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DIFFICULTIES IN THE ACQUISITION OF LINEAR ALGEBRA CONCEPTS

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Abstract. Research shows that linear algebra is not an easy course to teach to first year university science and mathematics students. Around the world many students struggle to grasp the ideas in linear algebra, which although they may appear simple, are very powerful, with inner depth. This paper describes a study with first year mathematics students at The University of Auckland who completed a questionnaire containing some conceptual questions examining geometric, matrix and algebraic representation of linear algebra, along with a questionnaire on their attitudes to the course. Results suggest that there are student difficulties concerning understanding definitions, a tendency toward a procedural approach rather than a conceptual one, and an apparent lack of representational versatility.

1. Background

Linear algebra may be regarded as a unifying and generalizing theory which is also a formal theory [**3**]. While it is true that linear algebra can simplify the solution to many problems, this is only true for those who are very familiar with the subject area. In contrast the first year university student who has no prior understanding of the course has a long way to go before being able to see the whole picture. For them the course is very intense, with ideas and definitions introduced very rapidly, with little connection to what they already know in mathematics. In recent years researchers (see e.g., [**2**]) have investigated the kinds of difficulties that students experience in first year university linear algebra courses. One proposal to account for the problems is that, unlike calculus, linear algebra is generally the first course students encounter which is based on mathematical theory, built systematically from the ground up. This makes the course highly demanding cognitively, and it can be a frustrating experience for both teachers and students. While some believe that the course is taught too early, Dubinsky's [**4**] view is that students can develop their conceptual understanding by doing problems and making mental constructions of mathematical objects and procedures.

Two inseparable sources of difficulties with the linear algebra course, identified by Dorier and Sierpinska ([**2**], p. 256), are "the nature of linear algebra itself (conceptual difficulties) and the kind of thinking required for the understanding of linear algebra (cognitive difficulties)". Dorier *et al.* [**3**] claim that while many students fear linear algebra because of its abstract, esoteric nature, many teachers also suffer because of the abstruse reasoning involved. Historically, many of the concepts of linear algebra found their final form after several iterations of applications of linear techniques, and with little apparent unification. Hence, it is not surprising

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that many students have real difficulties with the definitions of such concepts. An important principle enunciated by Skemp ([**11**], p. 32), and illustrating one major problem with definitions, is that "Concepts of a higher order than those which a person already has cannot be communicated to him by a definition, but only by arranging for him to encounter a suitable collection of examples." Thurston [**13**] supports this idea, describing how, when he presented a particularly difficult result, listeners were much more interested in his descriptions of the concepts than they were in formal definitions and proofs.

A second major difficulty students face with linear algebra is the inherent nature of its multiple representations. Kaput ([**9**], p. 524) explains the importance of enabling manipulation of mathematical concepts both within and between different representation (or notation) systems, such as algebraic notations, graphs, tables, ordered pairs, etc. The value of this inter-representational activity leading to improved conceptual understanding has also been espoused by Even ([**5**], p. 105), who describes "flexibility in moving from one representation to another" as promoting rich cognitive relationships and improved conceptual understanding. In addition to this crucial inter-representational activity, Thomas [**12**] has also emphasised the importance of interacting with various appropriate representations in a conceptual as well as a procedural manner during mathematics learning.

In linear algebra there are at least three different representational forms that students have to be able to form links between. These have been described by Hillel [**8**], as: the general theory (abstract vector algebra); the specific theory of $\mathbb{R}^n$ (matrices); and the geometry of 2- and 3-space (geometric vectors). He also addresses conceptual difficulties associated with translating ideas across these representational boundaries. His categories broadly correspond to the three modes of reasoning suggested by Sierpinska [**10**], which she calls the visual geometric, arithmetic of vectors and matrices, and the structural language of vector spaces and linear transformations. Hillel ([**7**], p. 234) specifically links student problems to representation, stating that "being able to move from one representation to another is a source of difficulties for the students."

In seeking to solve this representational dilemma, Harel [**6**] cautions on introducing geometry first since students do not easily move to the general case, but persist in seeing the geometric object as the actual mathematical object rather than a representation of it. He concludes that we should "be careful not to move students up hastily from $\mathbb{R}^n$ to more general vector spaces...In elementary linear algebra, there should be one world - $\mathbb{R}^n$ - at least during the early period of the course." (*ibid*, p. 185). A solution recommended by the LACSG [**1**] is to make linear algebra courses matrix-oriented. However, Dubinsky [**4**] takes issue with this recommendation, believing that it may lead to courses which are full of computational procedures but will be neither applications-oriented nor assist students to build conceptual understanding.

