UNDERSTANDING LINEAR ALGEBRAIC EQUATIONS VIA SUPER–CALCULATOR REPRESENTATIONS

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For many students, understanding of linear algebraic equations primarily accrues from working in a single, symbolic representation. It is not until later that they study graphs and attempts are made to link these to the previously studied equations. This paper reports results of a study where super–calculators were used with 13 year–old Korean students, none of whom had used calculators in mathematics lessons before, in an attempt to make explicit links between symbolic, tabular and graphical representations of equations. The results show that the students did build some improved deep links between the domains and that the experience of using the calculators was generally a positive one for them.

Background

It appears that often the focus of mathematics for secondary students has been on learning which encourages them to build knowledge specific to particular problems. Further, much of this learning has emphasised procedural methods rather than concepts. For example, in early algebra sometimes students learn how to simplify, expand, factorise and solve, without understanding the meaning of the processes or the nature of the objects (Tall *et al*., 1999) upon which they are acting. While procedural knowledge is important it is prone to be learned instrumentally (Skemp, 1979).

There is wide agreement that our conceptual structures or schemas are a key determinant of our progress in mathematics. It seems clear then, that the richer one's construction of schemas in a given domain in terms of conceptual associations or C–links (Skemp, 1979), the greater the potential for future expansion and linking to other domains of mathematical knowledge. One way to accomplish this is by making explicit conceptual links between different representations of mathematics, described by Kaput (1987, p. 23) as involving "a correspondence between some aspects of the represented world and some aspects of the representing world." Kaput (1992, p. 524) has listed one of the four classes of mathematical activity in school as "translations between notation systems, including the coordination of action across notation systems." and explains the value of technology in enabling manipulation of mathematical concepts both within and between these different representations. However, more is required than the ability to translate between the representation systems. Students may have such a surface ability without an understanding of the deeper, conceptual links which are imbedded in the transformation between the representations (Greer & Harel, 1998). Chinnappan and Thomas (1999) have investigated the schemas of experienced teachers who employ modelling techniques, and have suggested a model with schematic conceptual links as the foundation for learning which relates representations, internally and externally.

This research investigated the value of making explicit links between three different representations of linear algebraic equations: the symbolic; the graphical; and the tabular forms. While these representations have often been used before, here the concept of equation, and in particular the deep idea of conservation of a solution to an equation under cross–representation transformations, was emphasised during the learning experience.

A consideration of the best way to approach this led to the graphic calculator, since all three dynamically related representations arise naturally in that context (Kaput, 1992). In addition, graphic calculators are more accessible to students than computers are in many schools (Kissane, 1995; Thomas, 1996) and this is a key advantage. However some have been sceptical of the value of the technology in secondary mathematics learning and continued research is needed in order to provide convincing evidence of how graphic calculators can be valuably employed in the mathematics curriculum. There is already a growing body of such research (e.g. Ruthven, 1990; Penglase & Arnold, 1996; Graham & Thomas, 1997) supporting their use in mathematics learning, but it is still restricted in terms of content area and use of the calculators' facilities.

Method

The research described here forms the Korean part of a study of students in New Zealand and Korea (a high performing country on the TIMSS results), whose aim is to investigate the use of super/graphic calculators to improve students' conceptual understanding of linear algebraic equations. This research was carried out during the period 5th–16th July, 1999.

Subjects: The study involved one class of 35 Form 4 students (aged 13–14 years) from a middle school in Seoul, Korea, who are not currently allowed to use any calculator in their classes, or for assessment, such as examinations. Thus while 27 of the subjects had used a calculator (but never in their standard mathematics lessons) none had ever used a graphic or a super–calculator. The students had previously covered simplification of algebraic expressions and solving linear equations during the school year. This enabled us to use them as a stand–alone single subject group to see what, if anything, they could gain by additional exposure to and linking with the alternative methods of approaching equations that the super–calculator permits.

Instruments: A module of work using the TI–92 super–calculator was prepared. This contained a description of the basic facilities of the TI-92 and then showed how, using a 'Press', 'See', and 'Explanation' format, linear equations can be solved in three different ways: algebraically, graphically, and numerically from a table of values. Two algebraic methods were given, using the TI–92 to solve the equation directly and also using a standard balancing algorithm. An illustrative section from the module showing the four methods for the equation $2x - 5 = 3x - 9$ are given in Figure 1 (note the section is incomplete and formatting has been changed). The fact that the solution is the same in each case was emphasised. Two parallel tests, divided into sections A and B and comprising different numerical values, were constructed as pre– and post–tests. Section A of these tests comprised standard textbooks questions such as: $5x - 8 = 3x + 2$; $m = 8 - 3m$; and $6 - 8n = -3 + n$.

