SIMPLE GAMES WITH APPLICATIONS TO SECRET SHARING SCHEMES

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A Thesis Submitted in Partial Fulfillment of
the Requirements for the Degree of
PhD of Mathematics

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Abstract

In many situations, cooperating agents have different status with respect to an activity. Often a coalition can undertake the activity only if sufficiently many agents, or agents of sufficient seniority participate in it. The concept of a simple game, introduced by von Neumann & Morgenstern (1944), is flexible enough to model a large class of such situations. In this dissertation, we consider two important classes of simple games called the classes of weighted simple games, and roughly weighted simple games, and apply the knowledge obtained from their study to make progress towards solving an important open problem in a branch of cryptography called secret sharing schemes. Secret sharing schemes were first introduced by Shamir (1979) and now widely used in many cryptographic protocols as a tool for securely storing information that is highly sensitive and highly important. Such information includes encryption keys, missile launch codes, and numbered bank accounts. A secret sharing scheme stipulates giving to each player a piece of information called 'share' so that only authorised coalitions can calculate the secret combining their shares together. The set of all authorised coalitions is called an access structure. Mathematically speaking, an access structure is a simple game. Different types of secret sharing schemes exist, and some of them are more efficient and secure than others. The most informationally efficient and secure schemes are called ideal, and these are obviously very sought after and therefore give rise to the question: Which access structures are ideal? Using game-theoretic methods, we contribute to the problem of characterising all ideal secret sharing schemes in the two aforementioned classes of weighted and roughly weighted simple games. We start with a study of two important classes of simple games in the ideal weighted setting, namely hierarchical and tripartite. Then we study the operation of composing simple games. We then apply our knowledge, and existing results by Beimel, Tassa, & Weinreb (2008) and Farràs & Padró (2010), to providing an 'if and only if' characterisation theorem for all ideal weighted simple games. Finally, we undertake a study of some ideal roughly weighted simple games. Here, firstly we generalise our result regarding which

hierarchical simple games are weighted to which hierarchical simple games are roughly weighted. Secondly, we answer the question of whether a tripartite simple game can be roughly weighted and nonweighted. Thirdly, we show that there exists an ideal roughly weighted simple game that is neither a hierarchical nor a tripartite simple game, showing that the classification of ideal roughly weighted simple games cannot be accomplished along the same lines as in (Beimel, Tassa, & Weinreb, 2008).

Acknowledgments

The author wishes to sincerely thank Associate Professor Arkadii Slinko for his supervision of this study, and for his vital support, guidance, inspiration and encouragement during the course of this work. Gratitude also goes to Professor Bill Zwicker and Dr. Tanya Gvodzeva for their very helpful comments and suggestions. I would also like to express my appreciation and gratitude to my family who has given great support and encouragement, without which, my task would have been much more difficult.

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Chapter 1

Introduction

1.1 Background and Motivation

1.1.1 Simple games and secret sharing schemes

In many situations, cooperating agents have different status with respect to an activity. Often a coalition can undertake the activity only if sufficiently many agents, or agents of sufficient seniority participate in it. The classic example of such situation is the United Nations Security Council voting on whether or not to impose sanctions on a country developing nuclear weapons. Nine members of the Security Council in total must vote in favour of the sanctions, including all five permanent members. The concept of a simple game, introduced by von Neumann & Morgenstern (1944), is flexible enough to model a large class of such situations. In the theory of simple games seniority of players is usually modeled by assigning to players different weights and setting a threshold so that a coalition is winning (can undertake the activity) if and only if the combined weight of its players is at least the threshold. Such a simple game is called *weighted* or *weighted threshold*. This is perfectly natural, for example, in the context of corporate voting when different shareholders may hold different number of shares. More generally, if two players in a simple game are such that they are either equal to each other in seniority [1], or one is strictly more senior than the other, then they are *comparable* to each other, and if any two players in the game are comparable, then the game is called *complete*. The class of complete games includes the class of weighted games and more.

Secret sharing schemes , first introduced by Shamir (1979) and also independently by Blakley (1979), and now widely used in many cryptographic protocols, is a tool designed for securely storing information that is highly sensitive and highly important. Such information includes encryption keys, missile launch codes, and numbered bank accounts. A secret sharing scheme stipulates giving to players 'shares' of the secret so that only authorised coalitions can calculate the secret combining their shares together. Different types of secret sharing schemes exist, and some of them are more efficient and secure than others. The most informationally efficient and secure secret sharing schemes are called *ideal*. The set of all authorised coalitions of a secret sharing scheme is known as the *access structure* (e.g., Simmons, 1990; Stinson, 1992). It can also be modeled by a simple game, for, mathematically speaking, an access structure is a simple game, and for simplicity, sometimes we will refer to them as 'games'.

Our work in this thesis is game-theoretic in nature, however, it is ultimately applied to the study of ideal secret sharing schemes, hence the title of the thesis. Our approach to games here will be based on a way of relating players to each other that we will explain in Chapter 2, called Isbell's desirability relation (see Isbell (1958), Maschler & Peleg (1966) and Taylor & Zwicker (1999)). It is a way of organising a simple game into different levels of seniority, where each level contains players who all share the same seniority. This we found has simplified things, and has enabled us to re-write many of the existing results in a more transparent way.

[1] The concept of seniority will be rigorously defined later.

1.1.2 The project of characterising ideal secret sharing schemes

Since ideal secret sharing schemes are the most informationally efficient and secure, and since not all secret sharing schemes are ideal (see Benaloh & Leichter (1990)), the project of characterising all access structures that carry an ideal secret sharing scheme, or in other words characterising ideal access structures, has been pursued by numerous people (e.g., Brickell (1989); Brickell & Davenport (1991); Beimel & Chor (1994); Padró & Sáez (1998); Golić (1998); Collins (2002); Ng (2006); Martí-Farré et al. (2006); Farras et al. (2007); Beimel, Livne, & Padró (2008); Beimel, Tassa, & Weinreb (2008); Herranz (2009); HSU ChingFang (2010); Farràs et al. (2011); Farràs & Padró (2011); Farràs & Padró (2012); Farràs et al. (2012)). It is now considered as one of the main open problems in the theory of secret sharing schemes. Finding a description of all access structures that can carry an ideal secret sharing scheme appeared to be quite difficult, and there has been two approaches to it.

The first approach, which may be called the top-down approach, employs Matroid Theory. A major milestone was achieved by Brickell & Davenport (1991) who showed that all ideal secret sharing schemes have matroids corresponding to them. Furthermore, matroids are either representable or not (see Oxley (1992) for definitions), and it has been found, also by Brickell & Davenport (1991), that all representable matroids realise ideal access structures. As for the nonrepresentable matroids, it was shown that in at least one case, the Vamos matroid, does not realise an ideal access structure (Seymour (1992)). So this approach is reduced to classifying those matroids that do realise ideal access structures. The papers by Beimel (2011) and Martí-Farré & Padró (2007) give a nice overview and discussion of the problem, and the developments made therein.

The other approach, which is the bottom-up approach, aims at studying and characterising ideal access structures in different known particular classes of secret sharing schemes: ideal weighted access structures, ideal roughly weighted access structures, ideal complete access structures etc. This approach is the focus of this thesis, and in this important direction, many successful attempts have been made, including the characterisations of graph based ideal access structures (Brickell & Davenport, 1991), bipartite access structures (Padró & Sáez, 1998), tripartite access structures (Collins, 2002), hierarchical access structures (Brickell (1989); Tassa (2007)), multipartite access structures (Farras et al., 2007), weighted access structures (Beimel, Tassa, & Weinreb, 2008), complete access structures (Farràs & Padró, 2010). Of great relevance to the work presented in this dissertation, is the study of ideal weighted access structures. The first significant advance in the study of ideal weighted access structures was achieved by Beimel, Tassa, & Weinreb (2008). They found that weighted ideal access structures are either composed of two smaller weighted ideal access structures, or they are indecomposable. In the latter case, those access structures belong to either of the two families of access structures called 'hierarchical' and 'tripartite'. Hierarchical access structures are of two types, disjunctive and conjunctive, both of which were previously shown to be ideal by Brickell (1989) and Tassa (2007) respectively. Tripartite access structures were also shown to be ideal in Beimel, Tassa, & Weinreb (2008). Moreover, Beimel, Tassa, & Weinreb (2008) classified those access structures that are both disjunctive hierarchical and weighted, this result was further improved upon by Gvozdeva, Hameed, & Slinko (2013). Farràs & Padró (2010) refined the results obtained by Beimel, Tassa, & Weinreb (2008). They gave a characterisation of ideal access structures in the class of complete access structures (which includes weighted access structures), and then using this result, Farràs & Padró (2010) gave a list of indecomposable ideal weighted access structures. In particular, they classified all indecomposable weighted hierarchical and tripartite access structures.

To summarise, Beimel, Tassa, & Weinreb (2008) and Farràs & Padró (2010) have shown that an ideal weighted access structure is either decomposable, or a hierarchical access structure, or a tripartite access structure. For a complete characterisation of all ideal weighted access structures, it remained to be seen what the necessary and sufficient conditions are for the composition of two ideal weighted access structures to be also ideal weighted access structure. One of our main contributions in this thesis is giving an 'if and only if' characterisation theorem for all ideal weighted access structures (ideal weighted simple games). In the next section, we take a closer look at the results of Beimel, Tassa, & Weinreb (2008) and Farràs & Padró (2010), since that will enable us to explain our results better.

1.1.3 Ideal weighted secret sharing schemes: outline of existing results

The main results in this direction were obtained by Beimel, Tassa, & Weinreb (2008) and Farràs & Padró (2010). Let us first start with the main achievements in (Beimel, Tassa, & Weinreb, 2008), we shall state their theorem below, but it can be summarised as follows: any ideal weighted simple game is a composition of indecomposable ideal weighted simple games, such that an indecomposable ideal weighted simple game is either a hierarchical simple game of at most two nontrivial levels, or it is a tripartite simple game.

Their main theorem concerning the summary above is stated below, it may contain some unfamiliar terminology to the reader at this stage, but will become clearer in light of the work of the next few chapters. We have slightly modified the statement of their theorem to the terminology of simple games (since our thesis is game-theoretic), rather than secret sharing, in which it was originally stated.

Theorem 1.1.1. *((Beimel, Tassa, & Weinreb, 2008), Theorem 3.5) Let* Γ *be an ideal weighted simple game without dummies defined on a set of players* P*. Then:*

(i) it is a disjunctive hierarchical simple game of at most two nontrivial levels, or

(ii) it is a tripartite simple game, or

(iii) it is a composition of Γ_1 *and* Γ_2*, where* Γ_1 *and* Γ_2 *are ideal weighted simple games defined on sets of players smaller than* P*.*

Another important achievement they made in the same paper, is the characterisation of weighted simple games in the class of disjunctive hierarchical simple games (see Claim 6.5 in Beimel, Tassa, & Weinreb (2008)).

Few years later, the paper by Farràs & Padró (2010) gave more refined information about indecomposable ideal weighted simple games. They observed that not all hierarchical and tripartite games are indecomposable, and listed only the indecomposable ones. The main achievements of their paper can be summarised as follows: any ideal weighted simple game is a composition of indecomposable ideal weighted simple games, such that an indecomposable ideal weighted simple game is either a hierarchical simple game of one of four types, or it is a tripartite simple game of one of three types.

Here is their main theorem regarding this summary.

Theorem 1.1.2. *((Farràs & Padró, 2010), Theorem 10.1) Let Γ be an ideal weighted simple game without dummies. Then one of the following four conditions holds:*

(i) it is a simple majority game (a 1-level hierarchical simple game), or

(ii) it is a bipartite simple game in one of the types $\boldsymbol{B}_1$, $\boldsymbol{B}_2$ or $\boldsymbol{B}_3$ (2-level hierarchical simple games), or

(iii) it is a tripartite simple game in one of the types $\boldsymbol{T}_1$, $\boldsymbol{T}_2$ and $\boldsymbol{T}_3$, or

(iv) it is a composition of two smaller ideal weighted simple games.

We have again slightly modified their statement to the terminology of simple games, rather than secret sharing terminology.

Now we are ready to explain the work, and state the results of this thesis.

1.2 Our work in this thesis

Our results contribute to two closely related directions, both of which are ultimately related to the characterisation of ideal simple games:

(1) Complete characterisation of ideal weighted simple games;

(2) Classification of weighted and roughly weighted games within the classes of disjunctive and conjunctive hierarchical games.

The link between the two directions is provided by the theorems of Brickell (1989) and Tassa (2007) who respectively proved that disjunctive and conjunctive hierarchical games are ideal.

As mentioned earlier, from existing results we know that ideal weighted simple games are either hierarchical or tripartite or composed of smaller ideal simple games. Therefore we will dedicate a chapter for each one of those classes. We will carry out a thorough study of each one, starting with hierarchical simple games and ending with composed simple games. Then the information gathered will culminate in our characterisation theorem in Chapter 6, which characterises ideal weighted simple games.

Thereafter, we consider a bigger class of simple games beyond the class of weighted games, which is called roughly weighted games (see Taylor & Zwicker (1999), p.78 and Gvozdeva & Slinko (2011)). We made headway in the direction of characterising ideal roughly weighted games. Indeed, characterising roughly weighted hierarchical games is the first necessary step towards characterising ideal roughly weighted simple games. Our main contributions in this thesis are as follows, we break them into two parts, the first is for ideal weighted simple games, and the second is for ideal roughly weighted simple games.

1.2.1 Ideal weighted simple games

(a) As mentioned earlier, there are two types of hierarchical organisation, disjunctive and conjunctive, of which the latter received less attention in the literature than the former. We give canonical representations for both of them (Theorems 3.2.1 and 3.2.8). We then prove that the two types are duals of each other (Theorem 3.3.1). Also, we study the intersection of hierarchical games of both types with weighted games. Although, for the disjunctive case, this has been done by Beimel, Tassa, & Weinreb (2008), we give an alternative proof which is slightly more general than the existing one. We derive an analogous result for the conjunctive type by duality (Theorems 3.6.2 and 3.6.3).

(b) The class of tripartite simple games also has two types, and we give canonical representations for both types (Propositions 4.2.1 and 4.2.3). Also, we take a thorough look at the list of indecomposable weighted games in this class, which was given in Farràs & Padró (2010), and thereby show that one of the tripartite simple games in their list can in fact be decomposed.

(c) Although introduced in the foundational work by Shapley (1962) and reinvented by Martin (1993), the operation of composition of simple games has received little attention in the game-theoretic literature so far, so we investigate properties of it.

We first provide some insights about compositions and decompositions of simple games in general, such as the associativity of the operation of composition (Proposition 5.1.6), and that there is up to an isomorphism a unique way to compose two complete games such that the composed game is also complete (Lemma 5.2.1). We also demonstrate with an example that even if the two weighted games are ideal, then their composition may still not be weighted (Example 5.2.5), contrary to what has been implicitly assumed by several authors. Finally, in Chapter 6 we apply our knowledge of compositions to achieve our characterisation theorem stated below.

(d) The characterisation theorem shows that any ideal weighted simple game G is a composition of the form

$$G = H_1 \circ \ldots \circ H_s \circ I \circ A_n \ \ (s \geq 0);$$

where H_i is of type $\mathbf{H}$ for each $i = 1, \ldots, s$. Also, I, which is allowed to be absent, is an indecomposable game of types $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{T}_1$, $\mathbf{T}_{3a}$ and $\mathbf{T}_{3b}$, and A_n is the anti-unanimity game on n players. Moreover, A_n can be present only if I is either absent or it is of type $\mathbf{B}_2$; in the latter case the composition $I \circ A_n$ is over a player of the least desirable level of I. Also, the above decomposition is unique [2].

1.2.2 Ideal roughly weighted simple games

In Chapter 7, we take on the task of extending our characterisation of ideal weighted simple games to ideal roughly weighted simple games. The class of roughly weighted simple games is larger than the class of weighted simple games, and yet roughly weighted games retain many nice properties of weighted games (Gvozdeva & Slinko, 2011). So it is a natural progression to move from the study of ideal weighted games to the study of ideal roughly weighted games. This new problem is far from being solved in this thesis, however, we made good advances in this direction as will be explained shortly. We start by making it clear that roughly weighted games can be either complete or incomplete (Example 7.2.1), unlike weighted games which are necessarily complete. We then take a further step and show that even when the roughly weighted games are ideal, they may still be complete and incomplete, we give an example of an ideal incomplete nonweighted roughly weighted game, and we prove its ideality using Matroid Theory (Example 7.2.7). But we only concern ourselves for the rest of the thesis with studying ideal complete nonweighted roughly weighted games.

Let us now outline the general strategy that one might employ for classifying ideal roughly weighted games. This is the same strategy as was used by Beimel, Tassa, & Weinreb (2008).

(RW1) The first step is to try to answer the question: Is it true that any roughly weighted simple game is a composition of indecomposable roughly weighted games?

(RW2) Classify ideal indecomposable roughly weighted games in the classes of hierarchical and tripartite games.

[2] The types of games $\mathbf{H}$, $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{T}_1$, $\mathbf{T}_{3a}$ and $\mathbf{T}_{3b}$ will be described in full in Chapter 6.

(RW3) Which then leaves us with the question: Are there any ideal indecomposable roughly weighted games which are neither hierarchical nor tripartite?

And our accomplishments in this direction are as follows.

(a) In addressing (RW1), we give a result, Lemma 7.1.2, showing that, with few pathological exceptions, any roughly weighted game, regardless of whether it is ideal or not, decomposes into a number of indecomposable roughly weighted games. Hence, apart from a few pathological cases, an ideal roughly weighted game is a composition of indecomposable ideal roughly weighted games.

(b) The characterisation of roughly weighted hierarchical games. In addressing (RW2), we give an 'if and only if' characterisation of nonweighted roughly weighted hierarchical simple games, both disjunctive and conjunctive (Theorem 7.3.1 and 7.3.2). We discover that hierarchical games in this class can have up to three nontrivial levels, in contrast to weighted hierarchical games which can have only up to two nontrivial levels. This already establishes that the class of roughly weighted ideal simple games is larger than that of weighted ideal simple games. Also, we give an example of an indecomposable nonweighted roughly weighted tripartite simple game, thus also establishing that the class of roughly weighted tripartite games is larger than that of weighted tripartite simple games (Example 7.4.2).

(c) We then answer the question in (RW3) in the positive. We give an example of a 4-partite (of four seniority levels) roughly weighted simple game which is ideal but is neither hierarchical nor tripartite (Example 7.5.3). We prove that it is ideal with the help of a result from (Farràs & Padró, 2010), which characterises all complete ideal simple games (see page 164). We also prove its indecomposability.

    This leaves us with the open question: How many seniority levels can an ideal roughly weighted game have?

Let us now briefly describe the techniques used in this thesis.

1.2.3 Techniques used

Our study in this thesis is focused firstly on weighted games, and secondly on complete roughly weighted games, therefore all games under consideration (with a few exceptions) are complete. The main techniques used in the thesis are game-theoretic. They can be summarised as follows.

(a) Since all games being studied are complete, then on a foundational level, our approach to each game and all players of that game, is based on Isbell's desirability relation. This is because in a complete game, every two players are comparable in terms of Isbell's desirability relation, and hence the game as a whole can be divided into a number of equivalence classes, or desirability levels, each level containing players sharing the same Isbell desirability. Using these desirability orderings then, we can define shift-minimal winning coalitions and shift-maximal losing coalitions, and these play pivotal roles in the steps to follow. Isbell's desirability relation is also used in many places to work out canonical representations for games.

(b) The technique of trading transforms (Taylor & Zwicker, 1999) is used to work out which games are weighted and which ones are not. Employing this technique requires a list of shift-minimal winning coalitions. In particular, all cases where the composition of two games is not weighted were identified using trading transforms.

(c) Various combinatorial methods are used to establish canonical representations for games. For example, we identify the restrictions on the parameters of hierarchical games, necessary and sufficient, for all desirability levels to be nontrivial. The canonical representation then becomes one main tool. For example, it is used to prove that disjunctive and conjunctive hierarchical games are duals of each other.

(c) In Chapter 3, we prove structural characterisations of disjunctive and conjunctive hierarchical games. This structural characterisation shows that

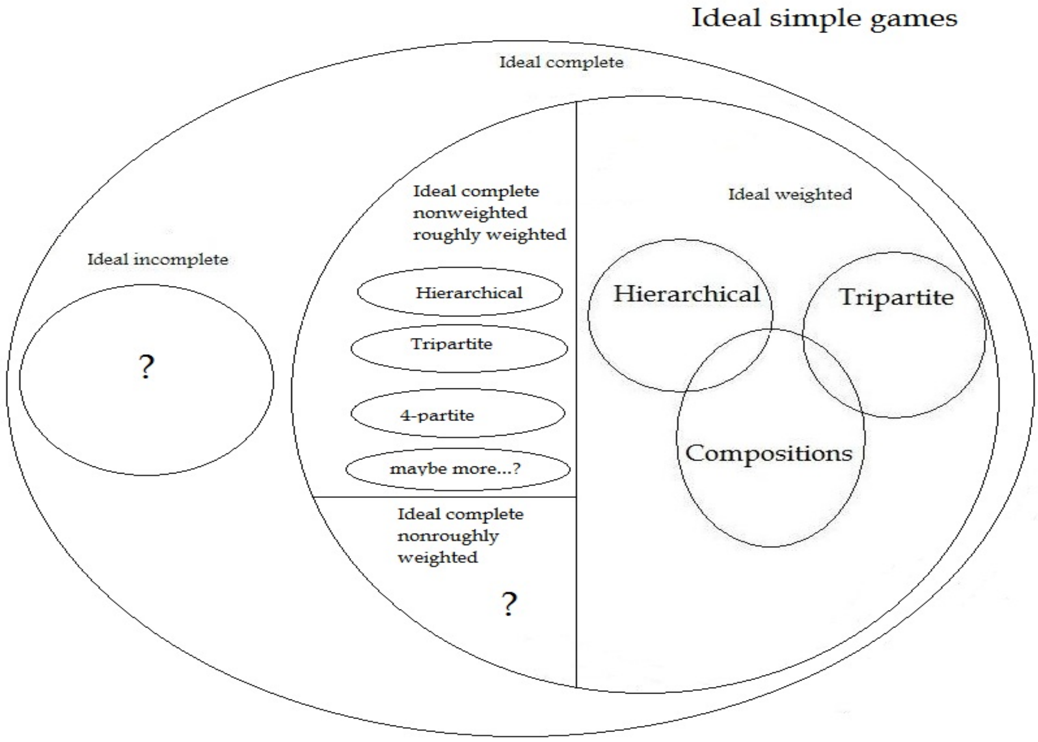

Figure 1.1: The main kinds of simple games considered in this thesis

a disjunctive hierarchical game is a complete game with a unique shift-maximal losing coalition, and a conjunctive hierarchical game is a complete game with a unique shift-minimal winning coalition. This becomes our main tool in Chapter 7, where we charactrise all nonweighted roughly weighted hierarchical games.

(d) Finally, in Chapter 7 we use Matroid Theory to prove the existence of an ideal incomplete nonweighted roughly weighted game.

Figure 1.1 gives an overview of the main simple games considered in this thesis.

The structure of the thesis is as follows:

Chapter 2 covers all the necessary basics for this thesis, both for simple games and secret sharing schemes. It is far from being an exhaustive reference for those two fields of study, but provides the reader with the necessary pre-requisites, and gives a general introduction to those two topics. For a more detailed reference on simple games, the reader is advised to look at (Taylor & Zwicker, 1999), and for secret sharing schemes (Stinson, 1992) and (Beimel, 2011), and the literature referenced there.

Chapter 3 discusses hierarchical simple games by firstly distinguishing its two types, their completeness, then establishing their canonical representations, the duality between the two types, structural characterisations and finally characterising their weightedness.

Chapter 4 is similar to Chapter 3 in its structure, but it is for tripartite simple games.

Chapter 5 contains results regarding the operation of composition of simple games, which are later used in characterising ideal weighted simple games.

In the first section of Chapter 6, we list the indecomposable ideal weighted simple games in details. And the remaining part of Chapter 6 is dedicated to proving one of the main results of this thesis: the characterisation theorem of ideal weighted games.

Chapter 7 contains material that makes advances in the direction of characterising ideal roughly weighted games that are not weighted. In particular, we fully characterise roughly weighted hierarchical games, both disjunctive and conjunctive.

Chapter 8 suggests few open problems and possibilities for future research.

Chapter 2

Preliminaries

2.1 Simple games in real life

A simple game is a mathematical object that is used in economics and political science to describe the distribution of power among coalitions of players (von Neumann & Morgenstern (1944), Shapley (1962)). Recently, simple games have been studied as access structures of secret sharing schemes (e.g., Padró et al. (2013); dela Cruz & Wang (2013); Farràs et al. (2012)), which will be discussed in Section 2.3. Simple games have also appeared in a variety of mathematical and computer science contexts under various names, e.g., monotone boolean functions (Korshunov, 2003) or switching functions and threshold functions (Muroga, 1971). Simple games are closely related to hypergraphs, coherent structures, Sperner systems, clutters, and abstract simplicial complexes. The term "simple" was introduced by von Neumann & Morgenstern (1944), because in this type of games players strive not for monetary rewards but for power, and each coalition is either all-powerful or completely ineffectual. However, these games are far from being simple.

A participant in a simple game is called *player*. An important class of simple games, well studied in economics, and a major topic in this thesis, is the class of weighted majority games which were briefly discussed in Chapter 1. However, it is well known that not every simple game has a representation as a weighted majority game (von Neumann & Morgenstern (1944)).

Let us now give a formal definition of a simple game. Throughout the thesis, we will denote the set $\{1, 2, \ldots, n\}$ as $[n]$. Also, for short, we may use the term game rather then simple game, and weighted game rather then weighted majority game.