In the light of the above background, the aim of the research study reported on here was to examine aspects of the conceptual understanding of linear algebra among first year university students. In this paper we present some issues involved in understanding of definitions and describe how the ability to see concepts in an inter-representational manner may relate to meaningful understanding.

2. Method

The case study was carried out in the first semester 2002 at the University of Auckland. The subjects were seventy stage one mathematics students taking a core mathematics paper (MATHS 152), covering both calculus and linear algebra, and designed for mathematics majors. The majority of students who enroll for this paper are familiar with basic linear algebra from a prerequisite paper. However, MATHS 152 introduces more advanced topics. The students spend about 6 weeks studying linear algebra and have about four hours of lectures plus an hour tutorial every week. There were 119 students enrolled in the paper, and of the 70-80 students attending the classes 70 students consented to complete a test and a questionnaire. These were a mixture of males and females, mainly 18-21 years old, with a majority of Asian students. The tests were completed during a one-hour tutorial, and since students receive a mark for attendance at these and for working through the tutorial problems, we were convinced that most students took the test seriously.

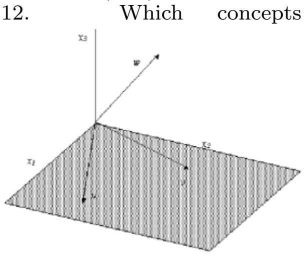
The instruments comprised a linear algebra test and an attitude questionnaire. The test was designed to assess student understanding of linear algebra in each of the geometric, matrix and algebraic representations (see Figure 1 for examples of the linear algebra questions). Unlike the paper's traditional questions, which often involve primarily procedural calculations, these questions principally concerned conceptual understanding of the ideas behind the theories. One of our primary motivations was to gather data on the role of student understanding and use of definitions in linear algebra. We suspected that many students did not learn or understand definitions, and hypothesised that this could be a key factor in their progress. Hence, in Question 1 students were asked to explain definitions of terms such as invertible matrix, eigenvector, linear combination and so on, in their own words, to evaluate understanding of the definitions rather than rote memorisation of words. Questions 2 and 4 involved matrices, and in particular conceptual understanding of the notion of basis and linearly independence.

Versatility, as demonstrated in the ability to shift attention between different representations was examined in Questions 8, 10 and 12. Question 8 was presented algebraically without matrices, whereas Question 10 required a geometric representation to be produced from information presented using matrices. Both questions 8 and 12(b) referred to eigenvectors and eigenvalues, but differed in the representation used. We were interested to compare the results in these questions with responses to the definition of the word eigenvector presented in question 1. In addition questions 3, 6, 7, 9 and 11 involved the application of understanding of conceptual ideas such as rank and invertible matrix or orthogonal vectors. The questionnaire involved 13 statements responded to using a five point Likert scale. Figure 2 shows relevant sample items from the questionnaire, which investigated student beliefs about definitions and areas of the paper they found problematic.

3. Results

The data from the seventy Maths 152 students who participated in the linear algebra test indicated some difficulty in coping with the definitions. Question 1 asked for definitions of linear algebra terms, requiring an answer in students' own words, rather than repeating a standard form of the definition. The results showed,

1. Explain each of the items in your own words. (DO NOT simply repeat a definition!)
(a) Invertible matrix
(b) Eigenvector
(c) Basis
(d) Spanning set
(e) Linear combination
(f) Subspace
(g) Linearly independent
2. Three vectors **a**, **b**, **c** form a basis for $\mathbb{R}^3$. Then either **a**, **b** or **c** must be $(1, 0, 0)$.
True ☐ False ☐ Explain.
6. Explain what can you tell about an $n \times n$ matrix if Rank $A = n$.
7. What can you say about the solutions of a system of linear equations represented by a 3×4 matrix A if Nul A=1 ?
8. If a linear transformation is represented by a matrix Q, and a vector P exists such that $QP = 3P$, what does the 3 tell us about this transformation?
10. The vector $\mathbf{z} = \begin{pmatrix} 7 \\ 4 \\ -3 \end{pmatrix}$ can be written as a linear combination of the vectors $\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}$. Draw a Geometric representation of this relationship.
12. Which concepts in linear algebra do these diagrams refer to?



