a year?

Discussion includes "authentic assessment", that is, assessment that mirrors the activities for which the course was a preparation. For example, law students having to argue their case in a mock court, English students being asked to review and edit reports or draft writing, mathematics students having to attack unseen problems in real time.

The appeal for such assessment is also curious because assessment is not part of the authentic experience. Authentic assessment is an oxymoron. A research mathematician does not have a senior researcher standing at his or her shoulder evaluating their attempts to formulate a conjecture or prove it. A writer may have their writing edited, but it is never graded. A lawyer does not get a score at the end of the year for their courtroom performance.

But wait. In the world outside university there is evaluation. We know when we have failed to prove a conjecture (or proved something trivial); we know when our short story is rejected; we know when our client gets sent down. And ultimately our professional lives are successful or not, and this bears consequences. However, here is the key point. This evaluation is ultimately our own. WE know when we have failed; we do not need others to tell us.

So not only is the educational experience particular in its insistence on externalised assessment, but also a severe consequence of authority-provided assessment is that we do not learn how to self evaluate and act sensibly on those evaluations.

The new delivery plan would have considerably reduced lecturer assessment. What it would have, however, is provision for students to monitor their own performance. Students need, and want, feedback, feedback, and more feedback. Much of this would be on-line and self-regulated. The rest would be personal in the engagements sessions. In early courses, self-monitoring student behaviour could be externally moderated by administrators, perhaps. But ultimately students will need to develop their own monitoring habits.

So, for first year, the new delivery plan will have no formal externalised assessment except for a broad grade given by the lecturer who takes their engagement session, and an automated multichoice test. Thus the hours spent by lecturers writing, checking, invigilating, and marking examinations and assignments. This is a significant saving. Is it equivalent to the 240 hours extra spent in student contact (an extra 10hours per week for twelve weeks)? Yes, it probably is: 3 assignments at 4 hours each to set; one test that takes 10 hours to set, 2 to check, 12 to invigilate, 15 to mark; one examination that takes 15 hours to set, 6 to check, 30 to mark and check. Total of 102 hours plus more assignment marking by tutors saved. So we are about 140 hours in excess of the traditional mode, but have an extra 0.25 of a lecturer. No significant difference. (An aside—if there is no examination, then the course could extend for an extra week or two—14% more time for each course).

These figures are based on the situation at The University of Auckland. Averaged across all first and second year undergraduate courses, the number of students is considerably less than 200 per lecture stream, and fewer students means significantly less time in the new plan, but very little change in the traditional plan. The numbers stack up surprisingly well.

Changing Culture

Finally, a word on the practicality of changing cultures.

Today's university students are "strategic" and have competing interests for time. So any assessment-free course may get ignored under the pressure of the demands of other courses. Hence the culture change will require the creation of a personal awareness and urgency of the need for mathematical thinking, experience, and expertise. This is a major difference from the content, goal-directed motivation of traditional courses: you must master these examples to pass this examination.

These are major culture changes. Are they possible? The transition from school to university has been researched as a significant break, one that causes problems for many students. But what if we made use of this discontinuity? The problem might change from one of size of break to re-setting of expectations—so long as we paid attention to informing students of the change, a break with school tradition, the taking on of personal responsibility, the new institutional habits could all be positive.

In my view the more difficult culture change will be that of the institution. Not only will there be different physical needs for learning environments, but also the support services and administrative requirements will change. For example, if a course has broad (or no) grades, how will other systems cope with that?

But most difficult is us. As lecturers, changing the habits of a life-time (those same habits as those of our own lecturers) will be nearly impossible. The challenge lies with us. Can we think outside the cube? Can we break with tradition simply on the basis of rational analysis of experience and educational research? We must.

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Team Based Learning in Undergraduate Mathematics

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In 2009, Judy Paterson introduced Team Based Learning (TBL) to a final year mathematics course (MATHS 302 "Teaching and Learning Mathematics"). Judy and I applied for a Teaching Improvement Grant from the University of Auckland to build TBL into MATHS 326 "Combinatorial Computing", and having been successful, are now in the process of following through. MATHS 326 covers a range of topics in combinatorics, with a focus on graph theory, colourings and block designs.

Team Based Learning is a carefully structured approach to building team learning into undergraduate teaching, for small and large classes (Michaelsen, Fink & Knight, 1997; Michaelsen, 1998). It differentiates between teams (which remain intact throughout a course) and groups (which may not). Its power comes from the high level of cohesiveness that can be developed within effective teams. The TBL approach has been used worldwide in courses from Architecture to Veterinary Clinical Pathology, and just about everything in between. The related TBL website [see references] contains very good videos and documentation explaining the structure and rationale of TBL course design.

There are two main shifts in teaching that occur with TBL. A shift from students *learning* concepts to *using* concepts, and a shift from students being *consumers* of knowledge to being *seekers* of knowledge: they become responsible for initial acquisition of content (through required readings), and work with their team to learn how to use content (in team tasks). Both of these sit well with my teaching philosophy. The TBL approach has advantages that appeal in mathematics teaching. Covering introductory definitions and concepts outside class in preparatory reading makes time for more in depth content in class. More importantly to my goals for the students, the synergy of a team working together on a mathematical problem is a good illustration of how research mathematicians collaborate.

I have not quite incorporated all of the documented TBL approach into the course. Students are not involved in setting the weights of various components of the course[2], and students do not formally rate each other's performance as part of the team. In the future, perhaps intra-team feedback for individuals would be useful; I am taking a wait-and-see approach on this for now. It is unclear whether intra-team feedback on teamwork would encourage or discourage students who under-contribute to teamwork.

[2] Component weightings are set with department approval before semester starts.

The 19 students were separated into three teams of approximately equal ability based on their academic transcripts, which they remained in through the semester[3]. The teams had varied backgrounds with a mix of pure mathematics, computer science and other majors. We discussed in class the difference between group work, which most of the students had experienced before in collaborative tutorials (Oates, 1999), and teamwork, where students remained in the same teams, and worked together throughout the semester. I expected at least a little resistance to fixing teams but there was none. The teams gelled well, with natural leaders taking control. The less well-prepared students interacted in the teamwork, and report than learned from their team-mates.

Each topic in a TBL course starts with a reading to be completed before the topic starts: for instance, I used "Group *Theory in the Bedroom*" (Hayes, 2005) as an accessible and interesting introduction to symmetry groups and permutations. Each reading introduces content that would require more than a lecture to cover traditionally. The first in-class contact for each topic is a Readiness Assurance Test (RAT), split into two parts.