Definition 2.1.1. Let $P = [n]$ be a set of players and let $\emptyset \neq W \subseteq 2^P$ be a collection of subsets of P that satisfies the following property, which is called the *monotonicity property*:

$$\text{if } X \in W \text{ and } X \subseteq Y, \text{ then } Y \in W. \tag{2.1.1}$$

In such case the pair $G = (P, W)$ is called a *simple game*, and the set W is called the set of *winning coalitions* of G. Coalitions that are not in W are called *losing*, and their set will be denoted by L. Also, if the removal of any player from a winning coalition makes the winning coalition losing, then it is called a *minimal winning coalition*.

Due to the property (2.1.1) the subset W is completely determined by the set W_{min} of minimal winning coalitions of G. A player who does not belong to any minimal winning coalition is called a *dummy*. Such player can be removed from any winning coalition without making it losing. We say that player p in a game is a *blocker* (also called a *veto* player, or a *vetoer*) if p belongs to every winning coalition. If all coalitions containing player p are winning, this player is called a *passer*. A blocker who is also a passer is called a *dictator*.

Let us look at few examples of simple games, in some of these examples we may use the word authorised instead of winning due to the nature of the example, but they mean the same thing.

Example 2.1.2. Let the set of players U consist of the various employees of a bank, where CEO stands for chief executive officer, GM stands for a general manager, and T stands for a teller. So that $U = \{CEO, GM_1, .., GM_3, T_1, .., T_5\}$.

Suppose that a certain major transaction can only be authorised by either the CEO, or any two general managers. Then the set of minimal authorised (winning) coalitions is $W_{\text{min}} = \{\{CEO\}, \{GM_1, GM_2\}, \{GM_1, GM_3\}, \{GM_2, GM_3\}\}$. So we can see that the CEO is a passer, and the tellers are dummies in this case.

What follows is another example of a simple game.

Example 2.1.3. The three top officials of some country, the president (P), the vice president (VP) and the minister of defense (MD) have nuclear briefcases, carried after them by security officers, so that any two of them can authorise a launch of a nuclear strike, but no one alone can do that. The set of players here is $U = \{P, VP, MD\}$. The set of minimal authorised coalitions is $W_{\text{min}} = \{\{P, VP\}, \{P, MD\}, \{VP, MD\}\}$. We can see that here we have no passers, and no dummies.

The simple game above is an example of a k-out-of-n game, meaning any k players from the total of n players in the game are winning, so the example above is a 2-out-of-3 one. This kind of a game is also called a *simple majority game*.

The next example is for the classic simple game of the United Nations Security Council (see for example Taylor & Zwicker (1999), p.9).

Example 2.1.4. Consider the United Nations Security Council voting on a resolution. There are 15 members of the Security Council, among which are five permanent members (PM), and 10 non-permanent members (NPM). A resolution is passed if nine members in total vote in favour of the resolution, including all five permanent members. So the set of players is $U = \{PM_1, .., PM_5, NPM_1, .., NPM_{10}\}$, and a typical minimal winning coalition will look like $\{PM_1, .., PM_5, NPM_1, .., NPM_4\}$.

The next example is the European Economic Community (see for example Taylor & Zwicker (1999), p.9).

Example 2.1.5. In 1958, the Treaty of Rome established the existence of a voting system called the European Economic Community. The voters in this system were the following six countries: France, Germany, Italy, Belgium, the Netherlands and Luxembourg. France, Germany and Italy were given four votes each, while Belgium and the Netherlands were given two votes and Luxembourg one. Passage required at least twelve of the seventeen votes. So in this case the minimal winning coalitions were $W_{\text{min}} = \{\{$France, Germany, Italy$\}, \{$France, Germany, Belgium, Netherlands$\}, \{$France, Italy, Belgium, Netherlands$\}, \{$Germany, Italy, Belgium, Netherlands$\}\}$. So we can see that Luxembourg was a dummy.

In Example 2.1.3, if we assign weight 1 to each of the P, VP and MD, and set the threshold to be 2, then we can see that it is a weighted majority game, since any two players have a combined weight of 2, which is equal to the threshold. But every player alone cannot win, since its weight is strictly less than the value of the threshold.

It is easy in Example 2.1.4 to see why this game is weighted: assign each permanent member weight 7, each non-permanent member weight 1, and set the threshold to be 39.

In Example 2.1.5, the game has been defined as a weighted game from the outset.

Let us end this section with an example of a nonweighted simple game (see Taylor & Zwicker (1999), p.10-11).

Example 2.1.6. (The System to Amend the Canadian Constitution). Since 1982, an amendment to the Canadian Constitution can become law only if it is approved by at least seven of the ten Canadian provinces, subject to the proviso that the approving provinces have, among them, at least half of Canada's population. A census (in percentages) for the Canadian provinces was: Prince Edward Island (PEI) had 1%, Newfoundland (New) had 3%, New Brunswick (NB) 3%, Nova Scotia (NV) 4%, Manitoba (Man) 5%, Saskatchewan (Sas) 5%, Alberta (Alb) 7%, British Columbia (BC) 9%, Quebec (Que) 29% and Ontario (Ont) had 34%. So the coalitions $S_1 = \{PEI, New, Man, Sas, Alb, BC, Que\}$, and $S_2 = \{NB, NS, Man, Sas, Alb, BC, Ont\}$ are minimal winning coalitions, because they both have exactly 7 provinces, and their populations surpass 50% of the total Canada's population. On the other hand, coalitions $T_1 = \{Man, Sas, Alb, BC, Que, Ont\}$ and $T_2 = \{PEI, New, NB, NS, Man, Sas, Alb, BC\}$ are both losing, because T_1 does not have 7 members and T_2 does not reach the 50% of the total Canada's population.

So we can see that the game above has two thresholds and two sets of weights. One threshold is 7 for the number of players in a coalition (provinces in this case), and the weight of each player in this case is 1. The other threshold is 50% (half of Canada's population in this case), such that the weight of each player here is the percentage of population of each province. This causes its nonweightedness (see Taylor & Zwicker (1999), p.10-11 for the details).

Next we look at some definitions and facts regarding important classes of simple games.

2.2 Properties, important classes and some operations in the theory of simple games

A distinctive feature of many games is that the set of players is partitioned into equivalence classes, and players in each of the equivalence classes have equal status. Meaning, in any coalition X (winning or losing) containing player x from some equivalence class E, x can be replaced with another player from E, without altering the fact that X was winning or losing. We formalise this in the next section.

2.2.1 Seniority of players and Multisets

Given a simple game G we define a relation $\sim_G$ on P by setting $i \sim_G j$ if for every set $X \subseteq P$ not containing i and j

$$X \cup \{i\} \in W \iff X \cup \{j\} \in W.$$

Lemma 2.2.1. $\sim_G$ *is an equivalence relation.*

Proof. This is Proposition 3.2.4 in (Taylor & Zwicker, 1999). $\square$

Given a game G on a set of players P, we may also define a relation $\succeq_G$ on P by setting $i \succeq_G j$ if for every set $X \subseteq P$ not containing i and j

$$X \cup \{j\} \in W \Longrightarrow X \cup \{i\} \in W.$$

The above is saying that player i is at least as influential, or at least as senior, as player j. In such a case, in every winning coalition X containing j but not i, we can remove j and replace it with i, and still have a winning coalition.

This relation is known as *Isbell's desirability relation* (see Isbell (1958), Maschler & Peleg (1966) and Taylor & Zwicker (1999)). We also define the relation $i \succ_G j$ as $i \succeq_G j$ but not $j \succeq_G i$. If $i \succ_G j$ then we say that the player i is *strictly more desirable* than player j. Coalitions prefer to have a player with higher desirability in their ranks than a player with a lower desirability, since having such a player will increase their chances of winning.

It is easy to see that Isbell's desirability relation is reflexive, and it is not difficult to show that it is transitive (see Taylor & Zwicker (1999), p.89-90).

We suggest analysing games with large classes of equivalent players with the help of multisets.

Definition 2.2.2. A *multiset* on the set $[m]$ is a mapping $\mu\colon [m] \to Z_+$ of $[m]$ into the set of nonnegative integers. It is often written in the form

$$\mu = \{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\} \tag{2.2.1}$$

where $k_i = \mu(i)$. This number is called the *multiplicity* of i in μ.

A multiset $\nu = \{1^{j_1}, \ldots, m^{j_m}\}$ is a *submultiset* of a multiset μ given in (2.2.1), iff $j_i \leq k_i$ for all $i \in [m]$. This is denoted as $\nu \subseteq \mu$.

The existence of large equivalence classes relative to $\sim_G$ allows us to compress the information about the game. This is done by the following construction.

Let now $G = (P, W)$ be a game and $\sim_G$ be its corresponding equivalence relation. Then P can be partitioned into a finite number of equivalence classes $P = P_1 \cup P_2 \cup \ldots \cup P_m$ relative to $\sim_G$ and suppose that $|P_i| = n_i$. Then we put in correspondence to the set of players P a multiset $\bar{P} = \{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}$. By doing so we identify those players in P that are equivalent and do not distinguish between them any further. We carry over the game structure to $\bar{P}$ by defining the set of submultisets $\overline{W} \subseteq 2^{\bar{P}}$ as follows: we assume that a submultiset $Q = \{1^{\ell_1}, 2^{\ell_2}, \ldots, m^{\ell_m}\}$ belongs to $\overline{W}$ if a subset of P containing ℓ_i players from P_i ($i = 1, 2, \ldots, m$), is winning in G. This definition is correct since the sets P_i are defined in such a way that it does not matter which ℓ_i players from P_i are involved. We will call $\bar{G} = (\bar{P}, \overline{W})$ the *multiset representation* of G.

As an illustration of the multiset representation and equivalence classes, recall Example 2.1.4 of the United Nations Security Council. In this game, we can see that in a winning coalition, a permanent member cannot be replaced by a non-permanent member, meaning permanent members are more senior. So the game of the United Nations Security Council could be written as $\bar{G} = (\bar{P}, \overline{W})$, where $\bar{P} = \{1^5, 2^{10}\}$, where equivalence class 1 is for permanent members, and equivalence class 2 is for non-permanent members. Also, $\overline{W}_{min} = \{\{1^5, 2^4\}\}$. Note how the multiset representation allows us to describe the winning coalitions in a compact form, for example 2^4 in $\{1^5, 2^4\}$ refers to any four members from equivalence class 2, which is more convenient than listing all combinations of four members from equivalence class 2.

Definition 2.2.3. A pair $\bar{G} = (\bar{P}, \overline{W})$ where $\bar{P} = \{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}$ and $\overline{W}$ is a system of submultisets of the multiset $\bar{P}$ is said to be a *simple game* on $\bar{P}$ if

$$\text{for any } X \in \overline{W} \text{ and any } X \subseteq Y \text{ it holds that } Y \in \overline{W}.$$

So the multiset representation of a simple game on a set of players P is a simple game on the multiset $\bar{P}$.

2.2.2 Complete simple games

A game G is called *complete* if $\succeq_G$ is a total preorder, i.e., reflexive, transitive relation for which every two players i and j are comparable with respect to $\succeq_G$, meaning either $i \succeq_G j$ or $j \succeq_G i$. But Isbell's desirability relation is reflexive and transitive, so in order to show that a game is complete, it is enough to show that every two players are comparable. Since $\sim_G$ is an equivalence relation, then a game is split into classes of equivalent players. For a complete game G we always enumerate equivalence classes so that $x \succ_G y$ if and only if $x \in P_i$ and $y \in P_j$ with $i < j$ (P_i and P_j are the equivalence classes from the previous section). Then in $\bar{G}$ we have $1 \succ_{\bar{G}} 2 \succ_{\bar{G}} \ldots \succ_{\bar{G}} m$, meaning players of equivalence class 1, or level 1, are more desirable than players of level 2 and so on.

In Example 2.1.3 the simple game had only one desirability level, hence it is complete. So the multiset of players can be written as $U = \{1^3\}$. Then the set of minimal authorised coalitions is $W_{\text{min}} = \{\{1^2\}\}$. As explained earlier, this is a 2-out-of-3 game.

Two simple games (P, W) and (P', W') are said to be isomorphic if there exists a bijection $\tau\colon P \to P'$ such that $X \in W$ if and only if $\tau(X) \in W'$. Isomorphisms preserve Isbell's desirability relation (Carreras & Freixas, 1996).

The following statement is immediate.

Proposition 2.2.4. *Two complete games are isomorphic if and only if they have the same multiset representation.*

This will be very useful later on when we deal with the multiset representations of hierarchical games, instead of the games themselves, so we can study them up to an isomorphism.

Other important notions that relate to Isbell's desirability relation, and that we will use extensively in this thesis, are those of a shift, a shift-minimal winning coalition and a shift-maximal losing coalition. Suppose a game G is complete. By a *shift* we mean a replacement of a player of a coalition by a less influential player which did not belong to it. Let us formalise it.

Definition 2.2.5. Let G be a complete simple game on a multiset $P = \{1^{n_1}, \dots, m^{n_m}\}$. Suppose a submultiset $A = \{\dots, i^{\ell_i}, \dots, j^{\ell_j}, \dots\}$ has $\ell_i \geq 1$ and $\ell_j < n_j$ for some $i < j$. We say that the submultiset $A' = \{\dots, i^{\ell_i - 1}, \dots, j^{\ell_j + 1}, \dots\}$ is obtained from A by a shift $i \mapsto j$.

Also, a winning coalition X is *shift-minimal* if every coalition contained in it and every coalition obtained from it by a shift is losing. A losing coalition Y is said to be *shift-maximal* if every coalition that contains it is winning and there does not exist another losing coalition from which Y can be obtained by a shift. These concepts can be immediately reformulated for games on multisets.

Example 2.2.6. In the United Nations Security Council example, the winning coalition $\{1^5, 2^4\}$ is a shift-minimal winning coalition, since any proper set of $\{1^5, 2^4\}$ is losing, and a single shift will produce the coalition $\{1^4, 2^5\}$ which is also losing.

Observe that in a complete game, since every two players are comparable, then any shift-minimal winning coalition can be obtained from some minimal winning coalition by one or more shifts, hence in a complete game shift-minimal winning coalitions fully determine the game.

A very important subclass of complete simple games is the class of weighted simple games mentioned earlier, in the next section we state its formal definition and notations.

2.2.3 Weighted simple games

Definition 2.2.7. A simple game $G = (P, W)$ is called a *weighted majority game* if there exist nonnegative real numbers $w_1, ..., w_n$, called *weights*, and a nonnegative real number q, called the *threshold*, such that

$$X \in W \Longleftrightarrow \sum_{i \in X} w_i \geq q.$$

We use the notation $[q; w_1, \dots, w_n]$ to describe the system of weights and threshold for a given weighted simple game (WSG). The weighted game of the United Nations Security Council can be represented as $[39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$. Also, the game of the European Economic Community is a weighted simple game with the representation $[12; 4, 4, 4, 2, 2, 1]$.

Observe that in a weighted game, if two players have the same weight, then they have the same desirability, and if one has weight greater than the other, then the former is at least as desirable as the latter. Hence every weighted game is necessarily complete, since every two players are comparable with each other in terms of weights, and so they are comparable in terms of desirabilities as well.

In secret sharing, weighted threshold access structures, introduced in (Shamir, 1979), are equivalent to weighted majority games.

Definition 2.2.8. We say that $\bar{G} = (\bar{P}, \overline{W})$ is a weighted majority game if there exist non-negative weights $w_1, ..., w_m$ and $q \geq 0$ such that a multiset $Q = \{1^{\ell_1}, 2^{\ell_2}, \dots, m^{\ell_m}\}$ is winning iff $\sum_{i=1}^{m} \ell_i w_i \geq q$.

If G is weighted, then it is well-known (see, e.g., Taylor & Zwicker, 1999, p.91) that we can find a weighted representation for which equivalent players have equal weights. Hence we obtain

Proposition 2.2.9. *A simple game $G = (P, W)$ is a weighted majority game if and only if the corresponding simple game $\bar{G} = (\bar{P}, \overline{W})$ is.*

Now we state some very useful facts about how to determine if a simple game is weighted or not.

The sequence of an even number $2j$ of coalitions

$$\mathcal{T} = (X_1, \dots, X_j; Y_1, \dots, Y_j) \tag{2.2.2}$$

is called a *trading transform* if the first set of j coalitions $X_1, \dots, X_j$ can be converted into the second set of j coalitions $Y_1, \dots, Y_j$ by rearranging players. In other words, for any player p the cardinality of the set $\{i \mid p \in X_i\}$ is the same as the cardinality of the set $\{i \mid p \in Y_i\}$.

Theorem 2.2.10. *(Taylor & Zwicker, 1999) A game $G = (P, W)$ is a weighted majority game if for no j does there exist a trading transform (2.2.2) such that $X_1, \ldots, X_j$ are winning and $Y_1, \ldots, Y_j$ are losing.*

Note that the converse of the statement above is also true, that is to say if a game is weighted then there does not exist a trading transform. This theorem allows to prove the existence of weights for a given game by means of a combinatorial argument. The following is an example of nonweightedness proved by spotting a trading transform.

Example 2.2.11. In a banking situation, in order to authorise a money order, you need the signatures of either two general managers, or four tellers, or one general manager and three tellers. So here we have two levels of seniorities, or desirabilities, the higher level of general managers L_1, and the lower level of tellers L_2. So suppose the multiset of players is $P = \{1^3, 2^5\}$, then the set of minimal winning coalitions is $W_{min} = \{\{1^2\}, \{2^4\}, \{1, 2^3\}\}$. Now, since $\{1^2\}$ and $\{2^4\}$ are winning, and $\{1, 2^2\}$ is losing, then we have the following trading transform, which shows that our game at hand is nonweighted.

$$(\{1^2\}, \{2^4\}; \{1, 2^2\}, \{1, 2^2\}). \tag{2.2.3}$$

Definition 2.2.12. (Gvozdeva & Slinko, 2011) Let $G = (P, W)$ be a simple game. A trading transform (2.2.2) where all $X_1, \ldots, X_j$ are winning in G and all $Y_1, \ldots, Y_j$ are losing in G is called *certificate of nonweightedness* for G.

For complete games the criterion can be made easier to use, by the following result.

Theorem 2.2.13. *(Freixas & Molinero, 2009) A complete game is a weighted majority game if and only if it does not have a certificate of nonweightedness (2.2.2) such that $X_1, \ldots, X_j$ are shift-minimal winning coalitions.*

Completeness can also be characterized in terms of the following trading transform.

Theorem 2.2.14. *(Taylor & Zwicker, 1999) A game G is complete if no certificate of nonweightedness exists of the form*

$$\mathcal{T} = (X \cup \{x\}, Y \cup \{y\}; X \cup \{y\}, Y \cup \{x\}).$$

This theorem says that completeness is equivalent to the impossibility for two winning coalitions to swap two players and become both losing. This property is also called *swap robustness*, and the above certificate is called a *certificate of incompleteness*.

2.2.4 Roughly weighted simple games

The second most important class that we look at in this thesis, which is larger than the class of weighted simple games, is that of roughly weighted simple games (RWSG), its formal definition is the following.

Definition 2.2.15. A simple game G is called *roughly weighted* if there exist non-negative real numbers $w_1, \ldots, w_n$ and a real number q, called the *quota*, not all equal to zero, such that for $X \in 2^P$ the condition $\sum_{i \in X} w_i < q$ implies X is losing, and $\sum_{i \in X} w_i > q$ implies X is winning. We say that $[q; w_1, \ldots, w_n]$ is a *rough voting representation* for G.

So in a roughly weighted game nothing can be said about coalitions whose weight is equal to the threshold. There can be both winning and losing ones. This concept proved to be useful as demonstrated by Taylor & Zwicker (1999). This concept realizes a very common idea in social choice that sometimes a decision rule needs an additional 'tie-breaking' procedure that helps to decide the outcome if the result falls on a certain 'threshold'.

This definition is based on a fundamental idea, rediscovered by Taylor & Zwicker (1999), but dating back to the unpublished Ph.D. thesis of Irving Gabelman (Gabelman, 1961). A *magic square* is a $k \times k$ matrix of integers for which there is a constant p, such that rows and columns sum up to p[1]. The idea is to use a magic square to construct a nonweighted game. Let us illustrate with the following example (see Taylor & Zwicker (1999), p.12).

Example 2.2.16. Consider the two following 3×3 matrices. The one on the left, that we call M, is for 9 players, and the one on the right is for their corresponding weights, such that the weight of each player is the integer in the position corresponding to the position of the player in the left matrix. The matrix on the right is a magic square with a constant sum $p = 15$:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix}$$

Using M, we construct a nine-player simple game G_M as follows: It has 9 players in total, their weights as explained above, and the quota is 15. Every coalition with total weight greater than 15 is defined to be winning, and every coalition with total weight less than 15 is defined to be losing. A coalition with total weight being exactly 15 is declared to be losing, unless it is a row, in which case we declare it to be winning. This game, with its rough voting representation $[15; 4, 9, 2, 3, 5, 7, 8, 1, 6]$, fits the Definition 2.2.15 of rough weightedness. Also, in the following certificate of nonweightedness, we transform the rows (winning coalitions) into the columns (losing coalitions), and hence we verify the game's nonweightedness according to Theorem 2.2.10:

$$(\{4, 9, 2\}, \{3, 5, 7\}, \{8, 1, 6\}; \{4, 3, 8\}, \{9, 5, 1\}, \{2, 7, 6\}).$$

Another simple example is the banking situation we encountered earlier in Example 2.2.11, which was shown to be nonweighted by the certificate (2.2.3). Recall that the set of minimal winning coalitions was $W_{min} = \{\{1^2\}, \{2^4\}, \{1, 2^3\}\}$. We assign weights to players as follows: the general managers are assigned weight

[1]Usually it is also required that both diagonals sum up to p as well, but we do not require this in our discussion here. Also, a magic square is *rigid* if each set S of entries that sums to p appears as either a row or a column.

of $\frac{1}{2}$ each, and the tellers are assigned weight of $\frac{1}{4}$ each, and the quota is set to be 1. It is easy to check that this game is roughly weighted with the rough voting representation $[1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$.

For deciding which games are roughly weighted and which ones are not, we have a similar concept to that of a certificate of nonweightedness.

Definition 2.2.17. A certificate of nonweightedness

$$\mathcal{T} = (X_1, \ldots, X_j, P; Y_1, \ldots, Y_j, \emptyset) \tag{2.2.4}$$

which contains the set of players P among winning coalitions and $\emptyset$ among losing ones is called a *potent* certificate of nonweightedness.

The following criterion of rough weightedness exists.

Theorem 2.2.18 (Gvozdeva & Slinko (2011))**.** *The game G with n players is roughly weighted iff for no positive integer j there exist a potent certificate of nonweightedness of length j.*

The paper by Gvozdeva & Slinko (2011) has discussions and many nice examples of games that are not roughly weighted. The following is one of them.

Example 2.2.19. We define the game $G = (P, W)$, where $P = \{1, 2, 3, 4, 5\}$ and the set of minimal winning coalitions $W_{min} = \{\{1, 2\}, \{3, 4, 5\}\}$. Then the trading transform

$$(\{1, 2\}^5, \{3, 4, 5\}^7, P; \{2, 3, 5\}^4, \{2, 3, 4\}^2, \{1, 3, 4\}^2, \{1, 4, 5\}^4, \emptyset)$$

is a potent certificate of nonweightedness. Indeed, all four coalitions $\{2, 3, 5\}$, $\{2, 3, 4\}, \{1, 3, 4\}, \{1, 4, 5\}$ are losing since they do not contain $\{1, 2\}$ or $\{3, 4, 5\}$.

Finally, we should mention briefly the observation that due to the fact that coalitions whose total weight is equal to the threshold can be winning or losing in a roughly weighted game, then it is not necessary that all roughly weighted games are complete. We can have either complete or incomplete roughly weighted games, but our focus in this thesis is on simple games that are complete. We shall return to and discuss this point in more details in Chapter 7.

Next, we describe the concept of duality in simple games, which will be very important later in the thesis.

2.2.5 Duality

We have already encountered two kinds of coalitions, the winning ones and the losing ones. But there is another kind that we can consider, which also plays a role in the real world. If the set of players of a game is P, then a coalition X is said to be *blocking*, if its complement $X^c = P \setminus \{X\}$ is losing. The coalition X may not be winning itself, but it can prevent its complement from winning. Thus, X is a blocking coalition if it corresponds to a collection of players that can prevent an issue from being passed.

Example 2.2.20. (Taylor & Zwicker (1999), p.14) Let G be the simple game on the set P that is made up of two households, the first one we call $H_1 = \{Jane, John\}$, and the second one being $H_2 = \{Marry, Mike\}$. In this game, a coalition is winning if and only if its intersection with each household is nonempty. Then $\{Jane, Marry\}$ is winning, but it is not blocking because its complement $\{John, Mike\}$ is also winning. On the other hand, both $\{Jane, John\}$ and $\{Marry, Mike\}$ are blocking coalitions that are not winning.

Using the notion of blocking we have the following.

Definition 2.2.21. Associated with each simple game G, there is a corresponding simple game G^d, called the *dual* of G, such that the winning coalitions of G^d are the blocking coalitions of G.

The definition above says that if we let L to be the set of losing coalitions of G, then the dual game of a game $G = (P; W)$ is defined as $G^d = (P; L^c)$, and $L^c = \{X^c \in 2^P | X \in L\}$. In other words, the winning coalitions of the game G^d dual to G are exactly the complements of losing coalitions of G. Note that $(G^d)^d = G$ (see (Taylor & Zwicker, 1999, Proposition 1.3.7)). Moreover, Isbell's desirability relation is self-dual, that is $x \succeq_G y$ if and only if $x \succeq_{G^d} y$ (see (Taylor & Zwicker, 1999, Proposition 3.2.8)). If we define the complement X^c of a submultiset $X = \{1^{l_1}, \ldots, m^{l_m}\}$ in $P = \{1^{n_1}, \ldots, m^{n_m}\}$ as the submultiset $X^c = \{1^{n_1 - l_1}, \ldots, m^{n_m - l_m}\}$, then all the duality concepts can be immediately reformulated for the games on multisets. Also, the operation $G \mapsto G^d$ of taking the dual is known to preserve weightedness and rough weightedness (Taylor & Zwicker (1999), Proposition 4.10.1(i), p.166).