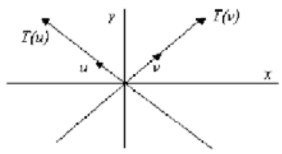


FIGURE 1. Examples of the linear algebra questions.

1. I understand most definitions in the 152 linear Algebra course.
3. I believe definitions are important in Linear Algebra.
4. I find it really hard to memorise too many definitions in Linear Algebra.
7. I would like to improve my linear algebra knowledge.
8. This university should offer a separate paper called "Introduction to Linear Algebra", which will explain the concepts a bit slower and more thoroughly.
9. I feel lost during the 152 linear algebra classes.
11. Understanding the concepts is the main difficulty of most students in linear algebra.

FIGURE 2. Sample items from the attitude questionnaire.

surprisingly, that 27% did not answer question 1 at all, and, as Table 1 shows, up to 60% were unable to provide a definition of some terms. Combining this with the low incidence of correct answers means that we can conclude that the students did not have a clear idea of these particular definitions.

TABLE 1. Percentages of Students Defining Terms in Question 1

Explain each of these items	No response (%)	Correct response (%)
(a) Invertible matrix	33.0	18.6
(b) Eigenvector	40.0	25.7
(c) Basis	45.7	2.9
(d) Spanning set	50.0	2.9
(e) Linear combination	44.3	12.9
(f) Subspace	60.0	5.7
(g) Linearly independent	35.7	15.7

While generally the students had real difficulties understanding or remembering conceptual definitions such as “spanning set” and “subspace” we discovered that they seemed to be more confident explaining terms such as “invertible matrix” and “linearly independent”. This may have been because they had seen them more often in their assignments or that they had been required to carry out calculations involving them. Confirming this link to procedural calculations, the most common responses students gave for the term “Invertible matrix” included: “It’s about finding the inverse of a matrix, by having identity matrix next to it.”, and “Can be row reduced to find an inverse.” Many students simply rephrased the given term stating that “An invertible matrix is a matrix that you can invert” or said it “has an inverse”. One student made it very clear that he was not able to explain the definitions but could only calculate using them, as shown in Figure 3.

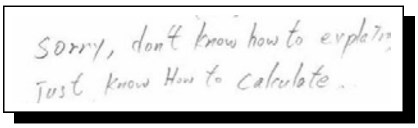


FIGURE 3. An answer to question 1 showing a procedural emphasis.

Similar results were also found regarding the rest of the terms in question 1. For example, in response to the term “eigenvector”, of the 60% who responded, 43% gave one of the following procedurally-based answers: a vector that when multiplied by a particular matrix will equal a multiple of itself; or $Ax = \lambda x$ where x is the eigenvector. A further example of the regression to procedures came in relation to this term, with 9% of the students indicating that an eigenvector is constructed from an eigenvalue. Figure 4 shows the response of one student who went further and tried to illustrate how to find the eigenvectors. The term “linear combination” also produced answers where students mainly translated the term into procedural ideas such as “one vector add one vector is equal to the other vector” or “the combination of vectors to become one vector”, demonstrating a lack of a real connection to the concept. Some of these difficulties with definitions were confirmed by a statistical analysis of the questionnaire. Here, 31.9% of students stated that they did not understand the definitions, although there was no evidence (χ^2=2.70,

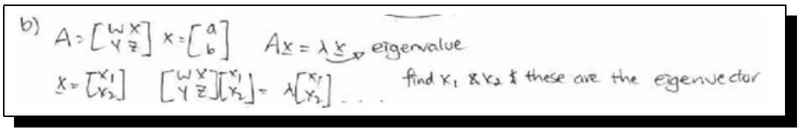


FIGURE 4. An example of a student response to the term eigenvector.

df=2, p=0.260) that the underlying population proportion of these was any larger than those who stated that they did understand them (26.1%). In addition 42% of students remained unsure of their understanding of definitions, which helps us appreciate why, acording to the linear algebra test, many students were struggling with understanding them. These results were corroborated by a number of student comments to open questions. It appears that many students realise that definitions are important and valuable ("I think it's quite important to understand the definitions, otherwise you are completely lost further on"), however they find them too difficult ("I think one of the hardest things to understand is the definitions... If you don't know the definitions, then you can't answer the question"), and some think that they don't need to know them fully to solve problems ("I can solve linear algebra questions even though I don't know the exact definition").