The first part of a Readiness Assurance Test is individual: an iRAT. Students spend fifteen minutes on eight multi-choice questions working individually, and hand in answers.

The second part of a Readiness Assurance Test is for the team: a tRAT. Following this, they have as long as they need (about twenty minutes) to answer the same questions *as a team* on an "Immediate Feedback" form: an iFAT (Figure 1).

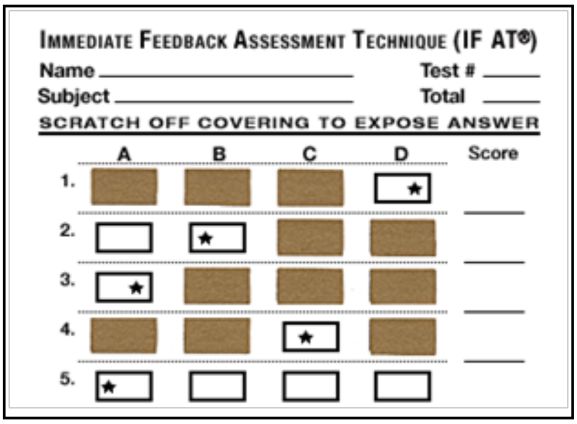

Figure 1. An example of an iFAT scratch sheet, with correct and incorrect team answers.

The team members work together to scratch their answers, revealing a star when they are correct. Finding the correct answer on the first, second or third attempt gains 4, 2 or 1 mark respectively, so a wrong answer is always

[3] TBL suggests relatively large teams of five to seven students, to share intellectual resources as equitably as possible.

followed by more team discussion, and reinforces the correct answer. Each tRAT counts for 0.5% of a student's overall grade (as does each iRAT). The RATs are formative assessments.

A typical question from the first RAT is: "What is the clique number, $\chi(G)$, of a tree?" with options: "(a) 1, (b) 2, (c) 3, (d) it depends"[4]. It combines required knowledge of cliques and trees (concepts covered in the reading), and is short, giving simple possible answers to the question.

Following the readiness assurance process, there is time at the end of the class to go over the key points that have been misunderstood. Marking the iRATs in class gives quick feedback on which concepts students have not understood – which have not always been in the areas I expected. A short explanation of these particular problem concepts ends the class. At this point, all students in the class have had the ideas of the preparatory reading consolidated, and their difficulties addressed.

The readiness assurance process of each topic is followed by a sequence of four or five "normal" lectures on subsequent content in the topic area. Some teaching time has been saved through the readiness assurance process, and now the more sophisticated mathematics can be approached. My goal for these lectures was building knowledge of the topic, which was required for the team to attack a team task set in a related area. Each of the five team-tasks count for 1% of a student's overall grade.

The first team task introduced the class to Vizing's Theorem, which gives a classification of graphs into two classes according to how their edges can be coloured. It turns out it is hard to find graphs which are in 'class 2'. The task asked for examples of various graphs in each class, and to explain *why* it is harder to find 'class 2' graphs. Teams submit answers to the task at the end of the class. Looking for real, accessible mathematics to challenge the teams led me to follow this task with a discussion in the next class of the Perfect Graph Theorem, which was proved in 2002[5]. I had not previously managed to bring such current mathematical research into an undergraduate mathematics course, but focusing on creating a mathematically "real" task led me down this path.

The teams approached the first task energetically, but perhaps a little haphazardly. One team in particular went off together on an interesting but nonetheless unproductive tangent, and struggled to complete the task on time. However there followed some discussion within the team about what went wrong and how they intended to do better next time, and they were much

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[4] The correct answer is "(d) it depends", which was a significant source of discussion within each team, and a point that was covered at the end of the first topic's RAT class.

[5] The Perfect Graph Theorem classifies all of the class 1 ("perfect") graphs of Vizing's Theorem, proving the Strong Perfect Graph Conjecture.

more focused on the remaining tasks. Another team seemed to accept too readily what one member deemed a sufficient answer, and their lack of rigour let them down somewhat. The quieter members of the teams began to find their voices and contribute more vocally to their team's efforts in the first task, and developed a little more as the course progressed.

It was surprising to observe that each team quickly became invested in the individual learning of its members. No team was observed to answer a tRAT question by voting – they always worked toward a consensus through discussion. Furthermore, there appears to be a genuine interest within each team to be sure all members understand an answer before it is scratched in a tRAT. This may be because our mathematics students have been exposed to collaborative tutorials in previous mathematics courses, and that working in fixed teams is not an uncomfortable further step.

Having implemented Team Based Learning in a small class, I am confident it would be possible to implement in a larger class. There is a larger amount of paperwork generated by the regular assessments, but it would not be unmanageable in a class with 100 or so students.

The structure of Team Based Learning energised my MATHS 326 class. It was very enjoyable teaching this way, and I think the students enjoyed learning this way, actively working in teams and achieving well. Although only 7.5% of the overall mark comes from team activities, the class took these assessments seriously and were engaged. The structure made the class more interesting by moving some of the basic definitions and concepts from in-class teaching to pre-class readings, so that deeper concepts could be covered in class.

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Guidance Tutoring — Guidance and Support in the First Steps of Studies

Terhi Tuulia Hautala
University of Helsinki

A new form of peer support was implemented at the Department of Mathematics and Statistics, University of Helsinki, in autumn 2002 to assist first-year students with their studies and to reduce the number of drop-outs. This new peer support system, known as guidance tutoring, has become an essential part of the department's curriculum for the first-year students.

The Starting Point

The Department of Mathematics and Statistics suffered from a massive loss of students at the beginning of their studies in mathematics; a large number of freshmen dropped out after their first semester or never even began their studies at the department. Only a handful of students continued on to advanced studies. Students considered the first year's curriculum too demanding, and lacked guidance and social interaction with other students and the staff. This resulted in poor success in studies and a lack of motivation, which in many cases lead to dropping out.

The developments implemented in the first-year courses, and especially into lectures, were unable to affect all areas of studying at the university, so something else was to be provided for the students outside the first-year courses that would bind them to the department and help them to study more successfully.

Background Theory

Lev Vygotsky's theory of the Zone of Proximal Development (ZPD), and especially the idea of capable peers, was used as a theoretical background for guidance tutoring. The Zone of Proximal Development has usually been applied to children and their learning, but now this theory was applied to the department's first-year students.

Vygotsky stated that learning was adapting skills, knowledge and ways of thinking which have developed in a certain culture. In this respect the Department of Mathematics and Statistics can be seen as a cultural environment to which students must adjust. Vygotsky's essential idea was that to understand an individual's psychological development, one must understand the network of social relations in which the individual develops and lives (Vygotsky, 1981).