2.2.6 Subgames and Reduced Games

There are two natural substructures of simple games that game theorists have considered, that of a subgame, and that of a reduced game.

Definition 2.2.22. Let $G = (P, W)$ be a simple game and $A \subseteq P$. Let us define subsets

$$W_{\text{sg}} = \{X \subseteq A^c \mid X \in W\}, \quad W_{\text{rg}} = \{X \subseteq A^c \mid X \cup A \in W\}.$$

Then the game $G_A = (A^c, W_{\text{sg}})$ is called a *subgame* of G and $G^A = (A^c, W_{\text{rg}})$ is called a *reduced game* of G. We shall refer to subgames and reduced games as *minors*.

A useful duality fact that we shall use in Chapter 7 is that if $A \subset P$, then $(G_A)^d = (G^d)^A$ and $(G^A)^d = (G^d)_A$ (see (Taylor & Zwicker, 1999), Proposition 1.4.8).

Example 2.2.23. Consider the simple game $G = (P, W)$, such that $P = \{1^5, 2^6\}$, $W = \{\{1^4\}, \{2^5\}\}$. If we let $A = \{1^3\}$, then the subgame G_A will have the multiset $A^c = \{1^2, 2^6\}$, and the set of minimal winning coalitions $(W_{\text{sg}})_{min} = \{2^5\}$. Also, the reduced game $G^A = (A^c, W_{\text{sg}})$ will have the multiset $A^c = \{1^2, 2^6\}$, and the set of minimal winning coalitions $(W_{\text{rg}})_{min} = \{\{1\}, \{2^5\}\}$.

2.3 Basics of Secret Sharing Schemes

The idea of a secret sharing scheme can be illustrated by the following example. In a bank, there is a vault which must be opened every day. The bank employs three senior tellers; but it is not desirable to entrust the secret combination for opening the vault to any one person. Hence, we want to design a system whereby any two of the three senior tellers can gain access to the vault, but no individual can do so. This problem can be solved by means of a secret sharing scheme. Shamir (1979) and Blakley (1979) have independently constructed the first secret sharing scheme called a threshold scheme; the former used polynomial interpolation, and the latter used projective geometries. Let t, n be positive integers, $t \leq n$. Informally, a (t, n)-threshold scheme is a method of sharing a secret s among a finite set U of n participants that are called *users*, in such a way that any t users can compute s, but no group of $t - 1$ or fewer users can do so. The value of s is chosen by a special actor called the *dealer*, whose role may be performed by a computer. The dealer is denoted by D and we assume $D \notin U$. When D wants to share s among the participants in U, he gives each user a piece of information called a *share*. The shares should be distributed in private, so no participant knows the share given to another user. At a later time, a subset $B \subseteq U$ of users will pool their shares in an attempt to compute the secret. If $|B| \geq t$, then they should be able to compute s as a function of the shares they collectively hold; if $|B| < t$, then they should not be able to compute s. So in the bank vault example above, we desire a $(2, 3)$-threshold scheme.

In practice, some other secret sharing schemes may be needed in which the number of users is not the only factor determining if those users can work out the secret or not, but it can be, for example, a combination of the number of users as well as their seniority.

Let us first define formally what an access structure is.

Definition 2.3.1. Let $U = \{u_1, \ldots, u_n\}$ be a set of users. An *access structure* is a monotone collection $\emptyset \neq \Gamma \subseteq 2^U$ of non-empty subsets of U, that is $B \in \Gamma$ and

$B \subseteq C$ implies $C \in \Gamma$. Sets in Γ are called *authorised*, and sets not in Γ are called *unauthorised*.

Associated with an access structure are the following terms. A set B is called a *minterm* of Γ if $B \in \Gamma$, and for every $C \subsetneq B$, the set C is unauthorised. A user u is called *self-sufficient* if $\{u\} \in \Gamma$. A user is called *redundant* if there is no minterm that contains it. An access structure is called *connected* if it has no redundant users.

To illustrate, consider Example 2.1.2 from page 30. The set of minterms, denoted by $\Gamma_{min} = \{\{CEO\}, \{GM_1, GM_2\}, \{GM_1, GM_3\}, \{GM_2, GM_3\}\}$. So the CEO is self-sufficient, and the tellers are redundant.

Before we proceed with the formal definitions and technical results, we note that the two theories of secret sharing schemes and simple games have intersections between them, and hence similar concepts, but they use different terminologies. It will be most helpful, therefore, to put the two terminologies into a one-to-one correspondence, as shown in Table 2.3. Also, note that the secret sharing schemes terminology used here, as well as the definitions, can be found in the literature such as (Shamir, 1979), (Stinson, 1992) and (Beimel, Tassa, & Weinreb, 2008).

We will now give a formal definition of a secret sharing scheme.

Definition 2.3.2. (Secret Sharing Scheme) Let $U = \{1, 2, \ldots, n\}$ be the set of users, and let $S_0, S_1, \ldots, S_n$ be finite sets where S_0 is the set of all possible secrets. Any subset

$$\mathcal{T} \subseteq S_0 \times S_1 \times \ldots \times S_n$$

is called a *distribution table*. If a secret $s_0 \in S_0$ is to be distributed among agents, then an n-tuple

$$(s_0, s_1, \ldots, s_n) \in \mathcal{T}$$

is chosen by the dealer at random uniformly among those tuples whose first coordinate is s_0, and then agent i gets the share $s_i \in S_i$. A *secret sharing scheme* is a

Simple Games	Secret Sharing Schemes
simple game	access structure
simple majority game	threshold access structure
weighted majority game	weighted threshold access structure
player	user
coalition	group of users
winning	authorised
losing	unauthorised
minimal winning coalition	minterm
passer	self-sufficient
dummy	redundant

Table : Terminology of the two theories compared

quadruple $\mathcal{S} = (U, W, \mathcal{T}, (f_X)_{X\in W})$, where U is the set of users, $W \subseteq 2^U$ is an access structure, $\mathcal{T}$ is a distribution table and for every $X = \{u_1, \ldots, u_k\} \in W$

$$f_X \colon S_{u_1} \times \ldots \times S_{u_k} \to S_0$$

is a function (algorithm) which satisfies $f_X(s_{u_1}, s_{u_2}, \ldots, s_{u_k}) = s_0$ for every $(s_0, s_1, \ldots, s_n) \in \mathcal{T}$. The family $(f_X)_{X\in W}$ is said to be the *family of secret recovery functions*. We note that, if $X = \{u_1, \ldots, u_k\} \in W$, in the distribution table there cannot be tuples $(s, \ldots, s_{u_1}, \ldots, s_{u_2}, \ldots, s_{u_k}, \ldots)$ with $s \neq s_0$.

Note that the distribution table, and hence the access structure, are public knowledge.

It follows from the definition above that if an authorised subset $X = \{j_1, \ldots, j_k\}$ of users pool their shares, then they can determine the secret applying the corresponding secret recovery function to their shares, i.e., $f_X(s_{j_1}, \ldots, s_{j_k}) = s_0$, whereas an unauthorised subset cannot. Furthermore, in addition to the aforementioned fact, if the secret sharing scheme is such that an unauthorised subset of users can get no information about the secret, then the scheme is called *perfect*. For a perfect scheme, the length of each share in bits cannot be shorter than the length of the secret (Karnin et al., 1983).

We now state a few more definitions, and also include few examples about secret sharing schemes. What follows are two examples of secret sharing schemes, the first example is for a perfect secret sharing scheme, and the second one is for an imperfect one. Also, for more discussion of the basics of secret sharing schemes, the reader is advised to look at (Stinson, 1992).

Example 2.3.3. Consider the secret sharing scheme with n users, such that the only authorised coalition will be the full set of users $U = \{1, 2, \ldots, n\}$. We take a sufficiently large field $\mathbb{F}$ and set $S_0 = \mathbb{F}$, so that it is infeasible to try all secrets one by one. We will also have $S_i = \mathbb{F}$ for all $i = 1, \ldots, n$. Given a secret $s_0 \in \mathbb{F}$ to share, the dealer generates $n-1$ random elements $s_1, \ldots, s_{n-1} \in \mathbb{F}$, and calculates $s_n = s_0 - (s_1 + \ldots + s_{n-1})$. Then he gives share s_i to user i. For all possible n-tuples $(s_0, s_1, \ldots, s_n)$ we will have $\sum_{i=1}^{n} s_i = s_0$, and the secret recovery function (in this case the only one) will be $f_U(s_1, \ldots, s_n) = s_1 + \ldots + s_n$.

And for the imperfect one we have the following.

Example 2.3.4. Let x_a and x_b be two relatively prime integers known to the two users a and b respectively, and let the domain of secrets be $\{0, \ldots, x_a \cdot x_b\} \subseteq \mathbb{Z}$, so the secret s is any integer between 0 and $x_a \cdot x_b$. We give share $s_a = s \bmod x_a$ to user a, and we give share $s_b = s \bmod x_b$ to user b. The only secret recovery function here is the algorithm based on the Chinese Remainder Theorem and the Euclidean algorithm. So a and b together can combine their shares and work out the secret. Now, even though each player alone cannot work out the secret, each of them has some partial information about it. It is easier for a or b to guess the secret than it is for a person from the street.

It was mentioned earlier that in every perfect secret sharing scheme, the length of each share in bits cannot be shorter then the length of the secret. This motivates the next definition.

Definition 2.3.5. (Ideal Access Structure) A secret sharing scheme with domain of secrets S is *ideal*, if it is perfect and the length of each share in bits is equal to

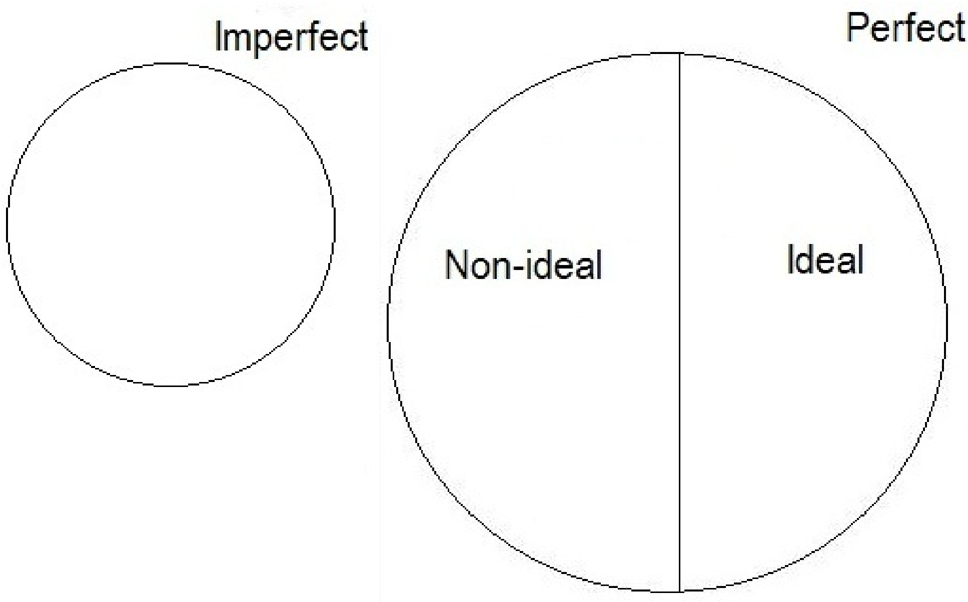

Figure 2.1: Perfect and imperfect secret sharing schemes

the length of the secret. An access structure Γ is said to be ideal if for some finite domain of secrets S, there exists an ideal secret sharing scheme realising it.

Example 2.3.6. Example 2.3.3 was of a perfect secret sharing scheme where the domain of the secrets was the same as the domains of the shares, meaning the lengths of the shares and the length of the secret are equal, so the scheme was ideal.

The definition above is for ideal access structures, but recall from the introduction that an access structure is a simple game, and keeping this in mind will clarify the concept of an ideal simple game. Since it is not possible to explain ideality with a purely game-theoretic perspective, a basic appreciation of secret sharing is required for understanding what an ideal simple game is. Also, in what follows we state an important theorem, which will be used in this thesis, that characterises ideal simple games in terms of their shift-minimal winning coalitions, it will help prepare the reader for the work to follow on ideal simple games.

Theorem 2.3.7 (Farràs & Padró (2010), Theorem 9.2)**.** *Let* Γ *be a complete simple game on* $P = \{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}$*. Also, let the set of shift-minimal winning coalitions be* $\{X_1, \ldots, X_r\}$*. Consider* $m_j =$ *max(supp*(X_j)*)*$, 1 \leq j \leq r$*, and suppose that the shift-minimal winning coalitions are listed in such a way that* $m_j \leq m_{j+1}$*. Then* Γ *is ideal if and only if*

(i) $m_j < m_{j+1}$ *and* $|X_j| < |X_{j+1}|$ *for all* $j = 1, \ldots, r-1$*, and*

(ii) $X_j^i \geq X_{j+1}^i$ *if* $1 \leq j \leq r-1$ *and* $1 \leq i \leq m_j$*, and*

(iii) if $X_j^i > X_r^i$ *for some* $1 \leq j < r$ *and* $1 \leq i < m_j$*, then* $n_k = X_j^k$ *for all* $k = i+1, \ldots, m_j$*.*

Finally, we end this section with another example of a secret sharing scheme, it is the famous Shamir scheme, which is known to be ideal. Shamir's scheme, realises, or implements, the kind of access structure known as *threshold access structure*, or k-out-of-n threshold access structure mentioned earlier, where at least k out of n users are needed to determine the secret.

Example 2.3.8 (Shamir, 1979)**.** Suppose that we have n users and the access structure is $W = \{X \subseteq A \mid |X| \geq k\}$, i.e. a coalition is authorised if it contains at least k users. Let $\mathbb{F}$ be a large finite field and we will have $S_i = \mathbb{F}$ for $i = 0, 1, 2, \ldots, n$. Let $a_1, \ldots, a_n$ be distinct fixed nonzero elements of $\mathbb{F}$, these are publicly known.

Suppose $s \in \mathbb{F}$ is the secret to share. The dealer sets $t_0 = s_0$ and generates randomly $t_1, \ldots, t_{k-1} \in \mathbb{F}$. He forms the polynomial $p(x) = t_0 + t_1 x + \ldots + t_{k-1}x^{k-1}$. Then he gives share $s_i = p(a_i)$ to user i (note that $a_0 = 0$ and $p(0) = s_0 = s$).

Suppose now $X = \{u_1, \ldots, u_k\}$ is a minimal authorised coalition. Then the secret recovery function is

$$f_X(s_{u_1}, \ldots, s_{u_k}) = \sum_{r=1}^{k} s_{u_r} \frac{(-a_{u_1}) \ldots \widehat{(-a_{u_r})} \ldots (-a_{u_k})}{(a_{u_r} - a_{u_1}) \ldots \widehat{(a_{u_r} - a_{u_r})} \ldots (a_{u_r} - a_{u_k})},$$

where the hat over the term means its non-existence. This is the value at zero of the Lagrange's interpolation polynomial

$$\sum_{r=1}^{k} p(a_{u_r}) \frac{(x-a_{u_1})\dots\widehat{(x-a_{u_r})}\dots(x-a_{u_k})}{(a_{u_r}-a_{u_1})\dots\widehat{(a_{u_r}-a_{u_r})}\dots(a_{u_r}-a_{u_k})},$$

which is equal to $p(x)$. We can write

$$\begin{bmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{k-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{k-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{k-1} \end{bmatrix} \begin{bmatrix} t_0 \\ t_1 \\ \vdots \\ t_{k-1} \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \tag{2.3.1}$$

Since all $a_1, \dots, a_n$ are different, any k rows of the matrix in (2.3.1) are linearly independent (and the corresponding determinant of the resulting $k \times k$ matrix is the well-known Vandermonde determinant). This is why any k users can learn all coefficients of $p(x)$, including its constant term (which is the secret) by solving the corresponding system of linear equations. We can write (2.3.1) as

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & a_1 & a_1^2 & \dots & a_1^{k-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{k-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{k-1} \end{bmatrix} \begin{bmatrix} t_0 \\ t_1 \\ \vdots \\ t_{k-1} \end{bmatrix} = \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_n. \end{bmatrix} \tag{2.3.2}$$

adding a new row (the dealer's row). Let us write it in matrix form as $H\mathrm{t} = \mathrm{s}$ and denote the rows of H as $\mathbf{h}_0, \mathbf{h}_1, \dots, \mathbf{h}_n$. Then the following is true: the span of a group of rows $\{\mathbf{h}_{j_1}, \dots, \mathbf{h}_{j_r}\}$ contains $\mathbf{h}_0$ if and only if $r \geq k$. So it follows that if the dealers row can be written as the linear combination $h_0 = \alpha_1 h_{j_1} + \dots + \alpha_r h_{j_r}$, then the secret can also be written as the linear combination $s_0 = \alpha_1 s_{j_1} + \dots + \alpha_r s_{j_r}$. This concludes the example showing how Shamir's scheme is used to implement a k-out-of-n threshold access structure.

Shamir's scheme is a prototype of what is called linear secret sharing schemes, which are ideal schemes (see Brickell (1989)). The main method of proving that an access structure is ideal is proving it to be linear. In particular, all indecomposable ideal weighted games that we shall look at in Chapter 6, were shown to be ideal by proving their linearity in (Farràs & Padró (2010), Theorem 9.2).

Shamir's scheme is also known to be related to Reed-Solomon codes (see McEliece & Sarwate (1981)).

Chapter 3

Hierarchical Simple Games

In this chapter and the next one, we will study two important classes of ideal complete simple games, namely the classes of hierarchical and tripartite games. This is because, as we mentioned earlier and stated in Theorems 1.1.1 and 1.1.2, these are the two classes that indecomposable ideal weighted simple games belong to according to (Beimel, Tassa, & Weinreb, 2008) and (Farràs & Padró, 2010).

We start with studying hierarchical simple games. There are two types of hierarchical simple games (HSGs) as we shall see, conjunctive and disjunctive. Since both types are ideal (Tassa, 2007; Brickell, 1989), they deserve to be studied in their own right. In this chapter we characterise all weighted hierarchical simple games. Later, in Chapter 7, we will also characterise all roughly weighted hierarchical simple games.

3.1 The two types of hierarchical simple games, definitions

In his pioneering paper Shamir (1979), suggested (independently from any literature on simple games) to model seniority of users by assigning nonnegative weights to them. However, this approach was not actively pursued, maybe because there is no guarantee that weighted simple games are ideal (see Beimel, Tassa, & Weinreb (2008) and references there). Instead, Simmons (1990) introduced the concept of a hierarchical simple game. Such a simple game stipulates that agents are partitioned into m levels, and a sequence of thresholds $k_1 < k_2 < \ldots < k_m$ is set, so that a coalition is winning if it has either k_1 agents of the first level, or k_2 agents of the first two levels, or k_3 agents of the first three levels etc. Consider, for example, the situation of a money transfer from one bank to another. If the sum to be transferred is sufficiently large, this transaction must be authorised by three senior tellers or two general managers. However, any two senior tellers and a general manager can also authorise the transaction. These games are called hierarchical *disjunctive* games, since only one of the m conditions must be satisfied for a coalition to be authorised/winning. If all the conditions must be satisfied, then the corresponding simple game is called hierarchical *conjunctive* game (Tassa, 2007). A typical example of a conjunctive hierarchical game would be the United Nations Security Council, where for the passage of a resolution all five permanent members must vote for it, *and* also at least nine members in total. This game has two levels.

Disjunctive and conjunctive hierarchical simple games have been proven to be ideal by Brickell (1989) and Tassa (2007) respectively. With one exception mentioned later, these two classes of hierarchical games have not been previously considered in the simple games literature. In this chapter we show that methods of the theory of simple games can make some proofs easier and more transparent. More precisely, our direct combinatorial approach is based on the technique of trading transforms discussed in Chapter 2. Using trading transforms we give a complete description of all weighted majority games among hierarchical games of both types.

So firstly, after the definition and examples and explaining why hierarchical games are complete, we give the canonical representation for both types of hierarchical games. Secondly, we prove the duality between disjunctive and conjunctive hierarchical games. Thirdly, we give a structural characterisation for disjunctive hierarchical games, by showing they are complete games with a unique shift-maximal losing coalition, then obtain a structural characterisation for conjunctive hierarchical games by duality, showing them to be complete games with a unique shift-minimal winning coalition. Finally, we characterise all weighted disjunctive hierarchical games, then characterize all weighted conjunctive hierarchical games by duality. We note that the class of complete games with a unique shift-minimal winning coalition was studied in its own right in (Freixas, 1997) and (Freixas & Puente, 1998, 2008) without any reference to hierarchical games.

Let us now start with the formal definition.

Definition 3.1.1. Suppose the set of players P is partitioned into m disjoint subsets $P = \cup_{i=1}^{m} P_i$, and let $k_1 < k_2 < \ldots < k_m$ be a sequence of positive integers, and let $\mathbf{k} = (k_1, \ldots, k_m)$. Then we define the game $H_\exists = (P, \mathbf{k})$ by setting the set of winning coalitions to be

$$W_\exists = \{X \in 2^P \mid \exists i \left(\left| X \cap \left(\cup_{j=1}^{i} P_j \right) \right| \geq k_i \right) \}.$$

and call it a *disjunctive hierarchical game*. For a sequence of thresholds $k_1 < \ldots < k_{m-1} \leq k_m$ (note that the last inequality may be nonstrict) we define its set of winning coalitions to be

$$W_\forall = \{X \in 2^P \mid \forall i \left(\left| X \cap \left(\cup_{j=1}^{i} P_j \right) \right| \geq k_i \right) \}.$$

We call the resulting game a *conjunctive hierarchical game* $H_\forall = (P, \mathbf{k})$.

The sets in the partition P will be considered ordered, so that, say, $P = Q \cup R$ and $P = R \cup Q$ will be two different partitions. Figure 3.1 illustrates the structure of a typical hierarchical simple game.

3.1.1 Completeness and examples

From the definition it follows that any disjunctive hierarchical game H is complete, let us explain. Recall that in order to prove the completeness of a game, it is enough to show that every two players are comparable with respect to Isbell's desirability relation, since transitivity and antisymmetry are obvious in this relation

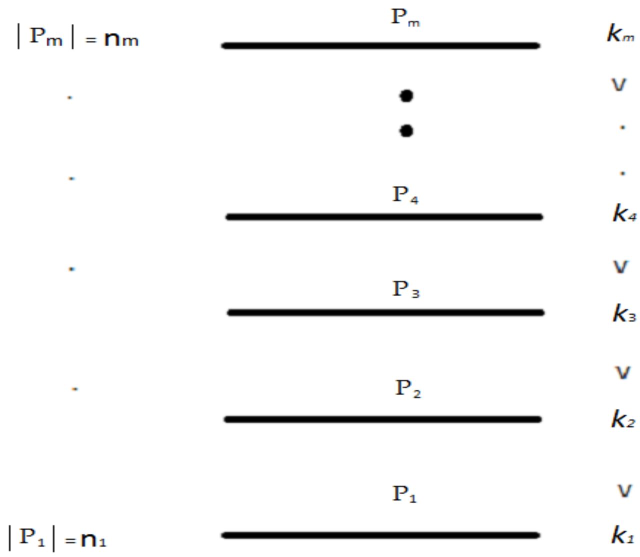

Figure 3.1: An m-level hierarchical simple game

(see page 36). Now, consider Definition 3.1.1. If a winning coalition X meets the ith-threshold requirement, then $|X \cap (\cup_{j=1}^{i} P_j)| \geq k_i$. It follows that a replacement of a player from P_j with a player from P_{j-1}, will still result in a coalition that meets the ith-threshold requirement, or may even meet a lower threshold requirement. But in both cases, the resulting coalition is winning. This shows that players from P_1 are at least as desirable as players from P_2, which in turn are at least as desirable as players from P_3, and so on. In other word, $P_i \succeq_H P_j$ for $i < j$, so any disjunctive HSG is complete. A similar argument shows that conjunctive hierarchical games are also complete.

Moreover, for any $i \in [m]$ and $u, v \in P_i$ we have $u \sim_H v$. So $\sim_H$ is the corresponding equivalence relation. We also let $|P_i| = n_i$, and denote the vector of the sizes of each P_i by $\mathbf{n} = (n_1, \ldots, n_m)$.

As an illustration of a disjunctive HSG, recall Example 2.2.11 where an electronic fund transfer of a large sum of money would be authorised by either two general managers, or four senior tellers, or one general manager and three senior tellers. There are 3 general managers and 5 senior tellers in total. So this is a disjunctive HSG of two levels: The general managers level P_1 with $k_1 = 2$, and the senior tellers level P_2 with $k_2 = 4$. In other words, we have $\mathbf{n} = (3, 5), \mathbf{k} = (2, 4)$.

As for conjunctive HSGs, we saw in Example 2.1.4 of the United Nations Security Council that a passage of a resolution requires the vote of at least 9 members in total, and all five permanent members. So here we have a conjunctive HSG of two levels such that $\mathbf{n} = (5, 10), \mathbf{k} = (5, 9)$, in other words, the permanent members make level P_1 with $k_1 = 5$, and the non-permanent members make level P_2 with $k_2 = 9$.

However, we cannot guarantee for an arbitrary partition and for arbitrary values of parameters in $\mathbf{k} = (k_1, \ldots, k_m)$ that the equivalence classes will be $P_1, \ldots, P_m$ and the multiset representation $\bar{H}$ of H will be defined on the multiset $\bar{P} = \{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}$, since it is possible to have fewer than m equivalence classes. Here is an example of redundancy in the description.