In contrast, section B addressed the students' conceptual thinking in solving equations, both within and across different representations. The concept of equivalent equations is also important and we wanted to know whether the students were able to conserve equation under addition of constant or variable quantities as used in method 1b), and could recognise equivalent ones without having to find solutions.

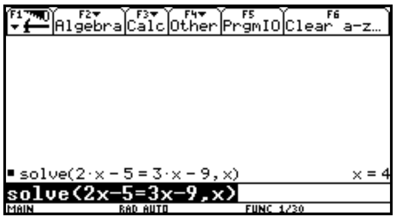
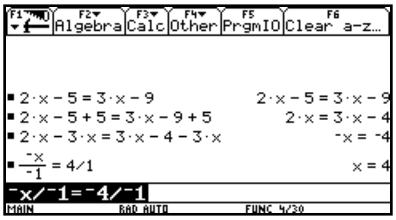
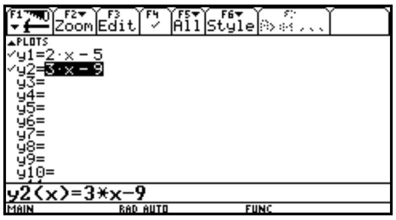
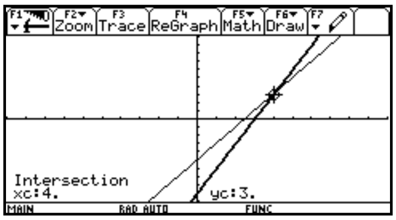
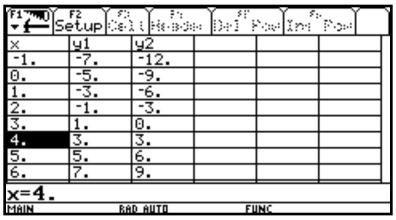
Method 1 a) ◆ [HOME] F2 1 2*x* − 5 [=] 3*x* − 9 , *x*) ENTER		$2x – 5 = 3x – 9$ is solved by an algebraic method. The b *x* tells the calculator to solve with respect to *x*. $x = 4$ is the value which makes both sides equal in value.
Method 1 b) 2*x* − 5 [=] 3*x* − 9 ENTER 2*x* − 5 + 5 [=] 3*x* − 9 + 5 ENTER 2*x* − 3*x* [=] 3*x* − 4 − 3*x* ENTER (-) *x* ÷ (-) 1 [=] (-) 4 ÷ (-) 1 ENTER		To find the value of *x*, we need to simplify the given expression step by step: If we add 5 to both sides, the expression is simplified to $2x=3x-4$. If we subtract $3x$, the expression is simplified to $–x = –4$. If we divide by $–1$, finally we get $x=4$
Method 2 ◆ [Y=] 2*x* − 5 ENTER 3*x* − 9 ENTER 1st curve? ENTER 2nd curve? ENTER Lower bound? 0 ENTER Upper bound? 6 ENTER	 	Here each side of the equation is defined as a function, using y1(*x*) and y2(*x*): y1(*x*)= $2x – 5$ y2(*x*)= $3x – 9$ Looking at the two graphs, we can see that they intersect at one point. 1st curve means y1(*x*), 2nd curve means y2(*x*). The lower and upper bound means the interval in which the intersection point is found. So the two graphs intersect at the point (4, 3). i.e. $x = 4$
Method 3 ◆ [TblSet] tblStart: –1 Δtbl: 1 ◆ [TABLE]		The point of intersection can be found using a table. Enter y1 and y2 as in method 2. When we look at the point $x=4$, we can see the values of the two functions are the same, and equal to 3.

Figure 1: A section of the module showing the layout and calculator screens

Within the symbolic representation we asked them questions (B1) such as:

Do the following pairs of two equations have the same solution? Give reasons for your answer.

a) $5x – 1 = 3x + 2$
$5x – 1 + 8 = 3x + 2 + 8$

b) $2 – 3x = x – 3$
$2 – 2x = 2x – 3$

We note that in both examples above the second equation may be seen by someone with the concept of equivalence of equations as a *legitimate transformation* of the first (by adding 8 or *x* to both sides) conserving the solution, although this may not be a *productive transformation* in terms of actually obtaining that solution.