Vygotsky formed the idea of the ZPD, describing it as follows:

> The distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers.
> (Vygotsky, 1978, p. 86)

Small group activity and peer tutoring were other bases for guidance tutoring. In addition, group work and peer tutoring have proved effective in helping students in their studies (Beasley 1997; Loos, Menzel & Poparad, 2004). The definition of peer tutoring is as follows: “people from similar social groupings who are not professional teachers helping each other to learn and learning themselves by teaching” (Topping 1996, p. 322). The theoretical advantages of peer tutoring have been linked to Vygotsky’s (1978) theory of the ZPD and to social and cognitive interaction with a capable peer. Thus peer tutoring benefits not only the tutee, but also the tutor. The act of tutoring itself involves further cognitive challenge in simplification and clarification.

Based on the ZPD and the idea of capable peers, development of the idea of a peer support system began. Working in a small group consisting of a student’s peers and a tutor provides the student with an opportunity to achieve better learning results regarding learning skills and mathematics. Group work promotes learning by encouraging discussions and debate, which in turn encourage the justification of ideas, the resolution of disagreements and the understanding of new perspectives (Webb, 1995).

The theory behind guidance tutoring was based on the following thinking: an individual has a certain ability to adapt and to learn new skills. If the individual works in social interaction with his peers, his ability to learn is better than working alone (Vygotsky, 1978). If a tutor, who is on a higher level of development, is added to this group, this ability improves (Topping, 1996).

Among first-year students, the interest and motivation to study was fading rapidly, as was clearly evident in the considerably smaller number of students participating in second-year courses than had participated in first-year courses the previous year. Through the peer support system, groups would work together to find the enthusiasm to study mathematics. One key to cultivating an interest in studies was providing students with experiences of success. At the beginning of studies, understanding new abstract concepts in mathematics was usually hard, and in this system the students were encouraged to participate actively in the learning process as a group. Social interaction significantly enhanced their understanding of new concepts. Peer tutoring gives students a chance to talk through problems they encounter in their learning with another student; they do not always need to do so with a teacher (Goodlad & Hirst, 1989).

The idea of co-operative learning in studying mathematics was also brought to guidance tutoring. The goal of cooperative learning is to commit students to participate actively in the learning process, to guide them to take responsibility for their own and for others' learning, to encourage them to learn together and to share information instead of competing against each other, to develop co-operation, social and problem-solving skills and to improve their self-esteem and motivation to study (Gillies & Ashman, 2003).

The system was also intended to promote learning to learn, so that the step from studying upper-secondary school mathematics to the university level would not be too difficult. Studying mathematics at the university requires possessing different kind of study skills than are usually used at the lower level of education. The student must master skills to make independent decisions and plans concerning his or her studies. Also the nature of university level mathematics makes it essential to develop abstract thinking and comprehension of mathematical concepts. Wright (1982, p. 36) describes learning to learn as follows: "the student has the confidence to develop a learning strategy closely related to why he or she is carrying out a particular learning activity at a particular time and to grasp a wider picture of why he or she is studying at the university and why specifically these courses." Although a student has to take responsibility for his or her actions, a certain amount of guidance in the beginning would do more good than harm. This guidance would provide the student with a picture of how to study mathematics that would make it easier for him or her to be in command of his or her own learning (Wright, 1982).

The Goals

Through guidance tutoring, our group aimed to achieve seven different goals. These goals were, firstly, that the small group activity would provide a place where tutors can observe students' learning; secondly, that these groups would serve as discussion platforms; and thirdly, that they would support learning to learn. The fourth goal was to encourage students to work together. The fifth goal was to encourage discussion on how to make the most of all different study situations the department has to offer and the sixth was to guide the students to find the motivation to study. The last goal was to have the tutors serve as contact people between the students and the staff.

These goals are essential regarding studying mathematics. The new students were not usually familiar with discussing mathematics with one other. They did not realize how different it is to study mathematics at a university level than in school and that they would have pay more attention to how and why they study.

The Recruitment and Training of Guidance Tutors

The tutors are selected through an application process and evaluated based on their social skills and whether they had the right conception of guidance tutoring and how motivated they were to become guidance tutors or to study their major. The number of guidance tutors has ranged from 10 to 18. The selected tutors usually had experience in small group activity. They may have been student tutors who guide first-year students for the first couple of weeks of their studies. Many such tutors study to become teachers, so some of them have also had some educational training. In addition, they usually have hobbies that relate to small group activities, Scout Association, for example, or sport teams.

The department provides training for tutors in guiding a small group before the autumn semester and the beginning of the tutoring sessions. These training sessions consist of discussions on the different roles in a group and how a band of students becomes a group. The training sessions are usually discussions; the selected tutors discuss the goals of guidance tutoring and the session topics for each semester together with the co-ordinator. The tutors are paired up and they discuss what expectations they have for the tutoring year, and the group of tutors use brainstorming to generate new ideas for guidance tutoring.

Tutoring Sessions

There are various weekly themes for tutoring sessions. The themes are selected so that they will support the students as well as possible and the timing of the themes is important, for example a lot of attention is focused on social interaction and developing study skills in the beginning of the academic year and later more attention is targeted on planning studies.

The motivation to study is found by granting students the feeling of success and by strengthening their own opinions about their own skills. This can be done by creating situations where the students can feel that they have achieved something (e.g. comprehension of a new concept in mathematics).

What and how to study is a very important topic of the tutoring session discussions. For most of the students, studying at the university is new and the study methods they have are usually from upper-secondary schools. In the sessions, the students discuss how studying at the university differs from studying at the upper-secondary level. Usually, the students find that much more reading and understanding and less mechanical counting must take place at the university level. The students notice that comprehension is the key to success, not learning by rote. Advice on which courses to take and when to take them is also needed so that the students will construct their mathematical knowledge wisely.

The Effects

A questionnaire was sent to all of the students at the department about their experiences with guidance tutoring and what effects it has had on their own studies in spring 2007. Approximately 80 per cent of the students who answered the questionnaire said that participation in guidance tutoring had positively affected their studies. Further, a substantial number of students stated that guidance tutoring had a beneficial impact on their social relationships. In particular, the development of their social network within the department was considered important.