Example 3.1.2. Let us consider $P = \cup_{i=1}^{3} P_i$ with $|P_1| = |P_2| = |P_3| = 3$. Let us also take $\mathbf{k} = (k_1, k_2, k_3) = (4, 5, 6)$, and let $H = H_\exists(P, \mathbf{k})$ be the corresponding disjunctive hierarchical game. For this game the condition $|X \cap P_1| \geq k_1$ is never satisfied. As a result we will have $x \sim_H y$ for every $x \in P_1$ and $y \in P_2$ leading to the multiset representation for this game on a multiset $\bar{P} = \{1^6, 2^3\}$. The same game could be obtained by taking the partition $P = P_1' \cup P_2'$ with $|P_1'| = 6$, $|P_2'| = 3$ and $(k_1, k_2) = (5, 6)$.

When different players from different classes have the same desirability, the corresponding classes get identified with each other, or *collapse* onto each other. In the example above, one level of the game was redundant and can be collapsed, so we need to impose certain conditions preventing redundancy and the collapse of different levels. A representation of a HSG in which no two sets in the partition of P are equivalent to each other, and no redundancy exists, will be referred to as *canonical representation*. In the next section we establish canonical representations for the two types of HSGs for that purpose.

3.2 Canonical Representations

Recall that in a complete simple game H on a multiset $P = \{1^{n_1}, \ldots, m^{n_m}\}$, players of type 1 are more influential than players of type 2 that are in turn more influential than players of type 3 and so on. In other words $1 \succ_H 2 \succ_H \ldots \succ_H m$. The concepts of a shift, a shift-minimal winning coalition and a shift-maximal losing coalition that were explained on page 37 will be very important in this section.

3.2.1 Disjunctive Hierarchical Games

Now we are ready for constructing a canonical representation for the disjunctive hierarchical games (DHGs).

Theorem 3.2.1. *Let $H = H_\exists(P, \mathbf{k})$ be a disjunctive hierarchical game defined on the set of players P partitioned into m disjoint subsets $P = \cup_{i=1}^m P_i$ with $n_i = |P_i|$ by a sequence of positive thresholds $k_1 < k_2 < \ldots < k_m$. Then $\sim_H$ has exactly m equivalence classes (which are then $P_1, ..., P_m$) if and only if*

(a) $k_1 \le n_1$, and

(b) $k_i < k_{i-1} + n_i$ for every $1 < i < m$.

Proof. As was discussed earlier, the players within each P_i are equivalent to each other, and $P_i \succeq_H P_j$ for $i < j$. Let us prove by induction that conditions (a) and (b) imply that, for every $i \in [m-1]$, there exists a shift-minimal winning coalition $M_i \subseteq \cup_{j=1}^i P_i$ of size k_i that intersects with P_i nontrivially.

Consider the condition (a). If $k_1 \le n_1$, then any k_1 players from P_1 form a winning coalition M_1 of size k_1 which ceases to be winning if we replace one of them with a player of P_j for every $j > 1$. We now use an inductive argument.

Suppose $i < m - 1$ and suppose that there exists a shift-minimal winning coalition $M_i \subseteq \cup_{j=1}^i P_i$, consisting of k_i players, and such that $M_i \cap P_i \neq \emptyset$.

Let us now construct the shift-minimal winning coalition M_{i+1} that satisfy the required conditions. We start by selecting an element $a \in M_i \cap P_i$ from M_i. The coalition $M_i' = M_i \setminus \{a\}$ is losing due to the minimality of M_i. By (b) we have $k_{i+1} - k_i + 1 \leq n_{i+1}$. This means that we can add to M_i' exactly $k_{i+1} - k_i + 1$ elements of P_{i+1} so that the resulting set X will reach the $(i+1)$th threshold and will therefore be winning. We now apply all possible shifts to X within the set $\cup_{j=1}^{i+1} P_j$ as long as it remains winning and take the resulting set as M_{i+1}. This will secure that the coalition M_{i+1} is a shift-minimal winning coalition. It is also clear from its construction that $M_{i+1} \cap P_{i+1} \neq \emptyset$. It is now easy to show that $P_i \not\sim_H P_j$ for every $i \neq j$. Suppose $i < j$ and $x \in M_i \cap P_i$ and $y \in P_j$. Then $(M_i \setminus \{x\}) \cup \{y\}$ is losing since it does not reach any threshold. Hence $x \succ_H y$ and $P_i \not\sim_H P_j$.

Let us now prove the converse, that is, if $P_i \not\sim_H P_{i+1}$, for all $i \in [m-1]$, then conditions (a) and (b) are satisfied. It is easy to see that $P_1 \not\sim_H P_2$ implies (a). Suppose now that for $i > 1$ we have $p \in P_i$, $q \in P_{i+1}$ and $p \not\sim_H q$. Then there exist $X \subseteq P$ such that $X \cup \{p\}$ is winning and $X \cup \{q\}$ is losing. This could only happen if the coalition $X \cup \{p\}$ reaches the ith threshold and does not reach any other threshold, i.e., when $|X \cap \cup_{j=1}^{s} P_j| \leq k_s - 1$ for $s \in [i-1]$ and $|X \cap \cup_{j=1}^{i} P_j| = k_i - 1$. In particular, we have $|X \cap \cup_{j=1}^{i-1} P_j| \leq k_{i-1} - 1$. Since $p \in P_i \setminus X$ we have $n_i - 1 \geq |X \cap P_i| \geq k_i - k_{i-1}$ and (b) is proved. □

Proposition 3.2.2. *Let $H = H_\exists(P, \mathbf{k})$ be a disjunctive hierarchical game as defined in Theorem 3.2.1. Then the game H does not have dummies if and only if $k_m < k_{m-1} + n_m$. If $k_m \geq k_{m-1} + n_m$ then P_m consists entirely of dummies.*

Proof. Let $k_m < k_{m-1} + n_m$, arguing as in the proof of Theorem 3.2.1 we would find a shift-minimal winning coalition M_m of cardinality k_m that would nontrivially intersect P_m. In this case players of P_m are not dummies as none of them can be removed from M_m. If $k_m \geq k_{m-1} + n_m$, then the last threshold is never achieved without already achieving some previous threshold, and in this case all players of P_m are indeed dummies. □

It is clear that whenever P_m consists of dummies we can always change the mth threshold to $k_m = k_{m-1} + n_m$. We will now always do that.

Definition 3.2.3. Let $H = H_\exists(P, \mathbf{k})$ be a disjunctive hierarchical game defined on the set of players P partitioned into m disjoint subsets $P = \cup_{i=1}^m P_i$ with $n_i = |P_i|$ by a sequence of positive thresholds $k_1 < k_2 < \ldots < k_m$. We will say that H is *canonically represented* by P and $\mathbf{k}$ if the conditions (a) and (b) of Theorem 3.2.1 are satisfied (i.e., $\sim_H$ has exactly m equivalence classes) and $k_m = k_{m-1} + n_m$ in case when P_m consists of dummies and $k_m < k_{m-1} + n_m$ otherwise.

Corollary 3.2.4. *Let $G = H_\exists(P, \mathbf{k})$ be a canonically represented m-level disjunctive hierarchical game. Then we have $n_i > 1$ for every $1 < i < m$.*

Proof. If $n_i = 1$ for some $1 < i < m$, then (b) in Theorem 3.2.1 cannot hold. □

This is also a consequence of Theorem 4.1(4) of Carreras & Freixas (1996).

Collapsing any existing redundant levels, if they existed, we may always assume that a disjunctive hierarchical game $H = H_\exists(P, \mathbf{k})$ is canonically represented. It will be convenient to denote the multiset representation of the disjunctive hierarchical game in Definition 3.2.3 as $\bar{H} = H_\exists(\mathbf{n}, \mathbf{k})$, where $\mathbf{n} = (n_1, ..., n_m)$ and $\mathbf{k} = (k_1, ..., k_m)$. We stress that this notation assumes that the representation of the game H was canonical.

Theorem 3.2.5. *Let $H = H_\exists(P, \mathbf{k})$ be a canonically represented disjunctive hierarchical game. Then the numbers $k_1, \ldots, k_m$ (the last one only when H has no dummies), and only they, are the possible sizes of shift-minimal winning coalitions in H. In particular, the multiset representation of H is uniquely defined by the game.*

Proof. In the proofs of Theorem 3.2.1 and Proposition 3.2.2 we have shown the existence of shift-minimal winning coalitions of sizes $k_1, ..., k_m$ (the last size was shown to exist only in absence of dummies). Suppose M is a minimal winning coalition. Let k_i be the smallest such that the ith threshold is reached by M. Then M contains at least k_i players from $\bigcup_{i=1}^{i} P_i$. If M contained any players from levels P_j for $j > i$, they could be removed without rendering M losing. Due to the minimality of M such elements do not exist. Similarly, if M contained more than k_i players from $\bigcup_{i=1}^{i} P_i$, some of them could be removed too. Due to the minimality of M we conclude that $|M| = k_i$. $\square$

We note that the first and the last (mth) levels are special. If $k_1 = 1$, then every user of the first level is self-sufficient (passer) and its presence makes any coalition winning and if $k_m \geq k_{m-1} + n_m$, then the mth level consists entirely of dummies.

3.2.2 Conjunctive Hierarchical Games

Now we derive a canonical representation for the class of conjunctive hierarchical games (CHGs). First recall the definition.

Definition 3.2.6 (Conjunctive Hierarchical Game)**.** Suppose that the set of players P is partitioned into m disjoint subsets $P = \cup_{i=1}^{m} P_i$, and let $k_1 < \ldots < k_{m-1} \leq k_m$ be a sequence of positive integers. Then we define the game $H_\forall(P, \mathbf{k})$ by setting the set of its winning coalitions to be

$$W_\forall = \{X \in 2^P \mid \forall i \left(\left| X \cap \left(\cup_{j=1}^{i} P_i \right) \right| \geq k_i \right) \}.$$

It is easy to come up with an example similar to Example 3.1.2, so we need to look for conditions on $\mathbf{n} = (n_1, ..., n_m)$ and $\mathbf{k} = (k_1, ..., k_m)$ which guarantee that the game $H_\forall(P, \mathbf{k})$ has indeed m levels.

Definition 3.2.7. Let $H = H_\forall(P, \mathbf{k})$ be a conjunctive hierarchical game defined on the set of players P partitioned into m disjoint subsets $P = \cup_{i=1}^{m} P_i$ with $n_i = |P_i|$ by a sequence of positive thresholds $k_1 < \ldots < k_{m-1} \leq k_m$. We will say that H is *canonically represented* by $\mathbf{n} = (n_1, ..., n_m)$ and $\mathbf{k} = (k_1, \ldots, k_m)$ if $P_i \not\sim P_j$ for every distinct $i, j \in [m]$ or equivalently $\sim_H$ has exactly m equivalence classes. We will denote the multiset representation of such a game $H_\forall(\mathbf{n}, \mathbf{k})$.

For conjunctive hierarchical games a theorem analogous to Theorem 3.2.1 can be proved.

Theorem 3.2.8. *Let $H = H_\forall(P, \mathbf{k})$ be a conjunctive hierarchical game defined on the set of players P partitioned into m disjoint subsets $P = \cup_{i=1}^m P_i$, where $n_i = |P_i|$, by a sequence of positive thresholds $k_1 < \ldots < k_{m-1} \leq k_m$. Then $P_1, \ldots, P_m$ are exactly the equivalence classes for $\sim_H$ if and only if*

(a) $k_1 \leq n_1$, and

(b) $k_i < k_{i-1} + n_i$ for every $1 < i \leq m$.

The last mth level consists entirely of dummies if and only if $k_{m-1} = k_m$.

Proof. If (a) and (b) are satisfied, then the coalition M such that $|M \cap P_1| = k_1$, and $|M \cap P_i| = k_i - k_{i-1}$ for $i = 2, \ldots, m$, exists in H. It is winning since all m thresholds are met. It is obviously shift-minimal since any shift leads to the violation of one of the thresholds. Moreover, due to (b) a shift replacing element of $M \cap P_i$ with an element of P_{i+1} is always possible. This immediately implies that $P_i \not\sim P_{i+1}$.

Suppose now that either (a) or (b) is not satisfied. If (a) is not satisfied, then the first threshold is never achieved and the game has no winning coalitions. This contradicts to the definition of a simple game. Suppose (b) is not satisfied and $k_i \geq k_{i-1} + n_i$ for some $1 < i \leq m$. We will prove that in such a case $P_i \sim P_{i-1}$. Suppose not, then there exists a winning coalition X which becomes losing coalition X' after a shift replacing element of P_{i-1} by an element of P_i. It can only become losing due to the fact that the $(i-1)$th threshold is no longer met for X'. Note that all remaining thresholds $i, \ldots, m$ are still achieved in the coalition X'. We can estimate the number of players in $X' \cap \bigcap_{j=1}^{i-1} P_i$ as

$$|X' \cap \bigcap_{j=1}^{i-1} P_i| \geq k_i - n_i \geq k_{i-1},$$

which contradicts the fact that the $(i-1)$th threshold is not achieved for X'. $\square$

As in the disjunctive case, the canonical representation of a conjunctive hierarchical game is unique. We observe a striking resemblance of the conditions in

Theorem 3.2.1 and Theorem 3.2.8. To explain this resemblance, in Section 3.3 below, we prove a theorem that establishes the duality between disjunctive and conjunctive hierarchical games.

3.3 Duality between disjunctive and conjunctive hierarchical games

The following result was mentioned in (Tassa, 2007, Proposition 4.1) without a proof. Since it is our main tool here we present it with a proof. In the proof it will be convenient to use multiset representations of hierarchical games.

Recall that $\mathbf{n} = (n_1, \ldots, n_m)$ and $\mathbf{k} = (k_1, \ldots, k_m)$ are fixed vectors of positive integers. So for any vector $\mathbf{k} = (k_1, \ldots, k_m)$, define the vector

$$\mathbf{k}^* = (n_1 - k_1 + 1, n_1 + n_2 - k_2 + 1, \ldots, \sum_{i \in [m]} n_i - k_m + 1).$$

Note that $\mathbf{k}^{**} = \mathbf{k}$.

Recall that a coalition is called blocking, if its complement with respect to the full multiset of players is losing. Also, recall from Definition 2.2.21, that G^d is the dual of G if the winning coalitions of G^d are the blocking coalitions of G.

Theorem 3.3.1. *Let $H = H_\exists(\mathbf{n}, \mathbf{k})$ be an m-level hierarchical disjunctive game. Then the game dual to H will be the conjunctive hierarchical game $H^* = H_\forall(\mathbf{n}, \mathbf{k}^*)$. Similarly, if $H = H_\forall(\mathbf{n}, \mathbf{k})$ is an m-level hierarchical conjunctive game, then $H^* = H_\exists(\mathbf{n}, \mathbf{k}^*)$. In other words,*

$$H_\exists(\mathbf{n}, \mathbf{k})^* = H_\forall(\mathbf{n}, \mathbf{k}^*), \qquad H_\forall(\mathbf{n}, \mathbf{k})^* = H_\exists(\mathbf{n}, \mathbf{k}^*).$$

Proof. We will prove only the first equality. As Isbell's desirability relation is self-dual, the game $H_\exists(\mathbf{n}, \mathbf{k})^*$ will involve the same equivalence classes as the original game $H_\exists(\mathbf{n}, \mathbf{k})$ and hence it will be defined on the same multiset $\{1^{n_1}, \ldots, m^{n_m}\}$. Let $\mathbf{k}^* = (k_1^*, k_2^*, \ldots, k_m^*)$. It is easy to see that $k_i^* < k_{i+1}^*$ is equivalent to $k_{i+1} <$

$k_i + n_{i+1}$ so we have $k_1^* < \ldots < k_{m-1}^* \leq k_m^*$ and $k_{m-1}^* = k_m^*$ if and only if $k_m = k_{m-1} + n_m$, that is the pair $(\mathbf{n}, \mathbf{k})$ satisfies the conditions of Theorem 3.2.1 if and only if the pair $(\mathbf{n}, \mathbf{k}^*)$ satisfies the conditions of Theorem 3.2.8. Consider a losing coalition $X = \{1^{\ell_1}, 2^{\ell_2}, \ldots, m^{\ell_m}\}$ in $H_\exists(\mathbf{n}, \mathbf{k})$. It satisfies $\sum_{j\in[i]} \ell_j < k_i$ for all $i \in [m]$. Then

$$\sum_{j\in[i]}(n_j - \ell_j) > \sum_{j\in[i]} n_j - k_i,$$

for all $i \in [m]$, and the coalition $X^c = \{1^{n_1-\ell_1}, 2^{n_2-\ell_2}, \ldots, m^{n_m-\ell_m}\}$ satisfies the condition $\sum_{j\in[i]}(n_j - \ell_j) \geq \sum_{j\in[i]} n_j - k_i + 1 = k_i^*$, for all $i \in [m]$. Therefore, X^c is winning in $H_\forall(\mathbf{n}, \mathbf{k}^*)$.

We also need to show that the complement of every coalition that is winning in $H_\exists(\mathbf{n}, \mathbf{k})$ coalition is losing in $H_\forall(\mathbf{n}, \mathbf{k}^*)$. Consider a coalition $X = \{1^{\ell_1}, 2^{\ell_2}, \ldots, m^{\ell_m}\}$ that is winning in $H_\exists(\mathbf{n}, \mathbf{k})$. It means that there is an $i \in [m]$ such that $\sum_{j\in[i]} \ell_j \geq k_i$. But then the condition

$$\sum_{j\in[i]}(n_j - \ell_j) \leq \sum_{j\in[i]} n_j - k_i < \sum_{j\in[i]} n_j - k_i + 1 = k_i^*$$

holds. Thus, the complement $X^c = \{1^{n_1-\ell_1}, 2^{n_2-\ell_2}, \ldots, m^{n_m-\ell_m}\}$ is losing in $H_\forall(\mathbf{n}, \mathbf{k}^*)$. $\square$

3.4 Hierarchical Subgames and Reduced Games

The operation of subgames and reduced games as they apply to HSGs will be very useful later on. Here we give the two basic propositions showing that subgames and reduced games of hierarchical games of a certain type are still hierarchical of the same type. We assume that all HSGs are canonically represented.

Proposition 3.4.1. *Let* $\mathbf{n}' = (n_1, \ldots, n_{m-1})$, $\mathbf{k}' = (k_1, \ldots, k_{m-1})$. *Then* $H' = H_\exists(\mathbf{n}', \mathbf{k}')$ *is a subgame of* $G = H_\exists(\mathbf{n}, \mathbf{k})$. *This subgame never has dummies and it does not have passers if* G *did not.*

Proof. Indeed, $H_{\exists}(\mathbf{n}', \mathbf{k}') = G_A$ for $A = \{m^{n_m}\}$. By Theorem 3.2.1 there always exists a minimal winning coalition which is contained in H' and has a nonempty intersection with the $(m-1)$th level. Hence the players of $(m-1)$th level are not dummies. $\square$

Proposition 3.4.2. *Let $\mathbf{n} = (n_1, \ldots, n_m)$, $\mathbf{k} = (k_1, \ldots, k_m)$ be such that $G = H_{\forall}(\mathbf{n}, \mathbf{k})$ is an m-level conjunctive hierarchical game. Suppose $k_1 = n_1$, $\mathbf{n}' = (n_2, \ldots, n_m)$, and $\mathbf{k}' = (k_2 - k_1, \ldots, k_m - k_1)$. Then $H_{\forall}(\mathbf{n}', \mathbf{k}')$ is the reduced game G^A, where $A = \{1^{n_1}\}$.*

3.5 Structural Characterisations

In this section we give structural characterisations for the two types of hierarchical games in terms of their shift-minimal winning and shift-maximal losing coalitions. First we start with DHGs.

3.5.1 A structural characterisation of Disjunctive Hierarchical Games

Theorem 3.5.1. *The class of disjunctive hierarchical simple games is exactly the class of complete games with a unique shift-maximal losing coalition.*

Proof. Without loss of generality we can consider only multiset representations of games. Let $G = H_{\exists}(\mathbf{n}, \mathbf{k})$ be an m-level hierarchical game. If $k_m < k_{m-1} + n_m$, then the following coalition is a shift-maximal losing one:

$$M = \{1^{k_1-1}, 2^{k_2-k_1}, \ldots, m^{k_m-k_{m-1}}\}.$$

Indeed, for every $i = 1, 2, \ldots, m$ it has $k_i - 1$ players from the first i levels, and so any replacement of a player with more influential one makes it winning. If $k_m \geq k_{m-1} + n_m$, then it has to be modified as

$$M = \{1^{k_1-1}, 2^{k_2-k_1}, \ldots, (m-1)^{k_{m-1}-k_{m-2}}, m^{n_m}\}.$$

There cannot exist any other shift-maximal losing coalition. Indeed, the first level has to lack exactly one player before reaching the first threshold, and the first two levels together have to lack exactly one player before reaching the second threshold and so on. Hence the uniqueness.

Suppose now that G is complete with the multiset representation on a multiset $P = \{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}$, where $1 \succ_G \cdots \succ_G m$, and has a unique shift-maximal losing coalition $M = \{1^{\ell_1}, 2^{\ell_2}, \ldots, m^{\ell_m}\}$. We claim that $\ell_i < n_i$ for all $1 \le i < m$. Suppose not, and $\ell_i = n_i$. We know that $i \succ_G i+1$. It means there exists a multiset X such that $X \cup \{i\}$ is winning but $X \cup \{i+1\}$ is losing. We first take X to be of maximum possible cardinality, and then do all possible replacements of a players from X with more desirable ones as long as the property $X \cup \{i\} \in W$ and $X \cup \{i+1\} \in L$ holds. This will make $X \cup \{i+1\}$ a shift-maximal losing coalition. Indeed, we cannot add any more elements to $X \cup \{i+1\}$ without making it winning, also replacing any element of $X \cup \{i+1\}$ with a more desirable one makes it winning as well. Since $X \cup \{i+1\}$ is not equal to M (the multiplicity of i is not at full capacity) we get a contradiction. Hence $\ell_i < n_i$. The coalition $M_j = \{1^{\ell_1}, \ldots, (j-1)^{\ell_{j-1}}, j^{\ell_j+1}\}$ must be winning for all j (including $j = 1$ when it is equal to $\{1^{\ell_1+1}\}$) because it can not be obtained from M by making shifts or taking subset (remember M is a unique shift-maximal losing coalition). Define k_i to be equal to $\ell_1 + \ldots + \ell_i + 1$ for every $i = 1, 2, \ldots, m$. Let us show that every coalition with k_i players from the first i levels is winning. If $i = 1$, then $k_1 = \ell_1 + 1$. The coalition $\{1^{k_1}\}$ cannot losing, otherwise it will be a subcoalition of a shift-maximal losing coalition, different from M.

Suppose now the statement is true for $i-1$. Let us consider a coalition $X = \{1^{s_1}, 2^{s_2}, \ldots, i^{s_i}\}$ with $s_1 + \ldots + s_i = k_i$, and prove that it is winning. If not, then by the induction hypothesis $s_1 + \ldots + s_{i-1} < k_{i-1}$. Let us make all possible replacements of players with more desirable ones that do not make X winning. Then we will have a coalition $Y = \{1^{\ell_1}, 2^{\ell_2}, \ldots, (i-1)^{\ell_{i-1}}, i^{\ell_i+1}\}$. Now, since the multiplicity of i in Y is greater than ℓ_i and no replacements of players with more desirable ones can be done any more, then Y is contained in a shift-maximal losing coalition different from M. This contradiction proves the statement.

Now if $\ell_m = n_m$ we set $k_m = k_{m-1} + n_m$, otherwise we set $k_m = \ell_1 + \ldots + \ell_m + 1$. It is easy to see that G is in fact $H_\exists(\mathbf{n}, \mathbf{k})$. $\square$

3.5.2 A structural characterisation of Conjunctive Hierarchical Games

In this section, we give a characterisation concerning CHGs that is analogous to the disjunctive hierarchical one of the previous section. First, we need the following lemma.

Lemma 3.5.2. *Let G be a complete simple game. Then S is a shift-maximal losing coalition of G if and only if its complement S^c is a shift-minimal winning coalition of the dual game G^*.*

Proof. Let S be a shift-maximal losing coalition in G. By the definition of the dual game, S^c is winning in G^*. Let us prove that it is a shift-minimal winning coalition there. Consider any other coalition X that can be obtained from S^c by a shift $i \to j$ in G^*. It means that there are players $i \in S^c$ and $j \notin S^c$ such that $j \prec_{G^*} i$ (which we know is the same as $j \prec_G i$) and $X = (S^c \setminus \{i\}) \cup \{j\}$. The complement of X is the set $X^c = (S \setminus \{j\}) \cup \{i\}$. Furthermore, $j \prec_G i$. Now S is obtained from X^c by the shift $i \to j$, hence the coalition X^c is winning in G, because there does not exist a losing coalition from which S can be obtained by a shift. Therefore, X is losing in G^*. Consider now a subset Y of S^c. The complement Y^c of Y is a superset of S. Hence, Y^c is winning in G and Y is losing in G^*. Thus, S^c is the shift-minimal winning coalition in G^*. The converse can be proved along the same lines and we leave this to the reader. $\square$

This will now lead us to the main characterization result.

Theorem 3.5.3. *The class of conjunctive hierarchical simple games is exactly the class of complete games with a unique shift-minimal winning coalition.*

Proof. Without loss of generality we can consider only multiset representations of games. By Lemma 3.5.2 the class of complete games with a unique shift-minimal winning coalition is dual to the class of complete games with a unique shift-maximal losing coalition in the sense that the dual of every game from the first class belongs to the second and the other way around. Theorem 3.5.1 claims that the second class coincides with the class of all disjunctive hierarchical games. By Theorem 3.3.1, the first class then is the class of all conjunctive hierarchical games. □

3.6 Weightedness

3.6.1 Characterising Weighted Disjunctive Hierarchical Games

Now we turn our attention to the characterisation of weighted games within the class of DHGs. We mentioned in several places that the characterisation of weighted DHGs was already achieved by Beimel, Tassa, & Weinreb (2008), however, Theorem 3.6.2 below, is a slightly more refined version, we shall explain why so shortly.