In addition the tests required the students to maintain the concept of solving an equation across representations, asking them to solve a symbolically presented equation in a graphical and a tabular domain (question B7) and to convert a graphical picture into a symbolic representation, as in Figure 2.

B6. Write a single equation in *x* which can be represented by the following diagram:

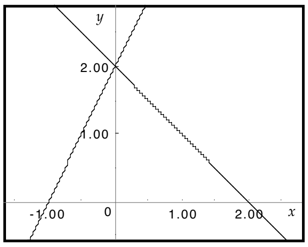
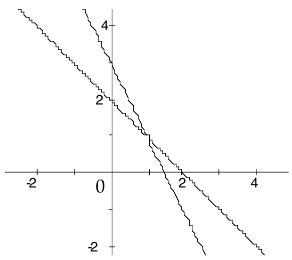


Figure 2: A question assessing understanding of transformation from a graphical to a symbolic representation

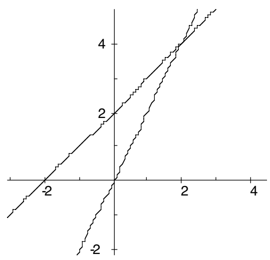
Students were also given a list of 6 symbolic, 3 graphical and 3 tabular representations and asked 'Which of the following are different ways of representing the same equations?' (see Figure 3). This tested their ability to conceptualise functional equality across representations.

B2. Which of the following are different ways of representing the same equations? Match the letters which correspond and write them in the boxes below.

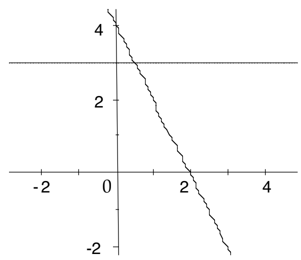
$x + 2 = 2x$	$4 - 2x = 3$	$-2x + 3 = 2 - x$
A	B	C
$5 - x = 2 - 2x$	$2x - 4 = 4x + 10$	$5 - 3x = -4$
D	E	F



G



H



I

x	*y*
–1	5
0	3
1	1
2	–1
3	–3

x	*y*
–1	3
0	2
1	1
2	0
3	–1

J

x	*y*
0	5
1	2
2	–1
3	–4
4	–7

K

x	*y*
0	4
1	2
2	0
3	–2
4	–4

x	*y*
0	3
1	3
2	3
3	3
4	3

L

Answer. ☐ ☐ ☐ correspond ☐ ☐ ☐ correspond

☐ ☐ ☐ correspond ☐ ☐ ☐ correspond

Figure 3. A question testing equivalent representations of equation forms

As well as the test each student was given a questionnaire covering their experience with the calculators (serving to triangulate the test results) and their understanding of the concepts associated with linear equations, and an attitude scale (using a 5 point Likert format) on their feelings towards calculators.

Procedure: The module (written in English and translated into Korean – the English version is shown in this paper) was initially given to the class teacher, who familiarised herself with the content. The first named researcher met twice with her to answer her questions and to make sure that she was comfortable with the calculator and the material. The teacher then taught the class for four lessons, two covering basic facilities of the calculators including introducing graphs and tables, the other two describing how to solve equations in different ways on the TI–92. The only previous experience the students had had was in solving equations algebraically.

Each student had access to their own TI–92 super–calculator, which they kept with them for the whole of the time of the study, including their time at home. During lessons the teacher stood at the front of the class, who sat in the traditional rows of desks, demonstrating each step while the students followed in the module and copied her working onto their own calculator. She employed a calculator viewscreen and projected the image using an overhead projector. However, the calculator commands were projected in English and it was necessary for the teacher to translate each one into Korean for the students, causing some confusion until they became more accustomed to the English commands. Each session lasted 45 minutes, and after the teacher's explanation, the students spent the rest of the session practising while the teacher circulated and assisted with any problems, and the last five minutes were used for a summary. At the end of fourth tutorial the students were given the post-test, followed by the attitude test and the questionnaire.