Guidance tutoring is not a study group, although the groups sometimes discuss the content of first-year courses. Rather, guidance tutoring combines student tutoring and study group activities, bringing out forms of peer tutoring and small group action. Guidance tutoring is an easy way to get to know other students, especially for those students who are not naturally so active in creating their own social network at the department. The survey conducted for the students who had participated in guidance tutoring highlighted the effect of guidance tutoring on social relations and further the impact it had on motivation to study. Here is a quote from the survey: “Originally, I was not planning on staying at the department at all. Had it not been for those social contacts made in my guidance tutor group, I would not be studying mathematics anymore.”

Guidance tutoring had a clear influence on how the students’ studies went during the first year, and this was reflected throughout their studies. The study techniques and skills learned in the groups remained in use after the first year. The students had also found that the interaction between them and the staff had increased and that the atmosphere at the department had become more student-focused and open. Students have become more active, and they participate more in the development of teaching at the department. They have also learned to plan their studies, so the road from freshman to Bachelor of Science and onwards to Master of Science has become clearer.

All the measures implemented to improve first-year studies at our department have affected how the students get on with their studies. The developments made in first-year courses (Oikkonen, 2009) have yielded better learning experiences and consequently, have motivated the students to study. The effects that guidance tutoring has had on first-year students add to these effects; guidance tutoring emphasises social interaction in learning mathematics, developing study skills and becoming an active member of the scientific community.

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Mathematics: What? Why? How?

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In this article I want to say what I think mathematics is and why, and even how, it might be presented at university level.

Mathematics: What? (Structure)

Anybody who has reached this point before would generally reach for the nearest dictionary and quote a definition of mathematics. But I'm too lazy to even look up the word on the web. Besides it wouldn't help. The result would be, at best, a one-dimensional view of something that is infinitely dimensional. Sorry to go as low as to paraphrase the Sound of Music, but "How do you find a word that means Maria?"

Let's start off slowly though. I guess that mathematics is a way of looking at the world and trying to sort out some of its problems. It seems to have presently accumulated an enormous number of facts, ideas and theorems. But, despite the way that maths is still taught in almost every classroom in the world, it is more than a collection of algorithms that, for some unknown reason appear to have to be known. There is also a creative side to the subject – a place where new maths suddenly appears, sometimes even miraculously. And the thing that we have tried to suppress from the general public is that it is created by human intervention.

In my own career[6] it seemed to me that there was a rough structure around the creation of new ideas and results. Writing this down in some ways doesn't help because any structure that can be put on a page will be inadequate for the grand task it is written down for. Nevertheless, it is the only way I can think of to communicate so here is my first attempt at a structure for the creative side of mathematics.

What you see in Figure 1 are the key points of a process. I'll try to weave these points together and show how they interact. But right at the start it is necessary to say that their interactions are in no way linear and often appear to be quite random.

To me there is no mathematics without a problem. Mathematics exists to solve problems. I don't see how you can do maths unless you are working on a problem. I also want to say that by problem, here, I mean something that it is not immediately obvious how to solve. OK, you may have some thoughts on the matter, some area of maths to look in for a solution, even some

[6] It should also be noted that there are many places where I have used the first person here when in fact I was working with a research group.

theorems that you might employ in the solution, but you can't just sit down and write out a solution. These are not exercises that we are dealing with where the method is known, these are genuine problems. Clearly if they are research problems, then nobody yet knows how to solve them.

Problem

Experiment

Conjecture

Proof Counter-argument Give up

Generalise / Extend For now/ For ever

Figure 1. An attempt to describe the structure of the creative process in mathematics.

A common combinatorial approach to solving problems is to experiment with the concept in hand. This may involve using a computer to generate examples that can be counted or examined in some way. It may involve generating examples by hand. But the aim of the exercise is to get to know the problem you are dealing with in more detail so that you can form some idea as to how a whole class of objects might behave. And when you have that idea you have a conjecture as to what might be going on.

If you are lucky the conjecture will be true and you can prove that it is true. But you won't know this beforehand, so it may well be false. In this indecisive position you will try to prove it but a proof may not come either because you there isn't one or you don't have the mathematical machinery to prove it. So you are caught in a wasteland where you alternate between looking for a proof and looking for a counterargument or a counterexample. Hopefully, any difficulty with a proof will lead to an idea for a counterexample and any difficulty in constructing a counterexample will give more fuel for the construction of a proof.

When you get a proof, you have a theorem and life looks very good. For me this was always a high point of the process. It was as if I had been holding my breath for weeks and finally I could breathe again. All the dead ends and frustrations along the way were over and forgotten and a goal had been achieved. However, a theorem only led to more work and in some ways a return to the start because it was only here that I could really contemplate generalisations and extensions. At this point it was essentially back to the beginning to see what bigger and deeper results could be found.

Of course, theorems take time and despite the best efforts, didn't always come. Indeed they often didn't come. I think that on balance that was good. I'm not sure that the whole business would have been as much fun as it was if I could have walked in to the office every morning and produced a theorem before lunch. It's comforting to know that I'm not the only mathematician who has had to give up. Like others before me I have given up forever on all the conjectures that were floating in my mathematical air. But the same is true for Fermat and Goldbach, and so on, and so on. But in Figure 1 I have allowed for giving up for now. That has always been a very useful technique. It's amazing how often, after banging my head against a mathematical brick wall for hours, turning off, even having to go to a meeting, provides some sort of distance from a problem, and while I'm trying to be professorial at my meeting, part of my brain, independent of my conscious efforts, seems to be gnawing at the problem. The Eureka feeling when it comes, the new idea seemingly out of nowhere, is a pretty good experience. And it's one that isn't only open to the fabled Archimedes at bath time.

Maybe this background brainwork is also responsible for the strange effect of the changing proof. When writing up a proof for publication it does sometimes happen that you discover a slip along the way and suddenly the proof evaporates. So the elation in getting a proof can be short lived. On the other hand, sometimes things get better rather than worse. It's amazing how often the original proof gives way to something that is much more elegant when it hits the journal pages. Sometimes this happens because of the intervention of a good referee but often a new approach just seems to present itself.

Mathematics: What? (Mathematics and Metacognition)

But there is much more to research than knowing the structure of the process that might lead to a result. In fact, despite the time I've spent over it, the process is not something that I think I was aware of in the heat of battle. It was just the normal process that I naturally went through. And although it looks as if there might be a reasonably linear route through the points of the structure, nothing could be far from the truth. I would scurry around from point to point where the problem led me. Experimentation, for example, might be employed at any stage; it was more than a guide to a conjecture. For example, if a proof wasn't working experimentation might help see what to try next. However, the goal is always clear. What you hope to do is to come out the end with at least a theorem. So this is where you are always heading, despite the twists and turns along the way.