It turns out that only a small number of hierarchical games are weighted, and the following is an example of a two-level hierarchical simple game that is not weighted.

Example 3.6.1. Consider the two-level disjunctive hierarchical game $G = H_\exists(\mathbf{n}, \mathbf{k})$ with $\mathbf{n} = \mathbf{k} = (2, 4)$. Then the following is a certificate of nonweightedness

$$(\{1^2\}, \{2^4\}; \{1, 2^2\}, \{1, 2^2\})$$

as the first two coalitions are winning and the remaining two are losing. Therefore the game is not weighted by Theorem 2.2.10.

A level of a disjunctive or conjunctive hierarchical game is said to be *trivial*, if it consists entirely either of dummies, or it consists entirely of passers in the disjunctive case, and blockers in the conjunctive case. The following theorem shows that every weighted disjunctive hierarchical game $H_\exists(\mathbf{n}, \mathbf{k})$ can have at most four levels where at most two of them are nontrivial. Moreover, if a trivial level of $H_\exists(\mathbf{n}, \mathbf{k})$ consists of passers or blockers, then it is the first level (most desirable), and if it consists of dummies, then it is the last one (least desirable).

Below, we state and prove our version of the characterisation for two reasons. Firstly, (Beimel, Tassa, & Weinreb, 2008) assumed the absence of dummies from the outset, this is because in the context of secret sharing schemes, accounting for dummies, who get meaningless shares, is of no practical importance. But in our general study of HSGs here we allow them. Secondly, the proof we provide is combinatorial that uses trading transforms (see Formula 2.2.2, the discussion associated with it and Theorem 2.2.10) which is easier to follow then the existing proof.

Theorem 3.6.2. *Let* $G = H_\exists(\mathbf{n}, \mathbf{k})$ *be an* m*-level disjunctive hierarchical simple game. Then* G *is a weighted majority game iff one of the following conditions is satisfied:*

(1) $m = 1$ *(in which case* G *is a simple majority game);*

(2) $m = 2$ *and* $k_2 = k_1 + 1$*;*

(3) $m = 2$ *and* $n_2 = k_2 - k_1 + 1$*;*

(4) $m \in \{2, 3\}$ *and* $k_1 = 1$*, that is, the game has two or three levels and the first level consists entirely of passers. In case* $m = 3$ *the subgame* $H_\exists((n_2, n_3), (k_2, k_3))$ *falls under (2) or (3);*

(5) $m \in \{2, 3, 4\}$*,* $k_m = k_{m-1} + n_m$*, that is, the game has up to four levels but the last level consists of dummies. The subgame* $H_\exists(\mathbf{n}', \mathbf{k}')$*, where* $\mathbf{n}' = (n_1, \ldots, n_{m-1})$ *and* $\mathbf{k}' = (k_1, \ldots, k_{m-1})$ *falls under one of the cases (1)–(4).*

Proof. We will prove this theorem using the combinatorial technique of trading transforms. If $k_m = k_{m-1} + n_m$, then users of the last level are dummies and they never participate in any minimal winning coalition. As a result, if there exists a certificate of nonweightedness

$$\mathcal{T} = (X_1, \ldots, X_j; Y_1, \ldots, Y_j) \tag{3.6.1}$$

with minimal winning coalitions $X_1, \ldots, X_j$, which exist by Theorem 2.2.13, then no dummies may be found in any of the $X_1, \ldots, X_j$, hence they are not participating in this certificate. Hence G is weighted if and only if its subgame $H_\exists(\mathbf{n}', \mathbf{k}')$, where $\mathbf{n}' = (n_1, \ldots, n_{m-1})$ and $\mathbf{k}' = (k_1, \ldots, k_{m-1})$ is weighted. So we reduce our theorem to the case without dummies, and in this case we have to prove that G falls under one of the cases (1)-(4). Let us assume that $k_m < k_{m-1} + n_m$.

If $k_1 = 1$, then every user of the first level is a passer since any coalition with participation of this player is winning. If a certificate of nonweightedness (3.6.1) exists, then player 1 cannot be a member of any set $X_1, \ldots, X_j$, since then it will have to be also in one of the $Y_1, \ldots, Y_j$ and at least one of them will not be losing. Hence G is weighted if and only if its subgame $H_\exists(\mathbf{n}', \mathbf{k}')$, where $\mathbf{n}' = (n_2, \ldots, n_m)$ and $\mathbf{k}' = (k_2, \ldots, k_m)$ is weighted.

This shows that a disjunctive hierarchical simple game is weighted if and only if its subgame without passers and dummies is weighted. Hence without loss of generality we can consider games without passers and dummies. This is equivalent to assuming $k_1 > 1$ and $k_m < k_{m-1} + n_m$.

The case $m = 1$ is trivial. Next we show that if at least one of the two conditions (2) and (3) is met, then G is weighted. So we assume that $m = 2$ and $k_1 \geq 2$. One can easily check that each game satisfying the second condition is weighted with weights $w_1 = \frac{1}{k_1}, w_2 = \frac{1}{k_2}$, and the quota q = 1 (these weights work for both cases $n_2 \geq k_2$ and $n_2 < k_2$). In the case the third condition is met, a game is weighted with weights $w_1 = \frac{1}{k_1}$ and $w_2 = \frac{1}{k_1 n_2}$.

Conversely, we show that if all conditions (1)-(3) fail, then G is not weighted. If $m = 2$, this means that $k_2 \geq k_1 + 2$ and $n_2 \geq k_2 - k_1 + 2$. In this case the game possesses the following certificate of nonweightedness:

$$(\{1^{k_1}\}, \{1^{k_1-2}, 2^{k_2-k_1+2}\}; \{1^{k_1-1}, 2^{\lfloor (k_2-k_1+2)/2 \rfloor}\}, \{1^{k_1-1}, 2^{\lceil (k_2-k_1+2)/2 \rceil}\}).$$

Since $n_2 \geq k_2 - k_1 + 2$, all the coalitions are well-defined. Also, the constraint $k_2 \geq k_1 + 2$ secures that $\lceil \frac{k_2 - k_1 + 2}{2} \rceil \leq k_2 - k_1$ and makes both multisets in the right-hand-side of the trading transform losing.

Now suppose $m \geq 3$, $k_1 \geq 2$ and we have no dummies. By Theorem 3.2.1 we have $k_1 \leq n_1$, $k_2 < k_1 + n_2$ and $k_3 < k_2 + n_3$. Suppose first that $k_3 \leq n_3$. Then, since $k_3 \geq k_2 + 1 \geq k_1 + 2 \geq 4$, the following is a certificate of nonweightedness.

$$(\{1^{k_1}\}, \{3^{k_3}\}; \{1^{k_1-1}, 3^2\}, \{1, 3^{k_3-2}\}).$$

Suppose $k_3 > n_3$. If at the same time $k_3 \leq n_2 + n_3$, then since $k_3 - n_3 < k_2$ we have a legitimate certificate of nonweightedness

$$(\{1^{k_1}\}, \{2^{k_3-n_3}, 3^{n_3}\}; \{1^{k_1-1}, 2, 3\}, \{1, 2^{k_3-n_3-1}, 3^{n_3-1}\}).$$

Finally, if $k_3 > n_3$ and $k_3 > n_2 + n_3$, then the certificate of nonweightedness will be

$$(\{1^{k_1}\}, \{1^{k_3-n_2-n_3}, 2^{n_2}, 3^{n_3}\};$$
$$\{1^{k_1-1}, 2, 3\}, \{1^{k_3-n_2-n_3+1}, 2^{n_2-1}, 3^{n_3-1}\}).$$

All we have to check is that the second coalition of the losing part is indeed losing. To show this we note that $k_3 - n_3 < k_2$ and $k_3 - n_2 - n_3 + 1 < k_2 - n_2 + 1 \leq k_1$. This shows that the second coalition of the losing part is indeed losing and proves the theorem. $\square$

3.6.2 Characterising Weighted Conjunctive Hierarchical Games

Now we can characterize weighted conjunctive hierarchical games. We will show that every weighted conjunctive hierarchical game $H_\forall(\mathbf{n}, \mathbf{k})$ can have at most four levels, where at most two of them are nontrivial. Moreover, if a trivial level of $H_\forall(\mathbf{n}, \mathbf{k})$ consists of blockers, then it is the first level, and if it consists of dummies, then it is the last one.

Theorem 3.6.3. *Let $G = H_\forall(\mathbf{n}, \mathbf{k})$ be an m-level conjunctive hierarchical simple game. Then G is a weighted majority game iff one of the following conditions is satisfied:*

(1) $m = 1$ *(in which case G is a simple majority game);*

(2) $m = 2$ *and* $k_2 = k_1 + 1$*;*

(3) $m = 2$ *and* $n_2 = k_2 - k_1 + 1$*;*

(4) $m \in \{2, 3\}$ *and* $k_1 = n_1$*, that is, the game has two or three levels and the first one consists entirely of blockers. In case $m = 3$, the reduced game $H_\forall(\mathbf{n}, \mathbf{k})^{\{1^{n_1}\}} = H_\forall(\mathbf{n}', \mathbf{k}')$ of G, where $\mathbf{n}' = (n_2, n_3)$ and $\mathbf{k}' = (k_2 - k_1, k_3 - k_1)$, falls under (2) or (3);*

(5) $m \in \{2, 3, 4\}$ *with* $k_m = k_{m-1}$*, that is the game has up to four levels but the last one consists entirely of dummies. Moreover, the reduced game $H_\forall^{\{m^{n_m}\}}(\mathbf{n}, \mathbf{k}) = H_\forall(\mathbf{n}', \mathbf{k}')$, where $\mathbf{n}' = (n_1, \ldots, n_{m-1})$ and $\mathbf{k}' = (k_1, \ldots, k_{m-1})$, falls under one of the (1) – (4).*

Proof. The theorem straightforwardly follows from Theorem 3.6.2, the duality between conjunctive hierarchical games and disjunctive hierarchical games and Propositions 3.4.1 and 3.4.2. $\square$

Chapter 4

Tripartite Simple Games

Tripartite simple games were defined and shown to be ideal in the influential paper by Beimel, Tassa, & Weinreb (2008). The importance of this class stems from the fact that any indecomposable (we will explain in Chapter 5 what indecomposable means) ideal weighted game is either hierarchical or tripartite. As in the case of hierarchical games, this class deserves to be thoroughly studied, and this study has not yet been done in the extant literature. So in this chapter, we undertake a thorough study of this class, which is for the most part parallel in structure to the study we did for hierarchical simple games. We say it is parallel for the most part because one difference the reader will notice, is that we will not characterise weighted tripartite games. Although we have an alternative proof for the characterisation of weighted tripartite games to the existing one by Farràs & Padró (2010), the fact that this characterisation already existed makes it unnecessary to include our alternative proof here.

In passing something should be said regarding the characterisation of multipartite games in general. A *multipartite* game is a game with different seniority levels where a coalition must satisfy one or more conditions at each seniority level in order to be winning. So a hierarchical game is an example of a multipartite game, and the tripartite games discussed in this chapter are another example, and there are more. A *bipartite* game is a 2-level multipartite game. Now one may wonder whether it is possible to characterise all multipartite games in general, this is an open problem, but we feel that with greater knowledge of the compositions of simple games (more on Chapter 5) an answer to this problem can be achieved.

4.1 The two types of tripartite simple games, definitions and examples

In this section we present tripartite simple games (TSGs), which are a generalization of bipartite simple games that were presented in (Padró & Sáez, 1998). In a bipartite game, the set of players is partitioned into two classes, and a coalition is winning if the sizes of its intersection with each of these classes satisfy some predefined conditions. In a tripartite game, the set of players is partitioned into three classes, and, similarly to bipartite games, a given coalition is winning if the sizes of its intersection with each of these classes satisfy some conditions. There are two types of TSGs involved in the study of ideal WSGs, and we should mention that the tripartite games treated here form a special class of tripartite games as invented by Beimel, Tassa, & Weinreb (2008), where the authors state that they are considering a 'specific' family of tripartite games.

The first type will be referred to as Δ_1, and the second type will be referred to as Δ_2. We start with the formal definition.

Definition 4.1.1. Let P be a set of n players, such that $P = A \cup B \cup C$, where A, B and C are disjoint and all nonempty [1]. Let m, d, t be positive integers such that $m \geq t$. Then the following is a TSG on P: a coalition X is winning in $\Delta_1(A, B, C, m, d, t)$ iff

$$|X \cap A| \geq t \text{ or } (|X| \geq m \text{ and } |X \cap (A \cup B)| \geq m - d).$$

[1] In (Beimel, Tassa, & Weinreb (2008), Definition 3.3) it says A and C are nonempty, meaning B is allowed to be empty. But if B is empty, then we get a decomposable bipartite game. So the case for B being empty does not add anything new to the list of indecomposable ideal weighted games, which are needed for characterising all ideal weighted games, as we show in Chapter 6. Therefore we exclude the possibility of B being empty from the definition of tripartite games.

Namely, a coalition X is winning in Δ_1 if either it has at least m players, $m-d$ of which are from $A \cup B$, or it has at least t players from A. If $|B| \leq d+t-m$, then the following is another TSG: a coalition X is winning in $\Delta_2(A, B, C, m, d, t)$ iff

$$|X \cap (A \cup B)| \geq t \text{ or } (|X| \geq m \text{ and } |X \cap A| \geq m-d).$$

That is, a coalition X is winning in Δ_2 if either it has at least m players, $m-d$ of which are from A, or it has at least t players from $A \cup B$.

We will sometimes abbreviate the two TSGs with Δ_1 and Δ_2 when no confusion may arise. We give two examples below, the first one is for Δ_1, and the second for Δ_2.

Example 4.1.2. An electrical engineering company has three groups of employees: 5 technical experts, 8 senior engineers and 8 junior engineers. Fixing a major power outage requires either four technical experts, or six employees, at least two of which are either from the technical experts group, or senior engineers group, or both. We let $A = \{\text{technical experts}\}, |A| = 5, B = \{\text{senior engineers}\}, |B| = 8$, and $C = \{\text{junior engineers}\}, |C| = 8$, so that the full set is $P = A \cup B \cup C$, and $m = 6, d = 4, t = 4$. Then a coalition X can fix the outage if either $|X \cap A| \geq 4$ or ($|X| \geq 6$ and $|X \cap (A \cup B)| \geq m-d = 2$), meaning this game can be modelled by the TSG of type Δ_1.

Example 4.1.3. Consider an alternative scenario to the one above, where $A = \{\text{technical experts}\}, |A| = 5, B = \{\text{senior engineers}\}, |B| = 2$, and $C = \{\text{junior engineers}\}, |C| = 8, P = A \cup B \cup C$, and $m = 6, d = 4, t = 4$, meaning $|B| = d + t - m = 2$. Here the two options of fixing a major power outage are: (1) Four employees, either from the technical experts group, or senior engineers group, or both. (2) Six employees, at least two of which are from the technical experts group. Then a coalition X can fix the outage if either $|X \cap (A \cup B)| \geq 4$ or ($|X| \geq 6$ and $|X \cap A| \geq m - d = 2$), meaning this game can be modelled by the TSG of type Δ_2.

Observe that all players within A are equally desirable to each other, and the same is true for all players within B and all players within C. But let us now take a closer look at the winning conditions of Δ_1 and Δ_2, in order to understand how the players from A, B and C relate to each other in terms of desirability, and therefrom prove that tripartite simple games are complete.

4.1.1 Completeness

We will first show that, in Δ_1, players of A are at least as desirable as players of B, which are in turn at least as desirable as players of C. In other words $A \succeq_{\Delta_1} B \succeq_{\Delta_1} C$.

A winning coalition of Δ_1 satisfies either $|X \cap A| \geq t$ or ($|X| \geq m$ and $|X \cap (A \cup B)| \geq m - d$). If there is no minimal winning coalition containing players from B, then B consists of dummy players, and it follows that $A \succeq_{\Delta_1} B$. Suppose now there is a minimal winning coalition of Δ_1 containing players from B, then it must satisfy $|X| \geq m$ and $|X \cap (A \cup B)| \geq m - d$. Now if we replace a player from B with a player from A, then the resulting coalition satisfies again the same winning condition. Hence $A \succeq_{\Delta_1} B$.

Similarly, if there is no minimal winning coalition containing players from C, then C consists of dummy players, since C cannot be empty by Definition 4.1.1. It follows that $B \succeq_{\Delta_1} C$. Suppose now there is a minimal winning coalition containing players from C, then it must satisfy $|X| \geq m$ and $|X \cap (A \cup B)| \geq m - d$. Now replace a player from C with a player from B, then the resulting coalition still satisfies the same winning condition, so $B \succeq_{\Delta_1} C$. But Isbell's desirability relation is transitive (see page 36), therefore $A \succeq_{\Delta_1} B \succeq_{\Delta_1} C$.

A similar argument shows that in Δ_2 we also have $A \succeq_{\Delta_2} B \succeq_{\Delta_2} C$. So in fact for $i \in \{1, 2\}$, we have

$$A \succeq_{\Delta_i} B \succeq_{\Delta_i} C. \tag{4.1.1}$$

This fact will be useful later on. We have now actually shown that both Δ_1 and Δ_2 are complete games.

However, with only Definition 4.1.1 and (4.1.1) above, there is no guarantee that desirability of players from different classes will be different. Let us illustrate this possibility with the following example.

Example 4.1.4. Consider $\Delta_1(A, B, C, m, d, t)$, where $A = \{a_1\}, B = \{b_1, b_2, b_3\}$, $C = \{c_1, c_2, c_3\}$, and $m = 4, d = 2, t = 2$. Now, since $|A| < t$, then a winning coalition meeting the requirement $|X \cap A| \geq t$ is never achieved, and therefore the only winning coalitions are those meeting the requirement $|X| \geq m$ and $|X \cap (A \cup B)| \geq m - d$. But the condition $|X| \geq m$ and $|X \cap (A \cup B)| \geq m - d$ is symmetric with respect to A and B, thus implying that the two classes A and B are equivalent.

We saw how in the above example that classes A and B became equivalent to each other. Many cases like the one above can occur. But we would like to identify conditions under which a tripartite game has three distinct classes, so no classes collapsing. As we shall explain in the next proposition, collapsing classes in the tripartite game results in fact in a 2-level hierarchical game, sometimes disjunctive and sometimes conjunctive. Therefore we need a canonical representation for each type, that allows us to identify the cases when A, B and C as the three different equivalence classes.

4.2 Canonical Representations

In order to identify the cases when A, B and C are three different equivalence classes, we will assume from now on that B is nonempty, for both types Δ_1 and Δ_2.

4.2.1 A Canonical Representation for games of type Δ_1

Proposition 4.2.1. *Given the tripartite simple game $\Delta_1(A, B, C, m, d, t)$, the following conditions must be satisfied in order for the game to have A, B and C as three distinct equivalence classes with respect to Isbell's desirability relation:*

(C1) $m > t$*;*

(C2) $m > d$*;*

(C3) $|A| \geq t$*;*

(C4) both $|A| + |B| + |C| \geq m$*, and* $|A| + |B| \geq m - d$*;*

(C5) $|C| > d$*;*

(C6) $m - d - |B| < t$*.*

Proof. In all that follows, assume that X is a minimal winning coalition in Δ_1, and let us refer to $|X \cap A| \geq t$ as the first winning condition. Similarly, we will refer to $|X| \geq m$ and $|X \cap (A \cup B)| \geq m - d$ as the second winning condition.

(C1) We know from the definition that $m \geq t$, so suppose $m = t$. Consider both the first and second winning conditions. Observe that the first winning condition for $t = m$ implies the second winning condition $|X| \geq m$ and $|X \cap (A \cup B)| \geq m - d$. This means that the first winning condition is redundant in this case. Since the second winning condition is symmetric with respect to A and B, then $A \sim_{\Delta_1} B$.

The fact that A and B become equivalent to each other means that together they form one class, call it AB. So the game at hand now has only two classes, AB and C. A winning coalition in this resulting game has to have at least $m - d$ players from AB and at least m players in total. It follows that the resulting game is a 2-level conjunctive HSG with $m - d$ being the first threshold, and m being the second threshold. The same applies to all cases; when classes A and B become equivalent, we always get a 2-level conjunctive HSG.

(C2) If $m \leq d$, then $m - d \leq 0$, and $|X \cap (A \cup B)| \geq m - d$ always holds. So the two winning conditions now become $|X \cap A| \geq t$ and $|X| \geq m$. But these are symmetric with respect to B and C. So it follows that $B \sim_{\Delta_1} C$.

The fact that B and C become equivalent to each other means that together they form one class, call it BC. So the game at hand now has only two classes, A and BC. A winning coalition in this resulting game has to have either at least t players from A, or at least m players in total. It follows that the resulting game is a 2-level disjunctive HSG with t being the first threshold, and m being the second threshold. The same applies to all cases; when classes B and C become equivalent, we always get a 2-level disjunctive HSG.

(C3) $|A| < t$ means the first condition is never satisfied, meaning only $|X| \geq m$ and $|X \cap (A \cup B)| \geq m - d$ can be met. Therefore $A \sim_{\Delta_1} B$, since $|X \cap (A \cup B)| \geq m - d$ is symmetric with respect to A and B, and we get a 2-level conjunctive HSG.

(C4) If any of $|A| + |B| + |C| \geq m$ and $|A| + |B| \geq m - d$ fails, then the second winning condition is never satisfied, so both classes B and C become classes consisting entirely of dummies, meaning $B \sim_{\Delta_1} C$, and we get a 2-level disjunctive HSG.

(C5) Suppose X contains players from C. Then it meets the second winning condition, but not the first. Since X is minimal, then $|X| = m$. If $|C| \leq d$, then $|X| \geq m$ implies $|X \cap (A \cup B)| \geq m - d$, so the winning conditions are in fact $|X \cap A| \geq t$ or $|X| \geq m$, which are both symmetric with respect to B and C. Therefore $B \sim_{\Delta_1} C$, and we get a 2-level disjunctive HSG.

(C6) Finally, if $m - d - |B| \geq t$, then the second winning condition $|X \cap (A \cup B)| \geq m - d$ implies $|X \cap A| \geq m - d - |B| \geq t$, meaning the first winning condition is already met, so the second winning condition is redundant. Therefore $B \sim_{\Delta_1} C$, and we get a 2-level disjunctive HSG. $\square$

Note that under certain circumstances, redundancy in the equivalence classes of TSGs may not only result in 2-level HSGs as shown above, but also in 1-level HSGs (see the discussion on canonical representation of HSGs in Chapter 3).

From now on, we assume that C1 - C6 from the above proposition hold, and we say in this case that Δ_1 is canonically represented. Next we show that in a canonically represented Δ_1, the three equivalence classes are indeed distinct, in other words they have strict desirability orderings, meaning $A \succ_{\Delta_1} B \succ_{\Delta_1} C$.

4.2.2 Seniority of levels in a canonically represented Δ_1

(1) **Class A is more senior than class B.**
Firstly, let us observe the following. We know from (C3) that $|A| \geq t$ and the first winning condition is not redundant, so there exists a minimal winning coalition with t players from A, call it X. If we replace a player from X with a player from B, the resulting coalition will have $t-1$ players from A, so it will fail the first winning condition. Also, the resulting coalition will have t players in total, but since $m > t$ by (C1), then it also fails the second winning condition, meaning players from B cannot be at least as desirable as players from A. But we also know from (4.1.1) that $A \succeq_{\Delta_1} B$, therefore we have shown that $A \succ_{\Delta_1} B$.

(2) **Class B is more senior than class C.**
By (C4) there exists a winning coalition containing players from A or B or both such that it meets the second winning condition. But if it does not contain players from B, then a replacement of a player from A with a player from B will still result in a coalition meeting the second winning condition, such that it now contains a player from B. So we can assume that a minimal winning coalition containing players from B exists, and it satisfies the second winning condition $|X| \geq m$ and $|X \cap (A \cup B)| \geq m - d$, and does not satisfy $|X \cap A| \geq t$. Since X is minimal, then we can assume that $|X| = m$ (otherwise we can drop an element of C which is contained in X and still have it winning). We know from (C5) that $|C| > d$. And if $|X \cap (A \cup B)| > m - d$, then we keep replacing players from B with ones from C until $|X \cap (A \cup B)| = m - d$. So now we have $|X \cap C| = d$. If by now we don't have players of B left in X, then we have $m - d$ players

from A, and we can replace a player from A with a player from B and still have a winning coalition. And if we now replace a player from B with one from C (possible since $|C| > d$) the resulting coalition is losing, showing that players from C cannot be at least as desirable as players from B. But since by (4.1.1) we have $B \succeq_{\Delta_1} C$, then it follows that $B \succ_{\Delta_1} C$.

(3) **Class C is the least senior**.
By combining (1) and (2) above, and given that Isbell's desirability relation is transitive (see Taylor & Zwicker (1999), p.89-90), then we see that class C is the least senior. Therefore, we have shown that

$$A \succ_{\Delta_1} B \succ_{\Delta_1} C.$$

4.2.3 Multiset representation of Δ_1

It follows from the above analysis regarding completeness and seniority levels that if the game is canonically represented, then it has three distinct levels of seniority. Also, let us from now on consider the three classes A, B and C to be the three levels of seniority $1, 2$ and 3 respectively, such that $1 \succ_{\Delta_1} 2 \succ_{\Delta_1} 3$. If we also let n_1, n_2 and n_3 to denote the total numbers of players of levels $1, 2$ and 3 respectively, then using the multiset notation, we can re-write the definition of Δ_1 as follows: Let $P = \{1^{n_1}, 2^{n_2}, 3^{n_3}\}$, and m, d, t be positive integers such that $m \geq t$. Then $P \supseteq X = \{1^{l_1}, 2^{l_2}, 3^{l_3}\}$ is winning in Δ_1 iff

$$l_1 \geq t \text{ or } (l_1 + l_2 + l_3 \geq m \text{ and } l_1 + l_2 \geq m - d).$$

4.2.4 The shift-minimal winning coalitions of Δ_1

Now, putting all the facts that we gathered about Δ_1 so far together, we can list all its shift-minimal winning coalitions. Recall that by a shift we mean a replacement of a player of a coalition by a less desirable player which did not belong to it. Also, a winning coalition X is said to be shift-minimal if every coalition contained in it and every coalition obtained from it by a shift are losing.