Results

The students had spent some time in normal lessons prior to the research learning how to solve equations in a symbolic arena. Hence it was not surprising that they had pre–test facilities of 86.7% and 69% on the two skills questions in section A, showing that before the study they were able to solve even difficult linear equations. However, for question 6 in section B (see Figure 2), not one student could relate either of the lines to an equation in x (this was considered by the teacher to be at too high a level for her students). Even when given a graph or tables and an accompanying symbolic equation as in question B7 (see Figure 4), very few students could solve the equation using the graphical or tabular representation, this question having a facility of 15.8%. On question B2, (Figure 3), students also had difficulty, with just 11.5% of answers correct. It seems, from this pre–test evidence, that students were good at solving equations symbolically, but were less good at translating between the symbolic and other representations, or using those representations to solve equations. They had problems relating the surface features of the domains and had not built an understanding of the deeper relationship based on conservation of equation solution under different representations of the functions.

After the calculator intervention there was no change in the students' skills at solving the equations (Section A: max score = 7; m_{pre}=5.61, m_{post}=5.54, t=0.16, n.s.) but they had improved significantly on some of the more testing questions in section B (max score = 35; m_{pre}=13.1, m_{post}=17.0, t=2.25, p<.05). However, the improvement was variable across the questions, as may be seen from Table 1, which gives the mean scores for each of the questions in sections A and B.

Each of questions B2, B6 and B7, where the students performed significantly better after the module, involved relating data between two or more representations (see Figures 2 and 3). While the improvement in question B6 was small, involving only 5 students, this was a more difficult question, needing further attention. Figure 4 shows the response of student L2 on question B7, where they were asked to solve the symbolic equation using the given graph or table. On the pre–test he was unable to use the graph or table to solve the equation, but on the post–test he has made the link to a solution in each case, and understands why it is the solution.

Table 1: A comparison of pre– and post–test section A and B mean scores

Question Number (Max score)	Pre–test mean	Post–test mean	t	p
A1 (5)	4.33	4.09	–0.44	n.s.
A2 (2)	1.38	1.46	0.41	n.s.
B1 (18)	10.5	11.5	0.77	n.s.
B2 (4)	0.46	1.14	3.95	<.0005
B3 (2)	0.85	1.09	1.46	n.s.
B4 (1)	0.04	0	–1.79	n.s.
B5 (2)	0.35	0.53	1.46	n.s.
B6 (2)	0	0.17	2.24	<.05
B7 (6)	0.95	2.66	3.29	<.005

We note that for the graph question he also solves the equation symbolically, but does not do so for the table.

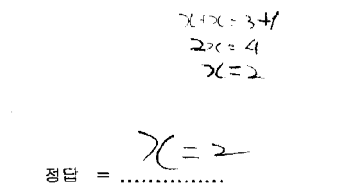
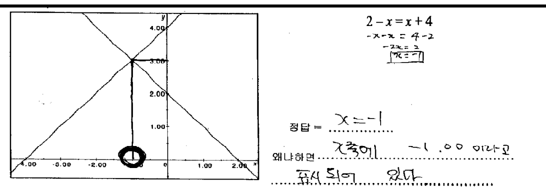
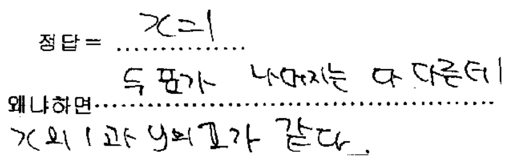
Pre–Test Solution	Post–Test Solution
	 [x equals –1, because –1.00 is indicated on the x-axis]
No Response	 [x = 1, because the values of y as 2 at the point x = 1 are the same from the two tables]

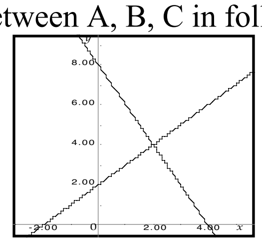
Figure 4: The B7 work of student L2 showing conceptual links between representations

The aspect of conceptual understanding, seeing the relationship between equations and their solutions across the representations, was further investigated in the students' questionnaire, where they were asked the following question:

Is there a relationship between A, B, C in following diagrams? If so, then what is it?