What else is there? Clearly there are all the known results of mathematics to draw on. The more of these you know, the more likely you are to come out at the end with something new. But maybe metacognition, thinking about your thinking, isn't so obviously a research partner. How do you muster your resources? How might the conditions of Blogg's Theorem be marshalled in the present situation to move you forward? How do you recognise that

Blogg's Theorem is the one that you might want? Is this the time to make a list? Is this approach any good? Maybe now is the time to abandon it and move in a different direction. If I could prove "that" would that lead to "that" and then give me "that"? Strategy, as well as process and mathematical knowledge, is constantly being applied. (See Figure 2.)

Metacognition Structure Known

Results

Figure 2. The 'mess' of research.

In cricket, people talk about 'being in the zone'. I think that this comes to a batsman after they have got set and are seeing the ball 'like a balloon'. In some sense then, they probably subsume control of the bat to the subconscious and play instinctively rather than consciously. Presumably at this stage they are totally focussed on the bowler and the ball.

I suspect that there are 'zones' too for mathematicians. Dunedin is not the warmest place in winter and we lived in a big old house that could have had more heating. I can remember 'waking up' many evenings after 3 or more hours at my desk, suddenly realising that I was cold. I hadn't been asleep; I had been deep into a problem. I was so focussed on what I was doing that I had totally neglected small matters of temperature. For me, at least, this submersion in a difficult problem was an important step towards its solution.

Mathematics: Why?

The question that then has to be asked about undergraduate mathematics is that if the subject progresses because researchers know both the results of the subject and the processes for its creation, why, in our teaching, do we spend so much time on only one of these aspects, the results, and very little time on the other? Why do we spend so much time on the regurgitation of the solution of specific types of problem rather than spending time on promoting the solving of problems? Of course, students need to know some mathematical results, you can't do any maths without them, but why have we been so slow to help them to develop in what is an important aspect of the subject overall – the practice of mathematics?

I assume that the normal reaction to this is that we get our students to solve problems all the time. It's the basis of all of our courses. But it seems to me that what we do is we present a range of standard problems and their solutions and are happy when students are able to replicate these solutions. We shy away from setting questions that are, in some sense 'open', and we avoid 'natural questions' while we move down set paths through traditional courses.

I think that we need to make more effort to introduce this second aspect of the subject for four reasons. First, solving problems is a fundamental part of mathematics. It is the motivation for the very existence of mathematics. So students should know about this side and that solving problems is not just about solving problems that everyone knows how to solve. Second, I know from experience with bright secondary students that they really enjoy that experience. This is good for them and their attitude towards the subject and good for the public face of maths. Third, when students leave us and go into careers they will need to solve problems, sometimes on a daily basis. Giving them one more differential equation in lectures isn't necessarily the best preparation for attacking new problems. Fourth, in some sense, tackling problems is a more human side of mathematics than the cold set of theorems and algorithms that are often presented, and presented as things to be learned.

I didn't really understand what mathematics was about until I was in the middle of my masters thesis. Suddenly it all made sense. Why was the 'secret' kept for so long? Others are on record as having realised what was going on later in their studies. Why do we hide the best, most interesting aspect of the subject from public gaze?

Mathematics: How?

Over the years I have tried to widen my approach to undergraduate teaching so that it went further than the presentation of techniques but I don't claim to have found perfection. Perhaps my best attempt was for the few years when I was responsible for pre-service primary students. There was a one-hour lecture every week but we got closest to mathematics in the three-hour 'workshop'. This enabled me to ask problems that developed the students feeling for basic number. In the workshops they were doing mathematics and hopefully understanding the number that they would soon teach.

In graph theory I always started with a few lectures that explored the difficult question: How many graphs are there with one vertex? After defining a graph as a set of dots and lines we looked at how many graphs we could get with various numbers of vertices. First, this was because these were natural questions to ask. Second, because the students went straight into solving problems and asking questions (such as "Can we have loops?"). Third, because by doing this they developed some intuition for the subject and started to collect some basic graphs to test ideas on. And fourth, because many of the basic ideas of graph theory simply fell out from what the students did. Things like degree of a vertex, connectivity, isomorphism and cycles just naturally needed to be considered on the way and led to the rest of the course.

But every subject in mathematics originally developed from scratch, even calculus. So there is no reason why similar approaches couldn't be taken with every subject.

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The Decimal System as a Topic in Transition From School to University

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At school, the model of the whole real number system is based on infinite decimals. However, the question "what is an infinite decimal exactly?" is usually avoided. At university, the real numbers are defined axiomatically. Thus a proper investigation of the infinite decimals is 'squeezed out' between the two institutional levels. Much educational research has been devoted to the perceived transition between school and university mathematics, but most is directed to general considerations. This communication aims to outline some of the mathematical options that exist to reinforce the understanding of infinite decimals, and how this understanding could fit in with an axiomatic approach. In particular, the concept of completeness of the real numbers is discussed. Hence we are treating a particular, but major, theme concerning the transition from school to university.

In the course of teaching numbers through the school years, students first learn natural numbers, with their usual operations. The operation of 'minus' eventually raises the need for the number 'zero' and negative integers. The operation of division ultimately leads to the idea of fractions. Fractions represent numbers, but only equivalence classes of fractions can be regarded as numbers. Often a particular representative is taken as a number, i.e. the fraction 'in its lowest terms'. Now the numbers that we have introduced up to now function perfectly as a self-contained body in the sense that the operations brought in are closed. But cognitively there seems a psychological demand that any line segment should have a length recognised as a number. As the length of the hypotenuse of a right triangle where the other two sides have length 1 cannot be expressed as a fraction, it is deemed that there must be other numbers apart from those that can be represented as a fraction. But what characterises these 'extra' numbers?

This is a question that is rarely raised seriously in school teaching. However, simultaneous with the development of learning about different kinds of numbers, a representation system is refined, namely the decimals. A digit scheme using a specific basis is a particularly efficient way to symbolize individual integers, and to describe operations implemented through certain algorithms. These algorithms typically are taught at primary school. Of course the 'default' basis taken in modern times is 'ten' (though for certain areas of mathematics, especially those related to programming and coding, it is more natural to adopt the binary system.) In fact, the retention of one particular basis can raise problems for students later when they meet other bases, but on the other hand this situation provides students an opportunity to obtain a deeper understanding on what the decimals involve. The introduction of the notion of 'negative integral powers' means that decimals can be extended from the integers to those 'numbers' that have a 'fractional part'. The integral part and the fractional part are separated by the decimal point; both parts are expressed by a finite string of digits. This is just a description of the symbolism $a_n a_{n-1} \ldots a_1 a_0 \bullet d_1 d_2 \ldots d_m$ denoting

$$a_n \text{x} 10^n + a_{n-1}\text{x}10^{n-1} + \ldots + a_1\text{x}10^1 + a_0\text{x}10^0 + d_1\text{x}10^{-1} + d_2\text{x}10^{-2} + \ldots + d_m\text{x}10^{-m}$$

where each of a_i and d_i are integers from zero to nine. Such finite decimals admit natural extension of the four fundamental operations expected for numbers. But the set of finite decimals has a 'defect'; it is not closed under division. For instance, $\frac{1}{3}$ cannot be expressed as a finite decimal.