For a canonically represented Δ_1, the first shift-minimal winning coalition is $\{1^t\}$, since we always have at least t players from level 1 by (C1). The second shift-minimal winning coalition has two possibilities, depending on the number of players in level 2. If $m - d < n_2$, then it is $\{2^{m-d}, 3^d\}$. That is because $n_3 > d$ by (C5), and there are at least $m - d$ players from levels 1 and 2 by (C4). But if $m - d \geq n_2$, then the second shift-minimal winning coalition is $\{1^{m-d-n_2}, 2^{n_2}, 3^d\}$, and it is a new minimal winning coalition due to $(C6)$. So these two possibilities for the second shift-minimal winning coalition give rise to two kinds of simple games, these are the two variations of Δ_1. An arbitrary game of the first kind will be referred to as T_{1a}, and an arbitrary game of the second kind will be referred to as T_{1b}, so that

$$T_{1a} \text{ has } \{1^t\} \text{ and } \{2^{m-d}, 3^d\} \text{ as shift-minimal winning coalitions;} \tag{4.2.1}$$

and

$$T_{1b} \text{ has } \{1^t\} \text{ and } \{1^{m-d-n_2}, 2^{n_2}, 3^d\} \text{ as shift-minimal winning coalitions.} \tag{4.2.2}$$

4.2.5 Ideality of Δ_1

The last property that we prove for Δ_1, is that it is an ideal simple game. This was already proved by Beimel, Tassa, & Weinreb (2008), but here we present a more straightforward proof based on the result by Farràs & Padró (2010), which we state below. Also, note that in (Farràs & Padró, 2010), it was shown that the tripartite games T_1, T_2 and T_3 of Theorem 1.1.2(iii) are ideal, whereas here we show that all games in the class of tripartite games are ideal. The result from (Farràs & Padró, 2010) that we will use, characterises ideal complete simple games in terms of properties of their shift-minimal winning coalitions. Before stating the result, we need some new notations, which we have slightly modified, for simplicity, from their original form. We will state the concepts and notations in general first, then apply them to our game Δ_1. Recall also the notation $[n] = \{1, 2, \ldots, n\}$.

Suppose we have a complete simple game of n desirability levels. For a shift-minimal winning coalition $X = \{1^{a_1}, 2^{a_2}, \ldots, n^{a_n}\}, a_i \leq n_i$ for all $i \in [n]$, let supp$(X) = \{i \in [n] : a_i \neq 0\}$. Suppose now that we have r shift-minimal winning coalitions $X_1, \ldots, X_r$. Then let $m_j = \max(\text{supp}(X_j)), j \in [r]$. Finally, let X_j^i be the number of players of type i in the shift-minimal winning coalition X_j.

To illustrate the above, suppose $n = 4$, and $X_1 = \{1^2, 2^3, 3^4\}$. Then supp$(X_1) = \{1, 2, 3\}, m_j = 3$. Also $X_1^1 = 2, X_1^2 = 3, X_1^3 = 4$ and $X_1^4 = 0$.

We are now ready to work with the characterisation theorem which we stated in Section 2.3. The statement of the theorem is also modified, we are re-writing it in a way that is consistent with the game-theoretic language and notation presented in this thesis.

Theorem 4.2.2 (Farràs & Padró (2010), Theorem 9.2)**.** *Let Γ be a complete simple game on $P = \{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}$. Also, let the set of shift-minimal winning coalitions be $\{X_1, \ldots, X_r\}$. Consider m_j = max(supp$(X_j)), 1 \leq j \leq r$, and suppose that the shift-minimal winning coalitions are listed in such a way that $m_j \leq m_{j+1}$. Then Γ is ideal if and only if*

(i) $m_j < m_{j+1}$ and $|X_j| < |X_{j+1}|$ for all $j = 1, \ldots, r-1$, and

(ii) $X_j^i \geq X_{j+1}^i$ if $1 \leq j \leq r-1$ and $1 \leq i \leq m_j$, and

(iii) if $X_j^i > X_r^i$ for some $1 \leq j < r$ and $1 \leq i < m_j$, then $n_k = X_j^k$ for all $k = i+1, \ldots, m_j$.

And now we can apply the above theorem to show that T_{1a} and T_{1b} are both ideal.

Proof of ideality of T_{1a} and T_{1b}. Let the shift-minimal winning coalitions be $X_1 = \{1^t\}$ and X_2 depending on whether we are dealing with T_{1a} or T_{1b} is $\{2^{m-d}, 3^d\}$ or $\{1^{m-d-n_2}, 2^{n_2}, 3^d\}$ respectively. Then supp$(X_1) = \{1\}$, meaning $m_1 = 1$. Also, supp$(X_2) = \{2, 3\}$ or $\{1, 2, 3\}$, but in both cases $m_2 = 3$. It follows that X_1 and X_2 are listed in such a way that $m_j \leq m_{j+1}$. We now check X_1 and X_2 against the three conditions of Theorem 4.2.2.

(i) This condition applies only to $j = 1$. But $m_1 < m_2$ and $|X_1| < |X_2|$, since $t < m$ by $(C1)$. So this condition holds.

(ii) This condition applies only to $j = 1 = i$. Now, $X_1^1 = t$, and $X_2^1 = 0$ or $m - d - n_2$. But $m - d - n_2 < t$ by Proposition 4.2.1(vi), so $X_1^1 > X_2^1$ and the second condition also holds.

(iii) Here the two conditions $1 \leq j < r$ and $1 \leq i < m_j$ are never both met, since if $j = 1$ then $m_j = 1$, implying that $i = 0$, contradicting $1 \leq i$.

Therefore both variants of Δ_1 are ideal.

4.2.6 A Canonical Representation for games of type Δ_2

We need to carry out an analysis for games $\Delta_2(A, B, C, m, d, t)$ similar to the one we did for Δ_1, since for Δ_2 the winning requirements are different. Recall that this is the case for $|B| \leq d + t - m$, and the two winning conditions are $|X \cap (A \cup B)| \geq t$, which we'll refer to as the first winning condition, and $|X| \geq m$ and $|X \cap A| \geq m - d$, which we'll refer to as the second winning condition.

Proposition 4.2.3. *Given a tripartite simple game $\Delta_2(A, B, C, m, d, t)$, the following conditions must be satisfied in order for the game to have A, B and C as three distinct equivalence classes with respect to Isbell's desirability relation:*

(C′1) $m > d$;

(C′2) $|A| + |B| \geq t$;

(C′3) both $|A| + |B| + |C| > m$, *and* $|A| \geq m - d$;

(C′4) $|B| + |C| > d$;

(C′5) $m > t$;

(C′6) $m - d < t$;

(C′7) $m - |C| < t$*;*

(C′8) $|C| \geq 2$*.*

Proof. Let X be a minimal winning coalition.

(C′1) If $m \leq d$, $|X \cap A| \geq m - d$ is trivially satisfied, so the winning conditions become $|X \cap (A \cup B)| \geq t$ or $|X| \geq m$, which are both symmetric with respect to A and B, thus $A \sim_{\Delta_2} B$.

The fact that A and B become equivalent to each other means that together they form one class, call it AB. So the game at hand now has at most two equivalence classes, AB and C. A winning coalition in this game has to have at least t players from AB or at least m players in total. It follows that the game is a 2-level disjunctive HSG with t being the first threshold, and m being the second threshold. The same applies to all cases: when classes A and B become equivalent, we always get a 2-level disjunctive HSG.

(C′2) This condition is needed for the first winning condition to be nontrivial. If it fails, then only the second winning condition can be met. But in the second winning condition $|X \cap A| \geq m - d$ is symmetric with respect to B and C, so we get $B \sim_{\Delta_2} C$.

The fact that B and C become equivalent to each other means that together they form one class, call it BC. So the game at hand now has at most two equivalence classes, A and BC. A winning coalition in this game has to have at least $m - d$ players from A and at least m players in total. It follows that the resulting game is a 2-level conjunctive HSG with $m - d$ being the first threshold, and m being the second threshold. The same applies to all cases: when classes B and C become equivalent, we always get a 2-level conjunctive HSG.

(C′3) If this condition fails, then either $|A| + |B| + |C| \leq m$, or $|A| < m - d$ or both. In the former, if $|A| + |B| + |C| < m$, then only $|X \cap (A \cup B)| \geq t$ can be met, which is symmetric with respect to A and B, so we get $A \sim_{\Delta_2} B$.

And if $|A|+|B|+|C| = m$, then in any winning coalition X such as $|X| = m$ all classes must be used fully, hence then the condition is also symmetric with respect to A and B, which again together with $|X \cap (A \cup B)| \geq t$ imply $A \sim_{\Delta_2} B$. If on the other hand $|A| < m - d$, then only $|X \cap (A \cup B)| \geq t$ can be met, again implying $A \sim_{\Delta_2} B$ as explained above, and we get a 2-level disjunctive HSG.

(C′4) If $|B| + |C| \leq d$, then $|X| \geq m$ implies $|X \cap A| \geq m - d$. So the two winning conditions are now $|X \cap (A \cup B)| \geq t$ or $|X| \geq m$, which are both symmetric with respect to A and B, meaning $A \sim_{\Delta_2} B$, and we get a 2-level disjunctive HSG.

(C′5) We know from Definition 4.1.1 that $m \geq t$, so we need to show that $m \neq t$. But if $m = t$, then since we are working from the definition of Δ_2, meaning $|B| \leq d + t - m$, then $|B| \leq d$. But here the first winning condition $|X \cap (A \cup B)| \geq t = m$ would imply $|X \cap A| \geq m - |B| \geq m - d$. And since the latter condition is symmetric with respect to B and C, then $B \sim_{\Delta_2} C$, and we get a 2-level conjunctive HSG.

(C′6) If $m-d \geq t$, then the second winning condition implies $|X \cap A| \geq m-d \geq t$, and hence $|X \cap (A \cup B)| \geq t$. Thus the second winning condition is redundant. But the first winning condition is symmetric with respect to A and B, so $A \sim_{\Delta_2} B$, and we get a 2-level disjunctive HSG.

(C′7) We know that the first winning condition does not require class C players. And if $m - |C| \geq t$, then the second condition actually implies the first, hence $A \sim_{\Delta_2} B$ as above, and we get a 2-level disjunctive HSG.

(C′8) Finally, if $|C| = 1$, then since from (C′7) above we know that we need $m - 1 < t$, then it follows that $m \leq t$, and we get $B \sim_{\Delta_2} C$ as shown in the proof of (C′5) above, and we get a 2-level conjunctive HSG. □

From now on, we assume that C′1 - C′8 from the above proposition hold, and in such a case we say that Δ_2 is canonically represented. Next we show that in a canonically represented Δ_2, the three equivalence classes have strict desirability orderings, meaning $A \succ_{\Delta_2} B \succ_{\Delta_2} C$.

4.2.7 Seniority of levels in a canonically represented Δ_2

(1) **Class A is more senior than class B.**
We know from (C′3) that we have a minimal winning coalition X meeting $|X| \geq m$ and $|X \cap A| \geq m - d$. We may assume that $|X| = m$ and $|X \cap A| = m - d$. Indeed, since X is minimal, then we can assume $|X| = m$, since we can always discard some players from B and C as by (C′4) we know that $|B| + |C| > d$. If X has $|X \cap A| > m - d$, then we can replace a player from A with one from $B \cup C$, and keep doing this until $|X \cap A| = m - d$. So let us assume now that our minimal winning coalition X has $|X| = m$ and $|X \cap A| = m - d$, meaning $|X \cap (B \cup C)| = d$. But since $|B| + |C| > d$, then either B is not used up fully in X, or C is not used up fully, or both. Suppose B is used up fully but not C, the other case is similar. If B is used up fully, then we can replace one player from B with one from C, X will remain winning as the second winning condition will still be satisfied. And now, a further replacement of a player from A with a player from B will result in a losing coalition, meaning players from B cannot be at least as desirable as players from A. But we also know from (4.1.1) that $A \succeq_{\Delta_2} B$, therefore we have shown that $A \succ_{\Delta_2} B$.

(2) **Class B is more senior than class C.**
By (C′2) there is a minimal winning coalition meeting the first winning condition that contains players from A or B or both, and it does not contain players from C. And if it contains only players from A, then replace a player from A with one from B and it will still be winning, such that it now contains a player from B. So assume we have X satisfying $|X \cap (A \cup B)| \geq t$ such that it contains players from B. If we now replace a player from B with one from C, the resulting coalition is losing, since it has size t and $m > t$ by (C′5). So players from C cannot be at least as desirable as players from

B. But we also know from (4.1.1) that $B \succeq_{\Delta_2} C$, therefore we have shown that $B \succ_{\Delta_2} C$.

(3) **Class C is the least senior**.
By combining (1) and (2) above, and given that Isbell's desirability relation is transitive, then we see that class C is the least senior. Therefore, we have shown that

$$A \succ_{\Delta_2} B \succ_{\Delta_2} C.$$

4.2.8 Multiset representation of Δ_2

It follows from the above analysis regarding completeness and seniority levels that if the game is canonically represented, then it has three distinct levels of seniority. So assuming the game is canonically represented, let us from now on consider the three classes A, B and C to be the three levels of seniority $1, 2$ and 3 respectively, such that $1 \succ_{\Delta_2} 2 \succ_{\Delta_2} 3$. If we also let n_1, n_2, n_3 to denote the total numbers of players of levels $1, 2$ and 3 respectively, then using the multiset notation, we can re-write the definition of Δ_2 as follows: Let $P = \{1^{n_1}, 2^{n_2}, 3^{n_3}\}$, and m, d, t be positive integers such that $m \geq t$ and $n_2 \leq d + t - m$. Then a submultiset $X = \{1^{l_1}, 2^{l_2}, 3^{l_3}\}$ is winning in Δ_2 iff

$$l_1 + l_2 \geq t \text{ or } (l_1 + l_2 + l_3 \geq m \text{ and } l_1 \geq m - d).$$

4.2.9 The shift-minimal winning coalitions of Δ_2

Now, putting all the facts that we gathered about Δ_2 so far together, we can list all its shift-minimal winning coalitions. For a canonically represented Δ_2, the first shift-minimal winning coalition is $\{1^{t-n_2}, 2^{n_2}\}$, since we always have at least t players from levels 1 and 2 by (C′2). The second shift-minimal winning coalition has two possibilities, depending on the number of players in level 3. If $n_3 \geq d$, then it is $\{1^{m-d}, 3^d\}$. This is because there are at least m players from classes A, B and C by (C′3), of which at least $m - d$ players are from A. But if $n_3 < d$, then the second shift-minimal winning coalition is $\{1^{m-d}, 2^{d-n_3}, 3^{n_3}\}$, which is a new coalition by (C$'$5). So these two possibilities for the second shift-minimal winning coalition give rise to two kinds of simple games, these are the two variations of Δ_2. An arbitrary game of the first kind will be referred to as T_{2a}, and an arbitrary game of the second kind will be referred to as T_{2b}, so that

$$T_{2a} \text{ has } \{1^{t-n_2}, 2^{n_2}\} \text{ and } \{1^{m-d}, 3^d\} \text{ as shift-minimal winning coalitions;} \quad (4.2.3)$$

and

$$T_{2b} \text{ has } \{1^{t-n_2}, 2^{n_2}\} \text{ and } \{1^{m-d}, 2^{d-n_3}, 3^{n_3}\} \text{ as shift-minimal winning coalitions.} \quad (4.2.4)$$

It is now recognised that there are four variations, T_{1a}, T_{1b}, T_{2a} and T_{2b} of TSGs.

Next we look at ideality of Δ_2.

4.2.10 Ideality of Δ_2

Proof of ideality of T_{2a} *and* T_{2b}. Let the shift-minimal winning coalitions be $X_1 = \{1^{t-n_2}, 2^{n_2}\}$ and X_2 depending on whether we are dealing with T_{2a} or T_{2b} is $\{1^{m-d}, 3^d\}$ or $\{1^{m-d}, 2^{d-n_3}, 3^{n_3}\}$ respectively. Then $\text{supp}(X_1) = \{1, 2\}$, meaning $m_1 = 2$. Also, $\text{supp}(X_2) = \{1, 3\}$ or $\{1, 2, 3\}$, but in both cases $m_2 = 3$. It follows that X_1 and X_2 are listed in such a way that $m_j \leq m_{j+1}$. We now check X_1 and X_2 against the three conditions of Theorem 4.2.2.

(i) This condition applies only to $j = 1$. But $m_1 < m_2$ and $|X_1| < |X_2|$, since $t < m$ by $(C'5)$. So this condition holds.

(ii) This condition applies only to $j = 1 = i$. Now, $X_1^1 = t - n_2,$ and $X_2^1 = m - d$. But Δ_2 is the case for $n_2 \leq d + t - m$, so $t - n_2 \geq m - d$, meaning $X_1^1 \geq X_2^1$. So the second condition also holds.

(iii) Here the two conditions $1 \leq j < r$ and $1 \leq i < m_j$ are both met when $j = 1 = i$, and since, as shown in (ii) above, $X_1^1 \geq X_2^1$, then this condition does not apply.

Therefore both variants of Δ_2 are also ideal.

Chapter 5

The Composition of Simple Games

The idea of composing simple games has a long history. Shapley (1962) introduced the most general type of composition, Martin (1993) and Beimel, Tassa, & Weinreb (2008) adapted a particular case of it to the needs of secret sharing schemes (without knowing about Shapley's construction). Their motivation was that sometimes all players can be classified into 'strong' players and 'weak' players, so that the game can be decomposed into the main game that contains strong players, and the auxiliary game which contains weak players.

It was shown by Beimel, Tassa, & Weinreb (2008) that if a weighted simple game is ideal, then it is either a disjunctive 2-level hierarchical, tripartite or a composition of two smaller ideal WSGs. This result was later refined by Farràs & Padró (2010). In the latter, they observed that since one of the possibilities of having an ideal WSG is that of a composition, then one should classify only indecomposable ideal WSGs. So in the latter paper, they gave a list of indecomposable ideal WSGs.

Thus the first stage of characterising ideal WSGs was the classification of indecomposable ideal WSGs. The second stage, which we now have to address, is to answer the question: When does the composition of two ideal weighted simple games also result in an ideal weighted simple game? The answer to this question is crucial for obtaining an 'if and only if' characterisation of ideal WSGs. The reason for this is the fact that the class of ideal weighted games is not closed under compositions.

So in this chapter and the next, we will undertake a thorough study of compositions of games in general, and of compositions of ideal WSGs in particular. Chapter 6 will be the culmination of this study, where we will give an 'if and only if' characterisation theorem for all ideal weighted simple games.

5.1 Definition and examples

The composition of simple games is an operation by which a new game is generated from two or more given ones. Shapley (1962) was the first to introduce it, under the name 'compound simple games'. A special case of it was re-discovered and investigated later in (Martin, 1993), and then used in (Beimel, Tassa, & Weinreb, 2008). Given two simple games G and H over the sets of players P_G and P_H respectively, the composition is constructed over an arbitrary player $g \in P_G$. The winning coalitions in the composition are the winning coalitions from G that do not contain player g, together with the winning coalitions of G that contain g, with g being replaced by a winning coalition of H. This operation is an important one, it played a pivotal role in the characterisation of ideal WSGs given by Beimel, Tassa, & Weinreb (2008) and Farràs & Padró (2010).

First we start with the formal definition of compositions.

Definition 5.1.1. Let $G = (P_G, W_G)$ and $H = (P_H, W_H)$ be two simple games. Let $g \in P_G$, and $P_C = P_G \cup P_H \setminus \{g\}$. Then the composition $C = (P_C, W_C)$ of G and H over g is given by

$$W_C = \{X \subseteq P_C : X_G \in W_G \textit{ or } (X_G \cup \{g\} \in W_G \textit{ and } X_H \in W_H)\},$$

where $X_G = X \cap P_G$ and $X_H = X \cap P_H$.

Namely, $X \subseteq P_C$ is winning in this simple game if either $X_G = X \cap P_G$ is winning in W_G, or $X_G \cup \{g\}$ is winning in W_G and $X_H = X \cap P_H$ is winning in W_H.

We shall denote the composition over player g from G by $G \circ_g H$. It is not difficult to see that if $|P_G| = 1$, then $G \circ_g H = H$ and, if $|P_H| = 1$, then $G \circ_g H \cong G$, so both compositions are trivial. Therefore, to separate trivial decompositions from nontrivial ones, we need $\min(|P_G|, |P_H|) \geq 2$. To formalise the concept, we give the following definition.

Definition 5.1.2. If C can be represented as a composition $C = G \circ_g H$, where $g \in P_G$ and $\min(|P_G|, |P_H|) \geq 2$, then C is *decomposable*, and if it cannot, then it is *indecomposable*.

In $G = (P_G, W_G)$, we let $W_G^{\min}$ denote the set of minimal winning coalitions of W_G. Observe the following basic fact.

Proposition 5.1.3. *Let G, H be two games defined on the disjoint set of players and $g \in P_G$. Then for $C = G \circ_g H$, the set of minimal winning coalitions is*

$$W_C^{min} = \{X \mid X \in W_G^{min} \text{ and } g \notin X\} \cup \{X \cup Y \mid X \cup \{g\} \in W_G^{min} \text{ and }$$

$$Y \in W_H^{min} \text{ with } g \notin X\}.$$

Proof. Follows directly from the definition. $\square$

Example 5.1.4. Let $G = (P_G, W_G)$ be the first simple game such that $P_G = \{1, 2, 3\}, W_G^{\min} = \{\{1, 2\}, \{2, 3\}\}$. And let $H = (P_H, W_H)$ be the second simple game such that $P_H = \{4, 5, 6\}, W_H^{\min} = \{\{4\}, \{5, 6\}\}$. Then the composition $C = G \circ_3 H$ over player 3 from P_G gives the game on the set $P_C = \{1, 2, 4, 5, 6\}$, and the following set of minimal winning coalitions $W_C^{\min} = \{\{1, 2\}, \{2, 4\}, \{2, 5, 6\}\}$.

Example 5.1.5. Let $G = (P, W)$ be a simple game and $A \subseteq P$ be the set of all vetoers in this game. Let $|A| = m$. Then $G \cong U_{m+1} \circ_u G^A$, where u is any player of U_{m+1}. So any game with vetoers is decomposable.

Next we prove associativity.

Proposition 5.1.6. *(Associativity) Let G, H, K be three games defined on three disjoint sets of players, and $g \in P_G$, $h \in P_H$. Then*

$$(G \circ_g H) \circ_h K = G \circ_g (H \circ_h K),$$

that is the two compositions are the same.

Proof. Let us classify the minimal winning coalitions of the game $(G \circ_g H) \circ_h K$. By Proposition 5.1.3 they can be of the following types:

- $X \in W_G^{\min}$ with $g \notin X$;
- $X \cup Y$, where $X \cup \{g\} \in W_G^{\min}$ and $Y \in W_H^{\min}$ with $g \notin X$ and $h \notin Y$;
- $X \cup Y \cup Z$, where $X \cup \{g\} \in W_G^{\min}$, $Y \cup \{h\} \in W_H^{\min}$ and $Z \in W_K^{\min}$ with $g \notin X$ and $h \notin Y$.

It can be seen that the game $G \circ_g (H \circ_h K)$ has exactly the same minimal winning coalitions. □

We also note the following interesting fact about the presence of dummies in the composition.

Proposition 5.1.7. *Let G and H be two games defined on disjoint sets of players P_G and P_H respectively, such that $C = G \circ_g H$, where $g \in P_G$. Then C has dummies if and only if either*

(i) g is a dummy in G, or

(ii) G has dummies different from g, or

(iii) H has dummies.

Proof. (i) If player g is dummy, then it does not participate in any minimal winning coalition of G, and therefore, it will not be a part of any minimal winning coalition of G, so by Proposition 5.1.3 the minimal winning coalitions of C are in

this case just the minimal winning coalitions of G. However, the full set of players is $P_C = P_G \setminus \{g\} \cup P_H$, meaning all players in P_H are now dummies in C.
(ii) If any player in G, other than g, is a dummy, then it will not participate in minimal winning coalitions of G, nor will it in minimal winning coalitions of C by Proposition 5.1.3.
(iii) If H has dummies, then they will not participate in minimal winning coalitions of H, nor will they in minimal winning coalitions of C, also by Proposition 5.1.3.
The converse is also clear. $\square$

It was mentioned in Chapter 3 that in the context of secret sharing schemes, accounting for dummies is of no practical importance [1]. Since our study of compositions here will be applied ultimately to ideal simple games, then we assume from now on that our simple games have no dummies. Therefore, by Proposition 5.1.7 above, none of the compositions of games will have dummies in them either.

We conclude this section by demonstrating the uniqueness of decomposition of games in the situation that will be further important. Note that a game where its only winning coalition is the full set of players is called a *unanimity* game.

Theorem 5.1.8. *Let H_{n_1,k_1} and H_{n_2,k_2} be two k-out-of-n games which are not unanimity games. Then, if $G = H_{n_1,k_1} \circ_{h_1} G_1 = H_{n_2,k_2} \circ_{h_2} G_2$, with G_1 and G_2 having no passers, then $n_1 = n_2$, $k_1 = k_2$ and $G_1 = G_2$. If $G = U_{n_1} \circ_{u_1} G_1 = U_{n_2} \circ_{u_2} G_2$ and G_1 and G_2 does not have vetoers, then $n_1 = n_2$ and $G_1 = G_2$.*

Proof. Suppose that we know that $G = H \circ_{h_1} G_1$, where H is a k-out-of-n game but not a unanimity game. Then all winning coalitions in G of smallest cardinality have k players, so k in this case can be recovered unambiguously. Then n can be also recovered. Indeed, since G_1 and G_2 have no passers, the set of all players that participate in winning coalitions of size k will have cardinality $n - 1$. So there

[1] It is easy to include dummies anyway: just assign meaningless shares to them.

cannot exist two decompositions $G = H_{n_1,k_1} \circ_{h_1} G_1$ and $G = H_{n_2,k_2} \circ_{h_2} G_2$ of G, where $k_1 \neq k_2$ with $k_1 \neq n_1$ and $k_2 \neq n_2$.