Hence, it was clear that the majority of these students were very uneasy about the definitions and felt better equipped to calculate answers by taking procedural approaches to problems than actually to think about the fundamental concepts. They also realised that because most questions in the course were procedural they did not really need to know the definitions in order to succeed.

The question remained as to how this problem with definitions and a propensity for procedures would influence students' representational versatility. The results showed that in question 10, where the data were presented in terms of matrices (vectors) and a geometric response called for, some students again preferred to take a procedural approach rather than draw a geometric representation of the required relationship. We wanted them to demonstrate an understanding that the vector **z** lies in the plane spanned by the vectors **u** and **v**. However, 31% of the students did not answer question 10 at all, and only 7% of students were able to show geometrically that the vectors should be in the same plane. The remainder either did not give a geometrical description at all but tried to solve it using matrices, or drew some kind of graph, but without the key element of being in the same plane. Figure 5a is an example taken from many who tried to solve the problem by using matrices. In contrast, Figure 5b shows the work of a student, among a small minority, who was able to translate the concept of a linear combination from natural language and matrices to a geometric representation, and explicitly describes the vectors as in the same plane. It appears that many students are not comfortable with moving between representations. In particular they get concerned when asked to use their geometric knowledge, choosing instead to fall back on the matrix calculations they have been taught. In this context, question 12 would have been a real challenge to such students. We were interested to see the interpretations of the geometric diagrams in terms of linear algebra concepts. However, 51% did not answer question 12 at all, and no one gave the answer "linearly independent" to part (a). Although there were students who did reasonably well in defining the

FIGURE 5. Representation of the linear combination relationship.

idea of linear independence in question 1, they were unable to reverse the process to recognise its relevance here. In part (b) only 11% of the students mentioned the word eigenvectors or eigenvalues. This seems to confirm the fact that, even though they may be familiar with the notions of linear independence and eigenvectors, and most likely can solve standard problems relating to these topics, they were not able to relate to them in a different representation. Unlike question 10, which presented matrices and required a geometric representation, question 8 concentrated on algebraic and symbolic representation of matrices. It referred to the same topic as question 12(b), namely eigenvectors, but differed in representation. We found that 25.7% did not answer this question, and of those who did the most common responses were, '3 is the eigenvalue' (57.7%), '3 times longer', 'expanded by 3', 'Q= 3', '3 is invertible', and 'QP is in the same direction as p but 3 times longer'. This shows that a good percentage of students managed to interpret the symbols in question 8 well, even though the words "linear transformation" may have hindered some students.

Question 2 addressed understanding of whether the standard basis for $\mathbb{R}^3$ is the only basis. Results showed that 10% did not answer the question at all, and 36% claimed that the statement is true (with 28% giving a reason). The explanations included the following types of surface-structure comments: '$\mathbb{R}^3$ has 3 rows'; and '$\mathbb{R}^3$ should have 3 entries'. On the other hand 53% claimed that the statement was false, with 38% giving quite good reasons based on linear independence, such as: 'It would be others as long as they can make $\mathbb{R}^3$'; 'that is just a common basis', 'it could be anything as long as they are linearly independent'; 'a basis doesn't need to be composed of (1,0,0)', 'as long as it is linearly independent and consists of 3 vectors each with 3 elements it will form a basis'.

4. Conclusions

In spite of widespread circumstantial evidence of a need to investigate student difficulties, and a number of intiatives regarding possible changes to the curriculum and ways of teaching and handling linear algebra, research on students' conceptual understanding of linear algebra is still fairly sparse. This small-scale study has provided evidence on a number of relevant issues. We have found that while students know that definitions are important, they do not like them, and do not seem to learn them or quote them. Not surprisingly, this means that they do not understand the meaning of definitions and are unable to apply them even in simple situations. Instead they often see linear algebra as the application of a set of procedures, which if learned will enable them to solve the given problems. Having such a tenuous grasp on the concepts of linear algebra means that students are not able to translate these concepts across different representations [**8**], and in particular they lack a geometric perspective of them, relating to $\mathbb{R}^n$. Our study appears to support the conclusion of Harel [**6**] that an emphasis on $\mathbb{R}^n$ in early linear algebra, with explicit links to geometry, might assist students to build a stronger conceptual basis for the subject.

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