Effecting the Division Algorithm n times for any natural number n, the fraction $\frac{1}{3}$ can be written as:

$$\underbrace{0.33\ldots3}_{\text{n digits}} + \frac{1}{3} \times 10^{-n}$$

Applying the Division Algorithm once more adds a further digit to the string of 3's already 'produced' and reduces the 'recompensing' term by a factor of ten. It is then rather conducive to assign to $\frac{1}{3}$ the new symbolism 0.3… where the dots suggest an infinite string of successive digits all taking the value of 3. It is important to realize that at this stage that 0.3… is just another way to say $\frac{1}{3}$; it does not have as yet an autonomous definition.

A similar situation occurs for all fractions (in lowest terms); on applying the Division Algorithm, either the fraction is 'converted' to a finite decimal or the digits eventually are repeated in successive 'blocks'. The reason for this is straightforward. Suppose $\frac{m}{n}$ is a fraction in its lowest terms, and does not have a finite decimal representation. Carrying out a 'new' step in the Division Algorithm applied on $\frac{m}{n}$ depends only on the remainder of the previous step. The number of values 'available' for the remainder is n–1. Hence before n digits are 'produced', a value for the remainders must be repeated, at which point a cycling behaviour must start in the digits. The resultant repeated block of digits is usually called the period (of the decimal representation of $\frac{m}{n}$), and the number of digits in the period as the length of the period. Applying the Division Algorithm indefinitely, it seems appropriate to give the fraction an alternative symbolism as an 'infinite' decimal:

$$a_n a_{n-1} \cdots a_1 a_0 \bullet d_1 .. \overline{d_i \cdots d_j}$$

where the digits that are over-lined indicate the period. Then the length of the period, say p, equals $j–i+1$. Then the expression above is given more explicitly by:

for any natural number $m \geq i$, $d_m = d_{((m-i)\bmod p)+i}$

Finite decimals evidently could be rewritten as 'infinite' decimals with a period of one digit of value zero, so all fractions (in their lowest terms) can be expressed as 'infinite' decimals. Taking this convention, the word 'infinite' in the term 'infinite decimals' is now redundant, so is omitted henceforth. A convincing simple argument suggests that, conversely, any decimal possessing a period represents a fraction. However, fractions (in their lowest terms) are not quite in a one-to-one correspondence to the set of periodic decimals; a finite decimal represents the same fraction as another decimal with a period of one digit of value 9. For example, the question whether $0.\overline{9} = 1.\overline{0}$ is resolved affirmatively simply by the fact that both sides represent the same fraction $\frac{1}{1}$. (If the period is not of the form of one digit of value 0 or 9, a periodic decimal is unique in representing a fraction.)

Showing that fractions are periodic decimals and vice-versa involves a mix of convention and fact. The fact is that as many digits as wished can be produced by the Division Algorithm, and so the period can be identified (in principle). The convention is that this information can be signified in the form of a decimal. The decimal representation is suggestive of a mental interpretation that the period is repeated indefinitely, despite the fact that there is only concrete meaning in finite repetition. This raises the issue of 'potential infinity' (against 'actual infinity'), an issue much elaborated on by researchers in the philosophy of mathematics and by mathematics educators (e.g., Moore, 1999; Dubinsky, Weller, McDonald and Brown, 2005).

When periodic decimals are first introduced, students ideally will have the curiosity to ask some questions. Is there any point in wondering whether you can give meaning to decimals that are not periodic? Should these be regarded as numbers also? If so, do decimals (including the periodic decimals) determine all numbers, and on what grounds would you make this judgment? If a decimal is not just an alternative symbolism for fractions, what defines it?

Again, these questions are rarely put to students. The usual teaching sequence in school is to decree that the set of decimals, or in other words the decimal system, constitutes the whole body of the real numbers. The numbers that are represented by periodic decimals are called rational numbers, to distinguish them from those that are not, which are called irrational numbers. The system of decimals becomes, in the class environment, the predominate model of the real numbers.

Despite this, typically at school little is done with this system. Decimals that are not periodic stay as vague entities. Many students learn by heart that $\pi = 3.14159\ldots$, with little understanding what the three dots signify, whereas the fact that π is irrational is rarely known or retained. True, a proof of this fact is beyond normal school work, but even the issue as a question is not raised. The activities done are usually restricted to exercises in converting fractions into periodic decimals and vice-versa, leaving the decimals corresponding to the irrationals virtually unexamined.

If the decimal system is not properly covered at school, the same holds at university where an axiomatic approach to the reals dominates. The topic usually is squeezed out of the curricula between the two levels. For example, Gardiner (1982) states that:

> But through the long division process, which we use to transform an ordinary fraction into a decimal fraction, frequently gives rise to infinite decimals, little if any time or effort is devoted either to investing these curious entities with ordinary meaning, or to clarifying the mathematical idea which justifies their representation as never ending decimals. Experience suggests that many undergraduates complete their studies of sequences, series, and limits in the calculus without ever realizing the light they shed on infinite decimals. (ibid, p. 70)

This is unfortunate, because a mature treatment of the decimals can be a good vehicle to motivate the axioms of the reals, especially in premeditating the axiom of completeness. Quite often the terms 'Calculus' and 'Real Analysis' are distinguished, the first conveying an intuitive understanding of what the real numbers are and from this what can be assumed about real functions, the second conveying a strictly axiomatic prescription of the reals where all properties of functions have to be proved however 'obvious' they may seem. It is often observed by educators that the transition between Calculus and Real Analysis is difficult for many students (e.g., Mamona-Downs, 2008). In this respect, it is reasonable to propose that teaching directed to an appreciation of the nature of infinite decimals should be of help. This proposition could be implemented within several different institutional settings; either as an item to insert in the school curriculum, as part of a 'primer' course at the start of students' undergraduate career, as background reading whilst taking a (first) undergraduate course in Real Analysis, or as a theme addressed in teaching training. (The latter means that schoolteachers of mathematics will be equipped with a reasonable background knowledge concerning decimals, which hopefully would be transmitted at class).