Let us consider now the game $G = U \circ_u G_1$, where U is a unanimity game. Due to Example 5.1.5 if G_1 does not have vetoers, then U consists of all vetoers of G and uniquely recoverable. $\square$

We leave the question of whether or not any decomposition is unique open.

5.2 Properties of complete games that are composed of smaller games

In this section, first we give some important general results concerning complete games that are composed of smaller games, then we give general results concerning weighted games that are composed of smaller games.

The first two lemmas of this section give a necessary and sufficient condition for a composition to be complete. These results will be very helpful when we come to characterising all ideal weighted games.

A simple game with a unique minimal winning coalition is called an *oligarchy*. Generally speaking, it is possible to have an oligarchy with players that do not participate in its unique minimal winning coalition, meaning those players are dummies. But we are excluding dummies in our study of ideal simple games, therefore all oligarchies that we deal with are unanimity games. Also, if every minimal winning coalition in a game is a singleton, then it is called *anti-unanimity* game, such that if it has n players, then it is denoted A_n.

The first lemma below will show that, with a few exceptions, if the composition is not over a player from the least desirable level of the first game, then the composition is not complete, and hence not weighted. Recall that we assume we have no dummies in the games.

Lemma 5.2.1. *Let G, H be two games the disjoint sets of players P_G and P_H respectively, and H is neither a unanimity nor an anti-unanimity game. If for two elements $g, g' \in P_G$ we have $g \succ g'$, then $G \circ_g H$ is not complete.*

Proof. As g is more desirable than g', there exists a coalition $X \subseteq P_G$, containing neither g nor g' such that $X \cup g \in W_G$ and $X \cup g' \notin W_G$. We may take X to be minimal with this property, then $X \cup g$ is a minimal winning coalition of G. Since g' is not dummy, there exist a minimal winning coalition Y of G containing g'. The coalition Y may contain g or may not. Firstly, assume that it does contain g. Since H is not a unanimity game and has no dummies, there exist two distinct minimal winning coalitions of H, say Z_1 and Z_2. Then we can find $z \in Z_1 \setminus Z_2$. Then the coalitions $U_1 = X \cup Z_1$ and $U_2 = (Y \setminus \{g\}) \cup Z_2$ are winning in $G \circ_g H$, and coalitions $V_1 = (X \cup \{g'\}) \cup (Z_1 \setminus \{z\})$ and $V_2 = Y \setminus \{g, g'\} \cup Z_2 \cup \{z\}$ are losing in this game, since $Z_1 \setminus \{z\}$ is losing in H and, $Y \setminus \{g'\}$ is losing in G. Since V_1 and V_2 are obtained when U_1 and U_2 swap players z and g', the sequence of sets $(U_1, U_2; V_1, V_2)$ is a certificate of incompleteness for $G \circ_g H$.

The second case, when Y does not contain g is similar. Let Z be any minimal winning coalition of H that has more than one player, which we know must exist since H is not an anti-unanimity. Let $z \in Z$. Then

$$(X \cup Z, Y; X \cup \{g'\} \cup (Z \setminus \{z\}), Y \setminus \{g'\} \cup \{z\})$$

is a certificate of incompleteness for $G \circ_g H$. □

As for the converse of the above lemma, we don't need the condition of H being neither a unanimity nor an anti-unanimity game.

Lemma 5.2.2. *Let G, H be two complete games on the disjoint sets of players P_G and P_H respectively, $g \in P_G$ be a player of the least desirable level in G. Then for the game $C = G \circ_g H$*

(i) for $x, y \in P_G \setminus \{g\}$ it holds that $x \succeq_G y$ if and only if $x \succeq_C y$;

(ii) for $x, y \in P_H$ it holds that $x \succeq_H y$ if and only if $x \succeq_C y$;

(iii) for $x \in P_G \setminus \{g\}$ and $y \in P_H$, we have $x \succeq_C y$. Moreover, if y is not a passer or vetoer in H, then $x \succ_C y$.

In particular, C is complete.

Proof. The first two cases are obvious. Let us prove (iii). We have $x \succeq_G g$ since g is from the least desirable class in G. Let us consider a coalition $Z \subset C$ such that $Z \cap \{x, y\} = \emptyset$, and suppose there exists $Z \cup \{y\} \in W_C$ but $Z \cup \{x\} \notin W_C$. Then Z must be losing in C, and hence $Z \cap P_G$ cannot be winning in G, but $Z \cap P_G \cup \{g\}$ must be winning in G. However, since $x \succeq_G g$, the coalition $Z \cap P_G \cup \{x\}$ is also winning in G. But then $Z \cup \{x\}$ is winning in C, a contradiction. This shows that if $Z \cup \{y\}$ is winning in C, then $Z \cup \{x\}$ is also winning in C, meaning $x \succeq_C y$. Thus C is a complete game.

Moreover, suppose that y is not a passer or a vetoer in H, we will show that $x \succ_C y$. Since g is not a dummy, then x is not a dummy either. Let X be a minimal winning coalition of G containing x. If $g \notin X$, then X is also winning in C. However, $X \setminus \{x\} \cup \{y\}$ is losing in C, since y is not a passer in H. Thus it is not true that $y \succeq_C x$ in this case. If $g \in X$, then consider a winning coalition Y in H not containing y (this is possible since y is not a vetoer in H). Then $X \setminus \{g\} \cup Y \in W_C$ but

$$X \setminus \{x\} \cup \{g\} \cup \{y\} \cup Y \notin W_C,$$

whence it is not true that $y \succeq_C x$ in this case as well. Thus $x \succ_C y$ in case y is neither a passer nor a vetoer in H. □

Now, although Lemma 5.2.1 is very useful, it is only partial because it excludes the possibility of the second game H being either a unanimity or anti-unanimity game. But we also have the following result for the case when H is an anti-unanimity game.

Lemma 5.2.3. *Let $G = (P, W)$ be a game where players $g, g' \in P$ is such that g is more desirable than nondummy player g'. Suppose also that we can find a* *trading transform*

$$(X_1, X_2; Y_1, Y_2)$$

such that

$$g' \notin X_1, \quad X_1 \cup \{g\} \in W, \quad X_1 \cup \{g'\} \in L;$$

$$g' \in X_2, \quad X_2 \cup \{g\} \in W, \quad X_2 \setminus \{g'\} \cup \{g\} \in L.$$

Then the composition $G \circ_g A_n$, $n \geq 2$, is not complete.

Proof. Let $a, b \in A_n$. We have the following certificate of incompleteness:

$$(X_1 \cup \{a\}, X_2 \cup \{b\}; X_1 \cup \{g'\}, X_2 \setminus \{g'\} \cup \{a, b\}). \tag{5.2.1}$$

This proves the lemma. □

5.2.1 Properties of weighted games that are composed of smaller games

It turns out that the necessary and sufficient condition of Lemmas 5.2.1 and 5.2.2 is necessary for a composition to be a weighted game, but it is not sufficient. This is what we show in the next corollary and the example following it.

Corollary 5.2.4. *Let G, H be two weighted games on disjoint sets of players P_G and P_H respectively, such that H is neither a unanimity nor an anti-unanimity game. If $C = G \circ_g H$ is weighted, then g belongs to the least desirable level of G.*

Proof. If g does not belong to the least desirable level of G, then C is not complete by Lemma 5.2.1. It follows that the composition is not weighted. □

Example 5.2.5. Consider as the first weighted simple game the disjunctive hierarchical game $H_\exists(n, k)$ such that $\mathbf{k} = (2, 3), \mathbf{n} = (2, 10)$. The set of shift-minimal winning coalitions is $\{\{1^2\}, \{2^3\}\}$, and it is a weighted game by Theorem 3.6.2(2). The second weighted simple game on $\{3^2, 4^3\}$ is a conjunctive hierarchical game $H_\forall(n, k)$ such that $\mathbf{k} = (1, 2), \mathbf{n} = (2, 3)$, so its only shift-minimal winning coalition is $\{3, 4\}$. It is also a weighted game by Theorem 3.6.3(2). We are composing over a player $g = 2$ from the second level of the first game, so the full multiset of the composition $C = G \circ_g H$ will be $P_C = P_G \cup P_H \setminus \{g\} = \{1^2, 2^9, 3^2, 4^3\}$, and the set of shift-minimal winning coalitions of C is $\{\{1^2\}, \{2^3\}, \{2^2, 3, 4\}\}$. This composition is not weighted due to the following certificate of nonweightedness:

$$(\{1^2\}, \{2^2, 3, 4\}; \{1, 2, 3\}, \{1, 2, 4\}).$$

Since HSGs, disjunctive and conjunctive, are ideal (Tassa, 2007; Brickell, 1989), then the above example has in fact shown that even when the two games being composed are ideal and weighted, their composition may still not be weighted.

The following proposition is also very useful.

Proposition 5.2.6. *If $C = G \circ_g H$ is a weighted simple game, then both G and H are also weighted simple games.*

Proof. Suppose first that we have a certificate of nonweightedness $(U_1, \ldots, U_j; V_1, \ldots, V_j)$ for the game H. Let also X be the minimal winning coalition of G containing g. Let $X' = X \setminus \{g\}$. Then

$$(X' \cup U_1, \ldots, X' \cup U_j; X' \cup V_1, \ldots, X' \cup V_j)$$

is a certificate of nonweightedness for C. Suppose now that $(X_1, \ldots, X_j; Y_1, \ldots, Y_j)$ is a certificate of nonweightedness for G and W is a fixed minimal winning coalition W for H. Define

$$X_i' = \begin{cases} X_i \setminus \{g\} \cup W & \text{if } g \in X_i \\ X_i & \text{if } g \notin X_i \end{cases}$$

and

$$Y_i' = \begin{cases} Y_i \setminus \{g\} \cup W & \text{if } g \in Y_i \\ Y_i & \text{if } g \notin Y_i \end{cases}$$

Then, since $|\{i \mid g \in X_i\}| = |\{i \mid g \in Y_i\}|$, the following is a trading transform

$$(X'_1, \dots, X'_j; Y'_1, \dots, Y'_j).$$

Moreover, it is a certificate of nonweightedness for C since all $X'_1, \dots, X'_j$; are winning in C and all $Y'_1, \dots, Y'_j$ are losing in C. So both assumptions are impossible. $\square$

Corollary 5.2.7. *Every weighted game is a composition of indecomposable weighted games.*[1]

The following result will also be useful in proving that certain simple games are indecomposable. Let us denote the k-out-of-n game by $H_{n,k}$ from now on.

Proposition 5.2.8. *A game $H_{n,k}$ for $n \neq k \neq 1$ is indecomposable.*

Proof. Suppose $H_{n,k}$ is decomposable into $H_{n,k} = K \circ_g L, K = (P_K, W_K), L = (P_L, W_L)$, where $n_1 = |P_K| \geq 2$ and $n_2 = |P_L| \geq 2$. If g is a passer in K, then it is the only passer, otherwise if there is another passer g' in K, then $\{g'\}$ is winning in the composition, contradicting $k \neq 1$.

We will firstly show that $n_2 < k$. Suppose that $n_2 \geq k$, and choose a player $x \in P_K$ different from g. Consider a coalition X containing k players from P_L, then X is winning in the composition, and it is also a minimal winning coalition in L. Now replace a player in X from P_L with x. The resulting coalition, although it has k players, is losing in the composition, because x is not a passer in K, and $k - 1$ players from P_L are losing in L. Therefore $k > n_2$.

We also have $|P_K \setminus \{g\}| = n - n_2 > k - n_2$. Let us choose any coalition Z in $P_K \setminus \{g\}$ with $k - n_2$ players. Note that it does not win with g as $|Z \cup \{g\}| = k - n_2 + 1 < k$ players. This is why $Z \cup P_L$ is also losing despite having k players in total, contradiction. $\square$

[1] As usual in Mathematics, we assume that if G is indecomposable, then it has a trivial decomposition into a composition of indecomposable games, i.e., $G = G$.

5.3 The composition of ideal simple games

As far as ideal simple games are concerned, they are closed under the operation of composition.

Lemma 5.3.1. *If $C = G \circ_g H$ such that g is not dummy, then C is ideal if and only if both G and H are ideal.*

Proof. See Lemma 8.1 in Beimel, Tassa, & Weinreb (2008). □

The following is immediate.

Corollary 5.3.2. *Every ideal weighted game is a composition of indecomposable ideal weighted games.*

Proof. Follows from combining Corollary 5.2.7 and Lemma 5.3.1. □

It was implicitly assumed in both (Beimel, Tassa, & Weinreb, 2008) and (Farràs & Padró, 2010) that the composition of any two ideal weighted games is also ideal weighted. We know from the above lemma that the composition will be ideal, but it doesn't have to be weighted, this was demonstrated in Example 5.2.5.

It turns out, as we shall prove, that G plays a pivotal role in determining when the composition is weighted. In the next chapter we will consider all possibilities for ideal weighted indecomposable G, and see which of those cases, when composed with H, result in an ideal weighted game. A list of the indecomposable ideal weighted games appeared in (Farràs & Padró, 2010). So our first task in the next chapter will be to describe all the indecomposable ideal WSGs in details.

Chapter 6

The Characterisation Theorem

Recall from Chapter 1 that the work carried out in this thesis is mainly motivated by one of the central problems of secret sharing schemes, which is to characterise all ideal access structures. But access structures are simple games, and one approach to this problem has been to characterise ideal simple games in particular known classes, such as the class of weighted simple games. Characterising ideal weighted simple games is the focus of this chapter.

Current results regarding the problem of characterising ideal weighted simple games by Beimel, Tassa, & Weinreb (2008) and Farràs & Padró (2010) show that if a game is ideal and weighted, then it is a composition of indecomposable ideal weighted simple games (see Corollary 5.3.2), where the indecomposable ideal weighted simple games have also been classified [1]. However, for a complete characterisation of ideal weighted simple games, we need an 'if and only if' theorem. The work carried out in this chapter will culminate in giving an 'if and only if' characterisation of ideal weighted simple games, which is Theorem 6.1.4.

Before discussing the main obstacle to obtaining a characterisation of all ideal weighted simple games, and before we describe our strategy for overcoming this obstacle, let us list the seven types of games that are given in (Farràs & Padró

[1]This classification, however, needs some minor adjustments, which we will address in Section 6.1.

(2010), p.234), that will be of a major importance in this chapter. Recall that T_{1a} and T_{1b} are tripartite games of type Δ_1 defined on page 82, T_{2a} and T_{2b} are tripartite games of type Δ_2 defined on page 89. The list of seven types of games, such that every ideal weighted and indecomposable game must belong to one of those types, is as follows.

The LIST

(CL1) Simple majority games.

(CL2) $\mathbf{B}_1$: Games of this type are hierarchical games $H_\forall(n,k), \mathbf{n} = (n_1, n_2), \mathbf{k} = (k_1, k_2)$, where $0 < k_1 < n_1$ and $k_2 - k_1 = n_2 - 1 > 1$.

(CL3) $\mathbf{B}_2$: Games of this type are hierarchical games $H_\exists(n,k), \mathbf{n} = (n_1, n_2), \mathbf{k} = (k_1, k_1+1)$, such that $1 < k_1 \leq n_1, k_2 \leq n_2$.

(CL4) $\mathbf{B}_3$: Games of this type are hierarchical games $H_\exists(n,k), \mathbf{n} = (n_1, n_2), \mathbf{k} = (k_1, k_1+1)$, such that $k_1 \leq n_1, k_2 > n_2 > 2$.

(CL5) $\mathbf{T}_1$: A game of this type is a weighted indecomposable T_{1a}. It satisfies the conditions $0 < m - d < n_2$ and $m - t = 1$ and $t > 1$ and $n_3 - 1 = d > 1$. So it has shift-minimal winning coalitions of the two forms $\{1^{m-1}\}$ and $\{2^{m-d}, 3^d\}$.

(CL6) $\mathbf{T}_2$: A game of this type is a weighted indecomposable T_{1b}. It satisfies the conditions $n_3 - 1 = d > 1$ and $m = t + 1$ and $n_2 > 0$. So it has shift-minimal winning coalitions of the two forms $\{1^t\}$ and $\{1^{m-d-n_2}, 2^{n_2}, 3^d\}$.

(CL7) $\mathbf{T}_3$: A game of this type, with shift-minimal winning coalitions of the two forms $\{1^{t-n_2}, 2^{n_2}\}$ and $\{1^{m-d}, 2^{d-n_3}, 3^{n_3}\}$ is written in a general form that covers a weighted indecomposable T_{2a} game, and a weighted indecomposable T_{2b} game. It satisfies $d - n_3 + 1 = n_2 > 0$ and $m - t = 1$ and $t - n_2 > m - d > 0, n_3 > 1$.

We shall refer to the list above simply as The LIST. The importance of The LIST, stems from the following theorem.

Theorem 6.0.3. *((Farràs & Padró, 2010), Theorem 21) If a weighted simple game is ideal, then it is one of the following:*

(i) a simple majority game (a k-out-of-n simple game), or

(ii) a bipartite simple game in one of the types $\boldsymbol{B}_1$, $\boldsymbol{B}_2$ or $\boldsymbol{B}_3$, or

(iii) a tripartite simple game in one of the types $\boldsymbol{T}_1$, $\boldsymbol{T}_2$ or $\boldsymbol{T}_3$, or

(iv) a composition of smaller ideal weighted simple games.

The above theorem heavily used the earlier results of (Beimel, Tassa, & Weinreb, 2008). Also, Theorem 6.0.3 shows that any indecomposable ideal weighted game cannot belong to any type of games outside the seven types of games: simple majority games, $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \mathbf{T}_1, \mathbf{T}_2$ or $\mathbf{T}_3$, but it does not claim that games belonging to those types are indecomposable. We will show, however, that games belonging to those types, with few exceptions, are indecomposable. But the proof of this will be deferred to Corollary 6.2.2.

Now, the main obstacle to obtaining a complete characterisation of all ideal weighted simple games, is that not any two ideal weighted games, when composed together, result in an ideal weighted game (see Example 5.2.5). Consider the composition $\Gamma = \Gamma_1 \circ_g \Gamma_2$, and recall from Lemma 5.3.1 that Γ is ideal if and only if both Γ_1 and Γ_2 are ideal. Moreover, current results mentioned above, inform us that both Γ_1 and Γ_2 have to be weighted in order for Γ to be weighted. In order to have an 'if and only if' theorem, we need to answer the question: what are the necessary and sufficient conditions for $\Gamma_1 \circ_g \Gamma_2$ to be ideal weighted? Our strategy for answering this question takes the following steps:

(1) We go through all seven types of ideal weighted simple games from The LIST, while making minor but necessary modifications to it. This is done in Section 6.1.

(2) Given that both Γ_1 and Γ_2 are ideal and weighted, we suppose that Γ_1 is indecomposable, while Γ_2 may be decomposable. And fixing Γ_1, we analyse

all possibilities of obtaining a weighted composition $\Gamma = \Gamma_1 \circ_g \Gamma_2$. This analysis reveals that there are only two cases that result in an ideal weighted composition (Theorem 6.2.1).

We know from Corollary 5.2.4 that given that Γ_2 is neither a unanimity nor an anti-unanimity game, player g must be from the least desirable level of Γ_1, otherwise Γ will not be weighted. So we divide this analysis into three parts:

(i) Assuming g is from the least desirable level of Γ_1, in Section 6.2.1 we give the only two cases that result in an ideal weighted composition Γ.

(ii) In Section 6.2.2, we go through the rest of the cases where g is still from the least desirable level of Γ_1, and show that none of those result in an ideal weighted composition Γ.

(iii) In Section 6.2.3, we consider the cases where g is not from the least desirable level of Γ_1, and Γ_2 is either a unanimity or an anti-unanimity game, and also show that none of these cases result in an ideal weighted composition Γ.

(3) Finally, putting together all the pieces from (1) and (2) above, we prove the main characterisation theorem.

First, we introduce some new notation for a type of games.

Recall that a simple majority game, or k-out-of-n game denoted $H_{n,k}$ for $1 \leq k \leq n$, is a game which has n players in total and it takes k or more to win. Also, recall that an anti-unanimity game on n players denoted A_n is one where its every player is a passer, so in fact $A_n = H_{n,1}$. Finally, the unanimity game on n players, denoted hereinafter as U_n, is a game where its only winning coalition is the full set of players, so in fact $U_n = H_{n,n}$. These three games are all 1-level hierarchical games, or alternatively, they can be characterised as the class of complete 1-partite games, i.e., the games with a single level of equivalent players.

6.1 Indecomposable Ideal Weighted Simple Games

In this section we shall go through the types of games from The LIST on page 104, and identify all indecomposable games in those types. For compactness we use the multiset representation of the games.

Remark 6.1.1. A *trivial game* is a game with only one player who forms the only winning coalition. And although a trivial game cannot be written as a composition of two nontrivial games, it will not be considered indecomposable, this is in line with similar mathematical conventions (e.g., not defining the number 1 as prime).

Theorem 6.1.2. *A game is ideal weighted and indecomposable if and only if it belongs to one of the following types:*

(1) ***H****: Games of this type are* A_2*,* U_2 *and* $H_{n,k}$ *where* $1 < k < n$*.*

(2) $\boldsymbol{B}_1$*: Games of this type are conjunctive hierarchical games* $H_\forall(n,k), \boldsymbol{n} = (n_1, n_2), \boldsymbol{k} = (k_1, k_2)$*, where* $n_1 > k_1 > 0$ *and* $k_2 - k_1 = n_2 - 1 > 1$*.*

(3) $\boldsymbol{B}_2$*: Games of this type are disjunctive hierarchical games* $H_\exists(n,k), \boldsymbol{n} = (n_1, n_2), \boldsymbol{k} = (k_1, k_1 + 1)$*, such that* $1 < k_1 \le n_1, k_2 \le n_2$*.*

(4) $\boldsymbol{B}_3$*: Games of this type are disjunctive hierarchical games* $H_\exists(n,k), \boldsymbol{n} = (n_1, n_2), \boldsymbol{k} = (k_1, k_1 + 1)$*, such that* $k_1 \le n_1, k_2 > n_2 > 2$*.*

(5) $\boldsymbol{T}_1$*: A game of this type is a weighted indecomposable* T_{1a}*. So it satisfies* $0 < m - d < n_2$ *and* $m - t = 1$ *and* $t > 1$ *and* $1 < d = n_3 - 1$*.*

(6) $\boldsymbol{T}_{3a}$*: A game of this type is a weighted indecomposable* T_{2a}*. So it satisfies* $t - n_2 > m - d$ *and* $d = n_3$ *and* $m = t + 1$ *and* $n_2 = 1$*.*

(7) $\boldsymbol{T}_{3b}$*: A game of this type is a weighted indecomposable* T_{2b}*. So it satisfies* $d - n_3 = n_2 - 1$ *and* $m - t = 1$ *and* $t - n_2 > m - d$*.*

The proof of the above theorem relies heavily on Theorem 6.0.3, and we will be referring to the types of games CL1 - CL7 from The LIST on page 104.

Remark 6.1.3. Note that $\mathbf{T}_1$ in Theorem 6.0.3(iii) is the same as $\mathbf{T}_1$ in Theorem 6.1.2(5). And games of type $\mathbf{T}_2$ in Theorem 6.0.3(iii) are decomposable as will be shown, so not included in Theorem 6.1.2. Also, $\mathbf{T}_3$ of Theorem 6.0.3(iii) is written in a general form that covers two possibilities, one for the indecomposable and weighted T_{2a} game, and one for the indecomposable and weighted T_{2b} game, the type of the former is called $\mathbf{T}_{3a}$, and the type of the latter is called $\mathbf{T}_{3b}$ in Theorem 6.1.2 items (6) and (7) respectively.

Proof. (1). A game $H_{n,k}$ where $1 < k < n$ is indecomposable by Proposition 5.2.8, and a game A_n is decomposable if $n > 2$ by the following

$$A_n \circ_h A_m \cong A_{n+m-1}$$

for any $h \in A_n$. So it follows that the only indecomposable anti-unanimity game is A_2. Also, U_n is decomposable if $n > 2$ by the following

$$U_n \circ_h U_m \cong U_{n+m-1}$$

for any $h \in A_n$. So it follows that the only indecomposable unanimity game is U_2. So A_2, U_2 and $H_{n,k}$ where $1 < k < n$ are all indecomposable. Moreover, they are ideal since they are 1-level hierarchical games, and they are weighted.

(2). Conjunctive HSGs of this type are weighted by Theorem 3.6.3(3). Also, in order for an indecomposable game to be of this type, then by CL2, it must have the conditions $n_1 > k_1 > 0$ and $k_2 - k_1 = n_2 - 1$. The only form of a shift-minimal winning coalition here is $\{1^{k_1}, 2^{k_2-k_1}\}$. However, note that in CL2, one of the conditions also says $n_2 - 1 > 0$, meaning $n_2 - 1 = 1$ is possible. But we will now show that $n_2 - 1 = 1$ in fact leads to a decomposition, hence it should be that $n_2 - 1 > 1$.

The decomposition is as follows: Assume $k_2 - k_1 = n_2 - 1 = 1$, so $n_2 = 2$ and $k_2 = k_1 + 1$, then we have $\mathbf{k} = (k_1, k_1 + 1), \mathbf{n} = (n_1, 2)$, and the shift-minimal winning coalition here is $\{1^{k_1}, 2\}$. Let the first game be $G = (P_G, W_G)$, $P_G = \{1^{n_1+1}\}, W_G = \{1^{k_1+1}\}, n_1 > k_1$, and let the second game be $H = (P_H, W_H), P_H = \{2^2\}, W_H = \{2\}$. Then composing the two games over player

$g \in P_G$ gives $W^{\min}_{G \circ_g H} = \{\{1^{k_1+1}\}, \{1^{k_1}, 2\}\}$, of which only $\{1^{k_1}, 2\}$ is shift-minimal. This proves that a game of type $\mathbf{B}_1$ is decomposable when $n_2 = 2$.