$x + 2 = -2x + 8$

A



B

x	y
–1	1
0	2
1	3
2	4
3	5

x	y
–1	10
0	8
1	6
2	4
3	2

C

This was very similar to question B2, where the pre–test facility was 11.5%, and the post-test 19%. In the questionnaire 17 (48%) of the students answered that the value of x is common, showing that they may have conserved solution of equation across the representations. It could be argued that they had merely calculated that in each case the solution was 2, without appreciating that the functions were the same. However, 9 (25%) of the students also answered that the expression A can be shown by the graph B and the table C, demonstrating that these had made the link. Further supporting this, 11 (31%) of the students mentioned later in their questionnaires, as an advantage of calculators, that through the use of the table, graph and algebra the solution to the equations could be more easily and quickly understood. In addition, when asked 'How many different ways can an equation be represented?' 21 (60%) replied 3, citing algebraic, table and graph. This was evidence that the tutorial had assisted construction of schematic understanding of surface and deep relationships between algebraic, tabular graphical representations of the concept of equality of linear functions.

Table 2: Breakdown of methods employed on the conservation of equation question B1

Pre-test Method	No solution* 39 (18.6%)	Solve 19 (9.0%)	Solve+Explain 109 (51.9%)	Explain only 43 (20.5%)
Post-test Method	No solution* 45 (21.4%)	Solve 23 (11.0%)	Solve+Explain 48 (22.9%)	Explain only 94 (44.8%)

* or incorrect solution

At first sight it appeared (see Table 1) that there was no improvement in the test question (B1) on the conservation of equation under transformations in the symbolic representation. However, as Table 2 shows, in the post–test many more equations were correctly solved by students in question B1 by considering the relationship between the two equations, without needing to solve the equations (as shown by the 'explain only' column). Student C, for example, had to symbolically solve equations of the type in question B1 a) and b) in the pre–test. However, in the post–test she was able to write that they have "the same solution because if you add the same value to both sides then the equation is the same", and "If you multiply both sides by the same value [4] then the equation is the same", without needing to solve either equation.

Student Attitudes: These students had never used a calculator in their mathematics learning, hence it was of great interest to find out their view of them. After their tutorials, when asked 'How do you feel about the TI–92 graphic calculator?', 22 (62%) of the students replied that they felt easy, comfortable, curious or were interested in using the calculator. When asked what difficulties they had encountered 17 (48%) of the Korean students replied that the commands on the calculator were difficult because they were in English. The attitude scale questions confirmed the positive view of calculators. Each question was scored with an integer from 1 to 5, and scores were reversed on negative questions so that in every case the higher the score the more positive the attitude to calculators. Overall their response to the calculators was significantly positive, with a mean score of 3.54 (t=5.16, p<.00005). Their mean scores showed that they clearly thought that calculators were valuable (I think the calculator is a very important tool for learning mathematics, 4.0), and made mathematics more interesting (More interesting mathematics problems can be done when students have access to calculators, 4.2). Further they were keen to use them more themselves (I want to improve my ability to use a calculator, 4.17), and thought that others should learn how to use them (All students should learn to use calculators, 4.0). It was interesting to see that they were not really influenced by commonly held views, such as the detrimental effect of calculators on mathematics skills (Using calculators will cause students to lose basic computational skills, 3.06; Students should not be allowed to use a calculator until they have mastered the idea or method, 3.23).

Conclusion

Our contention is that important conceptual links between the symbolic, graphical and tabular representations of functions can easily be lost if algebra is approached in a purely procedural manner. The value of graphic and super–calculators is that they may be used to assist teachers to make these links explicit, provided teachers are pedagogically alert to the deeper, underlying conceptual relationships, and preservation of the conceptual structure of the mathematics is central to their schemas. For example, one may approach the graphical solution of linear equations by providing the surface, procedural method of drawing the graph of each function and reading off the *x*–value of the point of intersection without explicitly tackling the deeper, functional relationships (see Greer & Harel, 1998 for other examples). The concept of linear function passes across four different representations in this method: algebraic, tabular, ordered pairs and graphical. To build rich relational schemas which contain internal representations of the external ones, students should experience the links and the sub–concepts of one–to–one, independent and dependent variable, etc. in each representation. In addition, the fact that two one–to–one functions coincide for a single value of the independent variable, and the conservation of these values across any representation is important. In this research study we have begun the process of testing the value of super–calculators such as the TI–92 in promoting the construction of relationships like these. This small–scale, uncontrolled study, provides some evidence of the value of the approach, with the students able to add conceptual schematic links to the knowledge they had built by studying equations purely symbolically. Whether the calculator is a significant factor, or the results could be duplicated without using them is of considerable interest, and we would welcome a study which sought to determine this.

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