The various settings might well affect the style of instruction, but a common aim certainly would be to 'legitimize' decimals as 'bona-fide' numbers, and having done this to explain how the ordinary elementary operations (add, subtract, multiply, divide) can be applied in the context of decimals with an infinite number of digits.

A status for periodic decimals has already been established; they represent the rationals with fixed procedures of conversion into fractions, and algebraic operations are permitted through our knowledge of the arithmetic of fractions. However, in order to have an integrated formulation of the real system, decimals that are periodic should be treated exactly in the same way as decimals that are not. Thus the special status of periodic decimals has to be temporarily suspended; once we have established the basis of the whole real system, it can be re-captured.

This does not mean that our experience with periodic decimals, especially finite decimals, will not influence our thinking concerning all decimals. Intuitively, a decimal is an infinite string of digits. For the periodic decimals, there is a process to generate as many digits as wished. This promotes a potential image of an infinite process. Non-periodic decimals, as a body distinguishing irrationals from the rationals, are not imagined in this way; for any randomly chosen irrational, there is an (unknown) allocation of a digit for each decimal place (without any sense of generation from previous digits). The mental image now is more allied to 'actual' infinity. However, it is natural to consider that a decimal is determined by its digits. So if a decimal is represented by $N.d_1d_2\ldots$, where N is an integer, and the d_i's undetermined, we can image the sequence of finite decimals $N.d_1$, $N.d_1d_2$, $N.d_1d_2d_3$, … as representing the decimal. Then a principle could be decided on that says such a sequence always converges to a number. Imposing such a 'principle' might seem to the student just as an act of assuming true what is desired; what justifies this? To counter this we might say that we have a preconceived idea that the number system exists but it is not as yet firmly determined; the principle acts in a way to finally seal a definitional basis for the reals. Having said this, the word 'principle' also has connotations of transparency that perhaps are lacking in this case. As sequences and their limiting behaviour are relatively sophisticated constructs and the assumption that decimals always converge might interfere with students' working practices and beliefs concerning limiting behaviour of real functions, this indeed might not be the most convincing way to justify decimals as being numbers. It is more natural to regard the statement 'a bounded increasing sequence converges' as expressing a property that has to be proved rather than expressing a tenet.

Thus we aim to find an alternative principle that might be accepted more readily. We associate $N.d_1d_2\ldots$ with a set of intervals $I_1 := (N.d_1, N.d_1 + 10^{-1})$, …, $I_i := (N.d_1\ldots d_i, N.d_1\ldots d_i + 10^{-i})$, … where an interval (a, b) indicates all the (putative) numbers between the rationals a and b. Clearly for $k>j$, $I_k \subset I_j$; for any natural n, $\bigcap_{i=1}^{n} I_i = I_n$, and the length of I_{n+1} is the tenth of that of I_n. If we take the 'infinite' intersection and compare it with the finite situation, we can imagine an infinite process for which the successive intervals collapse into a single element. The existence of this element of the infinite intersection, though, cannot be proved (unless the completeness property of the reals is defined in another way). It has to be regarded as a postulate, which is convincing enough to expect a consensus acceptance. Thus in this model ascribing a decimal as a number, the 'number' $N.d_1.d_2\ldots$ is a symbolism for the number arising from a particular infinite intersection.

But there are other options to interpret a decimal. For motivation let us take a particular number that we already know is irrational, say $2^{1/2}$. This, then, is represented as a decimal that does not have a period. Consider the set of all rationals that are less than $2^{1/2}$. By supposition, $2^{1/2}$ is an upper bound of this set. Now the decimal for $2^{1/2}$ 'cut off' at the $(i+1)^{\text{th.}}$ decimal place would be greater than $2^{1/2}–10^{-i}$ however large the natural i is taken; this means that $2^{1/2}$ is the least upper bound. This particular case could encourage us to formulate the whole number system in terms of the least upper bound of sets of rationals. However, we would have to invoke yet another principle; that for all bounded (non-empty) sets (of rationals) a least upper bound does exist.

Above we have mentioned three ways to give an expression for the numbers that 'lie beyond' the rationals. Each way depends on adopting a central principle that just has to be accepted. When you compare the three, it should be evident that they are all consonant to the same idea; hence they are collectively identified as different expressions of the same concept, i.e., completeness. There are other candidates beyond the three mentioned (see e.g. Artmann, 1988). (In particular, the idea of Dedekind cuts has obvious cognitive links with the dictate that least upper bounds exist, but conceptually it fits more comfortably with the image of the number line rather that the environment of the decimals.) However, each one has a form that is significantly different in character from the others. For this reason, perhaps it is only feasible to teach one model at school; if various ways of interpreting the decimals are introduced, the teacher would have the resultant task of reconciling one version to the others. In this respect, probably the version of completeness that would be most readily accepted by students is the one concerning nested intervals (and so this would be the most suitable to teach).

It is natural to regard the convergence of increasing, bounded above sequences as a theorem rather than a tenet; the existence of least upper bounds and Dedekind cuts bring in a set theoretical perspective where the role of finite decimals subsequently becomes obsolete.

However, there is a problem in obtaining models of the real numbers based on the decimals vis-à-vis a strict axiomatic approach. At school, numbers are taught in stages. In particular, only from having an understanding of the rationals do you build up the whole real system. The axiomatic approach taken at university, though, reverses this process; the axioms 'determine' the numbers as a unified body from 'the scratch', and special categories of numbers such as the rationals must be identified a-posteriori. This means that the expression of the axiom of completeness should not depend on a previous knowledge of rationals. This would force reformulations to our models of completeness stated before. In particular, removing the pre-supposition that rationals exist a-priori affects the assumption that the nested intervals corresponding to a decimal would lead to a single 'point'; it could possibly be an interval. As in passing from one interval to the next, the length of the interval would decrease by a factor of 1/10, although this possibility can be dismissed if you assume the Archimedean Property. Thus an expression for the completeness of the reals concerning the principle of nested sets has to be accompanied by the Archimedean Property.

At the risk of raising some philosophical difficulties, it would seem only sensible to try to confront students, in their later years of schooling, with the issue of completeness, and to do this based on the model of numbers that they are familiar with, i.e., the decimal system. This claim is strengthened by educational research that suggests measures should be taken to ease the transition between school mathematics and that taught at university; see for example Castella (2007). However, very few educational tracts explicitly deal with the concept of completeness (an exception is some work by Bergé, 2010); this is surprising as the real numbers are fundamental to both institutional levels. In particular, certainly some dialectic between the concept of density and completeness would be welcome at the school level. Another issue is that students at school take it for granted that numbers can be acted on by the operations plus, minus, multiplication and division. However, it is far from obvious how these operations are effected on (non-periodic) decimals; in order to tackle this problem one again has to examine more precisely what we mean by the decimals. If this is not attempted, an occurrence of incoherency appears within the usual school curriculum. In this communication, we will not expand on the theme of the arithmetic of the decimals; a good account may be found in Gardiner (1982), chapter II.11.

This short essay points out that a mathematical issue, i.e., what the real numbers are, is dealt with in a completely different framework at school from what it is at university, and discusses in mathematical terms how these frameworks could be reconciled to some degree. What we have not attempted is the following: to forward an educational programme to address this concern. I hope, though, that what we have written here will prompt other educators to undertake exactly this. Also, as the whole topic of the system of the decimals is left hanging at school, there are opportunities for educational research to take other directions than that suggested by this current dialogue. Even in the more concrete situation of periodic decimals, certain general patterns in the digits can be discovered (e.g., Brenton, 2008; Lewittes, 2007) that afford an excellent problem solving environment to test students' ability to apply or develop elementary results in number theory or group theory at the early undergraduate career. Another interesting angle would be to take the decimals as an exemplar of a topic relevant to the school/university transition; how should the resultant theory be presented as a blend of the practices of the two institutional levels?

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New Zealand Mathematical Society Colloquium

University of Otago, Dunedin, New Zealand

7th December - 9th December 2010

The annual Colloquium, for the first time under the auspices of the New Zealand Mathematical Society, will be hosted in 2010 by the Department of Mathematics and Statistics at the University of Otago.

The programme consists of five plenary sessions together with contributed talks representing, we hope, a wide range of contemporary pure and applied mathematics and related areas, such as mathematics history and education.

There will be a reception on Monday evening to welcome those attending; registration formalities can be completed then or on the following morning.

On Tuesday there will be a second reception at which posters will be exhibited and informally talked about by their authors. ANZIAM has kindly donated a prize for the best poster submitted by a student.

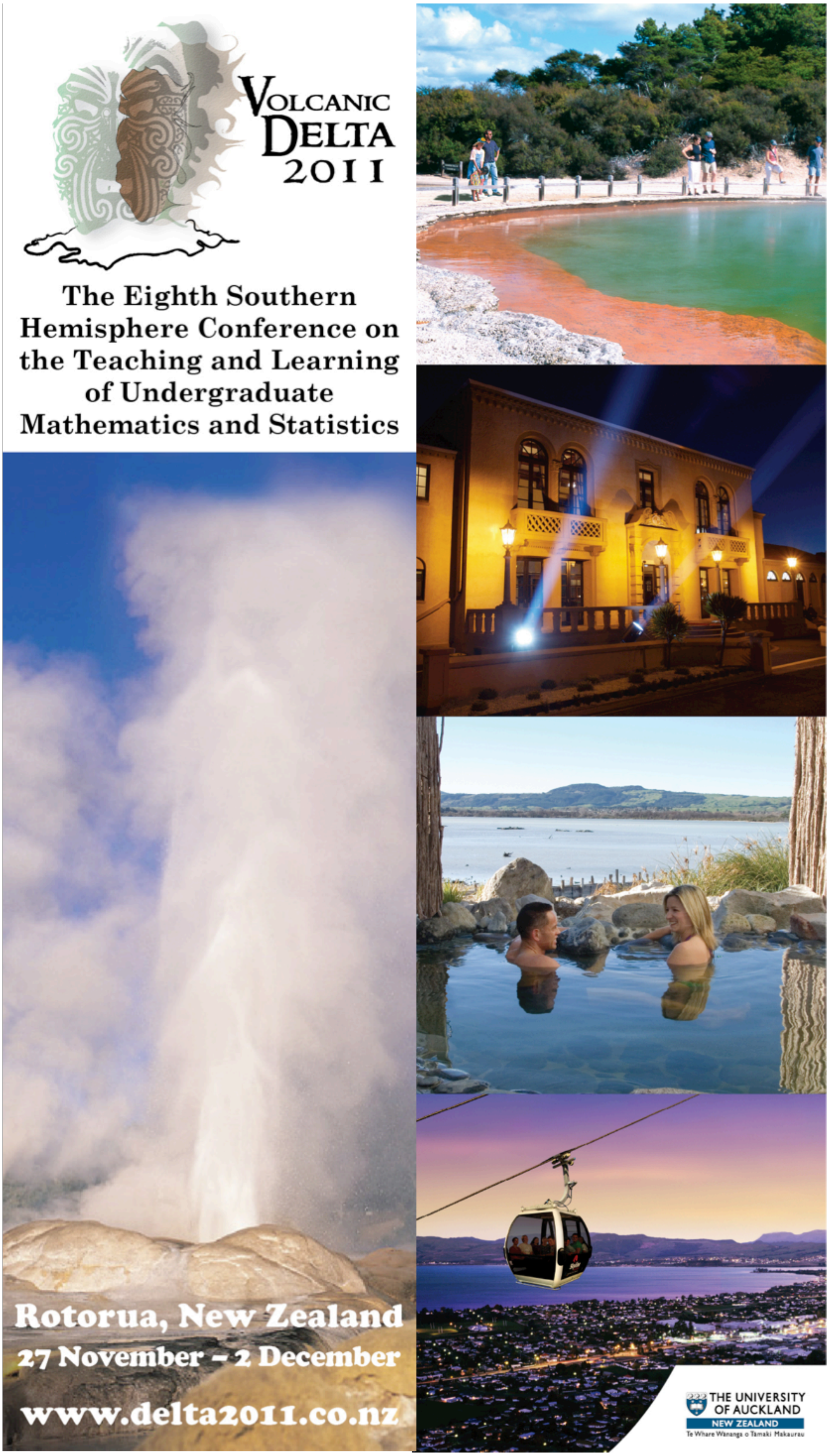
Excursions suitable for the energetic and the sedentary have been arranged for Wednesday afternoon. The Colloquium dinner will be held on Wednesday evening at the historic and picturesque Larnach's Castle on the Otago Peninsula. Transport to and from the dinner will be provided.

Further details are available from the website:
http://nzmathsoc.org.nz/colloquium/home.php

Please direct any enquiries to:
Ms Leanne Kirk lkirk@maths.otago.ac.nz

We very much look forward to seeing you there.

Peter Fenton,
Convener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