(3). Disjunctive HSGs of this type must have the condition $n_2 > 2$ for indecomposability by CL3 ($n_2 \geq k_2$ and $k_2 = k_1 + 1$ and $k_1 > 1$). The shift-minimal winning coalitions here have the forms $\{1^{k_1}\}$ and $\{2^{k_1+1}\}$.

(4). Disjunctive HSGs of this type must have the condition $n_2 > 2$ for indecomposability by CL4. The shift-minimal winning coalitions here have the forms $\{1^{k_1}\}$ and $\{1^{k_2-n_2}, 2^{n_2}\}$.

(5). Weighted and indecomposable TSGs of this type must satisfy $m - t = 1$ and $t > 1$ and $d > 1$ and $d = n_3 - 1$ by CL5, and also $m - d < n_2$ since games in this type are weighted and indecomposable T_{1a}. The two shift-minimal winning coalitions here have the forms $\{1^{m-1}\}$ and $\{2^{m-d}, 3^d\}$. Note that the T_{1b} game is not included here. This is because the T_{1b} game is decomposable as follows: Recall that in T_{1b} we have $m - d \geq n_2$. Also, $d < n_3$ by Proposition 4.2.1(C5), and the two shift-minimal winning coalitions for T_{1b} are of the forms $\{1^t\}$ and $\{1^{m-d-n_2}, 2^{n_2}, 3^d\}$.

The decomposition is as follows: Let the first game be $G = (P_G, W_G)$, $P_G = \{1^{n_1}, 2^{n_2+1}\}$ and the set of shift-minimal winning coalitions in W_G is $\{\{1^t\}, \{1^{m-d-n_2}, 2^{n_2+1}\}\}$. And let the second game be $H = (P_H, W_H)$, $P_H = \{3^{n_3}\}$, $W^{\min}_H = \{\{3^d\}\}$. The composition is over a player $g \in P_G$ from level 2. Then we can see that $T_{1b} = G \circ_g H$ on $P_G \setminus \{g\} \cup P_H$.

(6). Recall that in the game T_{2a} we have $n_3 \geq d$, and by CL7, a weighted and indecomposable tripartite game of type $\mathbf{T}_3$ must meet the conditions $d - n_3 + 1 = n_2 > 0$, meaning $d - n_3 + 1 \geq 1$, or $d \geq n_3$, so it follows that $d = n_3$ and $n_2 = 1$. We also need $m - t = 1$ and $t - 1 > m - d > 0$ by CL7. The two shift-minimal winning coalitions here have the forms $\{1^{t-1}, 2\}$ and $\{1^{m-d}, 3^{n_3}\}$.

(7). This is the case for the game T_{2b} where $n_3 < d$. A weighted and indecomposable tripartite game of type $\mathbf{T}_3$ must meet the conditions $d - n_3 = n_2 - 1$ and $m - t = 1$ and $t - n_2 > m - d$ by CL7. Also $n_3 > 1$ holds by Proposition 4.2.3(C′8). The two shift-minimal winning coalitions here have the forms

$\{1^{t-n_2}, 2^{n_2}\}$ and $\{1^{m-d}, 2^{d-n_3}, 3^{n_3}\}$.

The converse direction of the theorem will be proved in Corollary 6.2.2. □

By comparing the list in Theorem 6.1.2 with the list in Theorem 6.0.3, we see that there are only three differences: (1) The family of simple majority games of Theorem 6.0.3(i) is now less general, since it excludes many unanimity and anti-unanimity games. (2) In Theorem 6.1.2(2) we have $n_2 - 1 > 1$ instead of $n_2 - 1 > 0$. (3) The game T_{1b}, which is of type $\mathbf{T}_2$ in Theorem 6.0.3(iii), is excluded, since it is decomposable.

This completes our study of the seven types of indecomposable ideal weighted simple games, and using this list, we shall characterise all ideal weighted simple games, as stated in the following main theorem of this chapter.

Theorem 6.1.4. *G is an ideal weighted simple game if and only if it is a composition*

$$G = H_1 \circ \ldots \circ H_s \circ I \circ A_n \ \ (s \geq 0); \tag{6.1.1}$$

*where H_i is of type **H** for each $i = 1, \ldots, s$. Also, I, which is allowed to be absent, is an indecomposable game of types $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, $\boldsymbol{B}_3$, $\boldsymbol{T}_1$, $\boldsymbol{T}_{3a}$ and $\boldsymbol{T}_{3b}$, and A_n is the anti-unanimity game on n players. Moreover, A_n can be present only if I is either absent or it is of type $\boldsymbol{B}_2$; in the latter case the composition $I \circ A_n$ is over a player of the least desirable level of I. Also, the above decomposition is unique.*

Remark 6.1.5. Note that when studying the types $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$ mentioned above, and also the types $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{T}_3$, we will be referring to (Farràs & Padró, 2010), rather then their newer version (Farràs & Padró, 2012), for two reasons: (i) Although the theorem in (Farràs & Padró, 2012) speaks only of the types $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{T}_1$ and $\mathbf{T}_2$, they are essentially the same as $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{T}_3$ but written more compactly. Indeed, $\mathbf{B}_1$ in (Farràs & Padró, 2012) is the same as $\mathbf{B}_1$ in (Farràs & Padró, 2010), and $\mathbf{B}_2$ in (Farràs & Padró, 2012) is written in a general form that covers both $\mathbf{B}_2$ and $\mathbf{B}_3$ of (Farràs & Padró, 2010). Also, $\mathbf{T}_1$ in (Farràs & Padró, 2012) is written in a general form that covers both $\mathbf{T}_1$ and $\mathbf{T}_2$ in (Farràs & Padró, 2010), while $\mathbf{T}_2$ in (Farràs & Padró, 2012) is the same as $\mathbf{T}_3$ of (Farràs & Padró, 2010). (ii) There are lots of parameters at hand as we saw, and so the more detailed breakdown into $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{T}_3$ is better for our purposes.

In order to prove the above theorem, the next section will be dedicated to proving a crucial result, which is Theorem 6.2.1.

6.2 Compositions of ideal weighted indecomposable games

As was said in the beginning of this chapter, it was known that any ideal weighted game is a composition of indecomposable ideal weighted games. Indecomposable ideal weighted games were also practically known, and their slightly refined classification is given in Theorem 6.1.2. However, it was not known under what conditions the composition of two indecomposable ideal weighted games is also an ideal weighted game. This question will be answered in this section.

Suppose from now on that we have a composition $\Gamma = \Gamma_1 \circ \Gamma_2$, where both Γ_1 and Γ_2 are ideal and weighted, and Γ_1 is indecomposable. The plan now is to fix Γ_1 and analyse what happens when we compose it with an arbitrary ideal weighted game Γ_2. Since Γ_1 is ideal weighted and indecomposable, then it belongs to one of the seven types of games listed in Theorem 6.1.2. So we carry out the analysis case by case for all possibilities of Γ_1.

The main result of this section is the following.

Theorem 6.2.1. *Let $\Gamma = \Gamma_1 \circ \Gamma_2$ be a nontrivial decomposition, such that Γ_1 and Γ_2 are both ideal and weighted, and Γ_1 is indecomposable. Then Γ is ideal weighted if and only if either*

(i) Γ_1 *is of type* $\boldsymbol{H}$, *or*

(ii) Γ_1 *is of type* $\boldsymbol{B}_2$ *and* Γ_2 *is* A_n *such that the composition is over a player of level* 2 *of* Γ_1.

And by using the above theorem, we can show that games of the six types in Theorem 6.1.2 are indecomposable. Note that we say six instead of seven, this is because games of type **H** were shown to be indecomposable in the proof of Theorem 6.1.2.

Corollary 6.2.2. *None of the games of types $\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3, \boldsymbol{T}_1, \boldsymbol{T}_{3a}$ or $\boldsymbol{T}_{3b}$ are decomposable.*

Proof. We know that games of type **H** are indecomposable. Suppose a game, call it I, of one of the other six types is decomposable. Then by Theorem 6.2.1 we have either (i) $I = H \circ_h G$, where H is a game of type **H**, or (ii) $I = B_2 \circ_b A_n$, where B_2 is a game of type $\mathbf{B}_2$. Let us consider these two cases separately.

Case (i). Let $H = H_{n,k}, 1 \leq k \leq n$, where h is a player from H, and let us compose it with an arbitrary ideal weighted game $G = (P_G, W_G)$. Then the minimal winning coalitions in $H \circ_h G$ will either have k players from the most desirable level, or $k - 1$ players from the most desirable level together with a winning coalition of G.

Let us compare the shift-minimal winning coalitions of games of types $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$ (see page 109) with the shift-minimal winning coalitions of the composition $H \circ_h G$. Consider a game B_1 of type $\mathbf{B}_1$ first, where its shift-minimal winning coalition has the only form $\{1^{k_1}, 2^{k_2-k_1}\}$, such that $k_1 < n_1$. If $B_1 = H \circ_h G$, then either $H = U_2$ or not. Suppose it is, then in the multiset notation $U_2 \circ_h G$ has the shift-minimal winning coalition of type $\{1, 2^{k_2-k_1}\}$. But comparing $\{1, 2^{k_2-k_1}\}$ with $\{1^{k_1}, 2^{k_2-k_1}\}$ implies that $k_1 = n_1 = 1$, which contradicts the fact that $k_1 < n_1$ in $\mathbf{B}_1$. Suppose now that H is not U_2. Then some of the shift-minimal winning coalitions in $H \circ_h G$ contain players from its level 1 solely, and since $\mathbf{B}_1$ requires that there must be $k_2 - k_1$ players from its level 2 present in every shift-minimal winning coalition, then the composition $H \circ_h G$ cannot be a game of type $\mathbf{B}_1$.

Consider $\mathbf{B}_2$ with its two types of shift-minimal winning coalitions $\{1^{k_1}\}$ and $\{2^{k_1+1}\}$. Note that it does not have the requirement that a certain number of players from its level 1 must be present in a coalition in order to make it winning, but $H \circ_h G$ does have that requirement, so it cannot be a game of type $\mathbf{B}_2$ either.

Also, recall that a game of type $\mathbf{B}_3$ has shift-minimal winning coalitions of types $\{1^{k_1}\}$ and $\{1^{k_1-n_2}, 2^{n_2}\}$. So in order for a game of type $\mathbf{B}_3$ to be equal to $H \circ_h G$, the latter requires $k_1 - 1 = k_2 - n_2$, which contradicts the fact that $k_1 - 1 > k_2 - n_2$ in $\mathbf{B}_3$.

Now consider a game T_1 of type $\mathbf{T}_1$. The shift-minimal winning coalitions of T_1 have the two forms $\{1^t\}$ and $\{2^{m-d}, 3^d\}$ (page 109). It is easy to see that in order to have $T_1 = H \circ_g G$, we need $t = 1$ and $H = A_2$, but $t = 1$ contradicts the fact that $t > 1$ in $\mathbf{T}_1$.

Finally, a game T_3 of type $\mathbf{T}_{3a}$ has the two forms of shift-minimal winning coalitions $\{1^{t-1}, 2\}$ and $\{1^{m-d}, 3^d\}$. It follows that in order to have $T_3 = H \circ_g G$, we need $H = U_2$ and $t-1 = m-d$, which contradicts the fact that $t-1 > m-d$ in $\mathbf{T}_{3a}$. The case for $\mathbf{T}_{3b}$ is exactly the same (see top of page 110 for its shift-minimal winning coalitions).

Case (ii). Let B_2 be a game of type $\mathbf{B}_2$, where b is a player from its level 2, and let us compose it with A_n. Then the three forms of the minimal winning coalitions, in the multiset notation, in $B_2 \circ_b A_n$ are

$$\{1^k\}, \{2^{k+1}\}, \{2^k, 3\}. \tag{6.2.1}$$

It is clear that the composition $B_2 \circ_b A_n$ cannot be a game of any of the types $\mathbf{B}_1$, $\mathbf{B}_2$ or $\mathbf{B}_3$, since these types have only two desirability levels, whereas $B_2 \circ_b A_n$ has three.

Also, the shift-minimal winning coalitions in (6.2.1), which are $\{1^k\}$ and $\{2^k, 3\}$, are not the same as those of games of types $\mathbf{T}_{3a}$ or $\mathbf{T}_{3b}$ either. And in order for them to be the same as the ones of games of type $\mathbf{T}_1$ ($\{1^{m-1}\}$ and $\{2^{m-d}, 3^d\}$), we need $m - 1 = m - d$, meaning $d = 1$. But this will give a contradiction since $\mathbf{T}_1$ has the condition $d > 1$, so games of type $\mathbf{T}_1$ are also indecomposable. $\square$

Remark 6.2.3. Recall that if a game has a certificate of nonweightedness, then it is nonweighted, and since weighted games are complete games, then the criterion of nonweightedness can be made easier by finding a certificate of nonweightedness where the winning coalitions in the certificate are shift-minimal winning coalitions by Theorem 2.2.13. This important fact will be used in the proofs to follow.

Let us start with the two cases where the composition is always weighted.

6.2.1 Compositions that are ideal and weighted

The first proposition below is for Theorem 6.2.1(i), and the second proposition is for Theorem 6.2.1(ii). Recall that an indecomposable game of type **H** is either A_2, U_2 or $H_{n,k}$ for $1 < k < n$.

Proposition 6.2.4. *Let* $\Gamma_1 = (P_1, W_1), g \in P_1, \Gamma_2 = (P_2, W_2)$*, and let* $\Gamma = \Gamma_1 \circ_g \Gamma_2$*. If* Γ_1 *is of type* ***H*** *and* Γ_2 *is weighted, then* Γ *is weighted.*

Proof. Let $X_1, \ldots, X_m$ be minimal winning coalitions and $Y_1, \ldots, Y_m$ be losing coalitions of Γ such that

$$(X_1, \ldots, X_m; Y_1, \ldots, Y_m)$$

is a certificate of nonweightedness of Γ. Let $U_i = X_i \cap P_1$, then either $|U_i| = k$ or $|U_i| = k - 1$. However, if for a single i we have $|U_i| = k$, then it cannot be that all of the sets $Y_1, \ldots, Y_m$ are losing, as there will be at least one among them of cardinality at least k. Thus $|U_i| = k - 1$ for all i. In this case we have $X_i = U_i \cup S_i$, where S_i is winning in Γ_2. Let $Y_i = V_i \ \cup \ T_i$, where $V_i \subseteq P_1$ and $T_i \subseteq P_2$. Since all coalitions $Y_1, \ldots, Y_m$ are losing in Γ, then $|V_i| = k - 1$ for all i, implying all T_i are losing in Γ_2. But now we have obtained a trading transform $(S_1, \ldots, S_m; T_1, \ldots, T_m)$ for Γ_2, such that all S_i are winning and all T_i are losing in Γ_2. This contradicts the fact that Γ_2 is weighted. $\square$

Proposition 6.2.5. *Let* $\Gamma_1 = (P_1, W_1)$ *be a weighted simple game of type* $\boldsymbol{B}_2$*,* g *is a player from level* 2 *of* P_1*, and* Γ_2 *is* A_n*, then* $\Gamma = \Gamma_1 \circ_g \Gamma_2$ *is a weighted simple game.*

Proof. Since g is a player from level 2 of P_1, then Γ is a complete game by Lemma 5.2.2. Also, recall that shift-minimal winning coalitions of a game of type $\mathbf{B}_2$ are $\{1^{k_1}\}$ and $\{2^{k_1+1}\}$. We shall prove weightedness of Γ by showing that it cannot have a certificate of nonweightedness. In the composition, in the multiset notation, Γ has the following shift-minimal winning coalitions $\{1^{k_1}\}, \{2^{k_1}, 3\}$. So all shift-minimal winning coalitions have k_1 players from $P_1 \setminus \{g\}$. Also, since Γ_1 has two thresholds k_1 and k_2 such that $k_2 = k_1 + 1$, then any coalition containing more than k_1 players from $P_1 \setminus \{g\}$ is winning in Γ_1, and hence winning in Γ. Suppose now towards a contradiction that Γ has the following certificate of nonweightedness

$$(X_1, \ldots, X_n; Y_1, \ldots, Y_n), \tag{6.2.2}$$

where $X_1, \ldots, X_n$ are shift-minimal winning coalitions and $Y_1, \ldots, Y_n$ are losing coalitions in Γ. Let the set of players of A_n be P_{A_n}. It is easy to see that at least one of the coalitions $X_1, \ldots, X_n$ in (6.2.2) is not of the type $\{1^{k_1}\}$, so at least one of these winning coalitions has a player from the third level, i.e. from A_n. But since each shift-minimal winning coalition in (6.2.2) has k_1 players from $P_1 \setminus \{g\}$, then each losing coalition $Y_1, \ldots, Y_n$ in (6.2.2) also has k_1 players from $P_1 \setminus \{g\}$ (if it has more than k_1 then it is winning). Moreover, at least one coalition from $Y_1, \ldots, Y_n$, say Y_1, has at least one player from P_{A_n}. It follows that $(Y_1 \cap P_1) \cup \{g\} \in W_1$ and $Y_1 \cap P_{A_n}$ is winning in A_n. Hence Y_1 is winning in Γ, contradiction. Therefore no such certificate can exist. $\square$

In the next section we analyse the rest of compositions $\Gamma = \Gamma_1 \circ \Gamma_2$ in terms of Γ_1, where the composition is over a player from the least desirable level of Γ_1. We will show that none of them is weighted.

6.2.2 Compositions that are ideal and nonweighted: when compositions are over a player from the least desirable level of Γ_1

Here we will consider two cases:

(1) Γ_2 has at least one minimal winning coalition with cardinality at least 2.

(2) $\Gamma_2 = A_n, n \geq 2$.

We will start with the second case. Denote players of A_n by P_{A_n}.

Proposition 6.2.6. *If Γ_1 is an ideal weighted indecomposable simple game, such that g is a player from the least desirable level of Γ_1, then $\Gamma_1 \circ_g A_n$ is not weighted.*

Proof. Let Γ_1 be of type $\mathbf{B}_1$. The only shift-minimal winning coalition of Γ_1 is of the form $\{1^{k_1}, 2^{k_2-k_1}\}$, where $n_1 > k_1 > 0, k_2 - k_1 = n_2 - 1 > 1$.

Composing over a player of level 2 of Γ_1 gives, among other winning coalitions, in the multiset notation $\{1^{k_1}, 2^{k_2-k_1}\}$ and $\{1^{k_1}, 2^{k_2-k_1-1}, 3\}$. Thus the game is not weighted due to the following certificate of nonweightedness:

$$(\{1^{k_1}, 2^{k_2-k_1}\}, \{1^{k_1}, 2^{k_2-k_1-1}, 3\}; \{1^{k_1-1}, 2^{k_2-k_1+1}, 3\}, \{1^{k_1+1}, 2^{k_2-k_1-2}\}).$$

Since $k_2 - k_1 + 1 = n_2$ in $\mathbf{B}_1$ (see page 107), then the coalition $\{1^{k_1-1}, 2^{k_2-k_1+1}, 3\}$ is allowed.

Now consider $\mathbf{B}_3$ (composing $\mathbf{B}_2$ type of game with A_n gives a weighted game by Proposition 6.2.5). Its shift-minimal winning coalition are $\{1^{k_1}\}, \{1^{k_2-n_2}, 2^{n_2}\}$. Composing over a player of level 2 of Γ_1 gives among other winning coalitions $\{1^{k_1}\}, \{1^{k_2-n_2}, 2^{n_2-1}, 3\}$. Hence the game is not weighted due to the following certificate of nonweightedness:

$$(\{1^{k_2-n_2}, 2^{n_2-1}, 3\}, \{1^{k_2-n_2}, 2^{n_2-1}, 3\}; \{1^{k_2-n_2+1}, 2^{n_2-2}\}, \{1^{k_2-n_2-1}, 2^{n_2}, 3^2\}).$$

Note that $k_2 - n_1 + 1 < k_1 \leq n_1$ and $n_2 > 2$ in $\mathbf{B}_3$ (see page 107), so the coalition $\{1^{k_2-n_2+1}, 2^{n_2-2}\}$ is allowed.

Now consider $\mathbf{T}_1$, where the shift-minimal winning coalitions are $\{1^{m-1}\}$, $\{2^{m-d}, 3^d\}$. If we compose over a player of level 3 of Γ_1, then W_Γ^{smin}, in the multiset notation, includes $\{2^{m-d}, 3^{d-1}, 4\}$, and Γ is not weighted due to the following certificate of nonweightedness:

$$(\{2^{m-d}, 3^{d-1}, 4\}, \{2^{m-d}, 3^{d-1}, 4\}; \{2^{m-d+1}, 3^{d-2}\}, \{2^{m-d-1}, 3^d, 4^2\}).$$

The coalition $\{2^{m-d+1}, 3^{d-2}\}$ is valid, since $m - d + 1 \leq n_2$ (see page 82), and also $d > 1$ (see page 107).

Note that the certificate above uses only levels 2 and 3. So if we consider the subgame Γ_{1_A}, $A = \{1^{n_1}\}$, in other words the subgame consisting of levels 2 and 3 of Γ_1, then we see that the shift-minimal winning coalition in Γ_{1_A} is $\{2^{m-d}, 3^d\}$. This shows that Γ_{1_A} is actually a game of type $\mathbf{B}_1$. Indeed, this is because in Γ_{1_A}, a winning coalition needs to have at least m players (which corresponds to meeting threshold k_2 in $\mathbf{B}_1$), of which at least $m-d$ players are from level 2 (which corresponds to meeting threshold k_1 in $\mathbf{B}_1$). This observation will be useful later on.

Now consider $\mathbf{T}_{3a}$, where the shift-minimal winning coalition are $\{1^{t-1}, 2\}$, $\{1^{m-d}, 3^d\}$. If we compose over a player of level 3 of Γ_1, then W_Γ^{smin} includes, in the multiset notation, $\{1^{m-d}, 3^{d-1}, 4\}$, and Γ is not weighted due to the following certificate of nonweightedness:

$$(\{1^{m-d}, 3^{d-1}, 4\}, \{1^{m-d}, 3^{d-1}, 4\}; \{1^{m-d+1}, 3^{d-2}\}, \{1^{m-d-1}, 3^d, 4^2\}).$$

Since $m - d < t - 1$ and $d = n_3$ in $\mathbf{T}_2$ (see page 107), where $n_3 \geq 2$ by Proposition 4.2.3 (C′8), then $\{1^{m-d+1}, 3^{d-2}\}$ is a valid coalition.

Finally, consider $\mathbf{T}_{3b}$, where the shift-minimal winning coalition are $\{1^{t-n_2}, 2^{n_2}\}, \{1^{m-d}, 2^{d-n_3}, 3^{n_3}\}$. If we compose over a player of level 3 of Γ_1, then W_Γ^{smin} includes $\{1^{m-d}, 2^{d-n_3}, 3^{n_3-1}, 4\}$. The composition is not weighted due to the following certificate of nonweightedness:

$$(\{1^{m-d}, 2^{d-n_3}, 3^{n_3-1}, 4\}, \{1^{m-d}, 2^{d-n_3}, 3^{n_3-1}, 4\};$$

$$\{1^{m-d+1}, 2^{d-n_3}, 3^{n_3-2}\}, \{1^{m-d-1}, 2^{d-n_3}, 3^{n_3}, 4^2\}).$$

The coalition $\{1^{m-d+1}, 2^{d-n_3}, 3^{n_3-2}\}$ is losing because in $\mathbf{T_{3b}}$ we have $d - n_3 = n_2 - 1$ and also $t - n_2 > m - d$ (see page 107), meaning $(m-d+1)+(d-n_3) = m - d + 1 + n_2 - 1 \leq t - n_2 + n_2 - 1 = t - 1$. $\square$

The remaining cases for composing Γ_1 with Γ_2 over a player from the least desirable level of Γ_1, such that Γ_2 has at least one minimal winning coalition

X with $|X| > 1$ all have a common methodology, which is described by the following definition and lemma.

Definition 6.2.7. Let $\Gamma = (P, W)$ be a simple game and $g \in P$. We say that a losing coalition X is g-winning if $g \notin X$ and $X \cup \{g\} \in W$.

Lemma 6.2.8. *Let Γ_1 be a game for which there exist coalitions X_1, X_2, Y_1 and Y_2 such that*

$$(X_1, X_2; Y_1; Y_2)$$

is a trading transform, X_1 is winning, X_2 is g-winning and Y_1 and Y_2 are losing. Also, let Γ_2 be a game that is not an anti-unanimity game, then $\Gamma = \Gamma_1 \circ_g \Gamma_2$ is not weighted.

Proof. Let Z be a minimal winning coalition of Γ_2 which has at least two elements, and let $Z = Z_1 \cup Z_2$, where Z_1 and Z_2 are losing in Γ_2. Then it is easy to check that

$$(X_1, X_2 \cup Z; Y_1 \cup Z_1; Y_2 \cup Z_2)$$

is a certificate of nonweightedness for Γ. Indeed, X_1 and $X_2 \cup Z$ are both winning in Γ and $Y_1 \cup Z_1$ and $Y_2 \cup Z_2$ are both losing in Γ. $\square$

The above definition and lemma will be used for analysing the rest of the cases in this section, showing that the composed games are never weighted.

Proposition 6.2.9. *Let $\Gamma = \Gamma_1 \circ_g \Gamma_2$. If Γ_1 is of type $\mathbf{B}_1$, then Γ is not weighted.*

Proof. Let $\Gamma_1 = (P, W)$ be of type $\mathbf{B}_1$, so $\Gamma_1 = H_\forall(\mathbf{n}, \mathbf{k})$. The shift-minimal winning coalition of Γ_1 has the only form $\{1^{k_1}, 2^{k_2-k_1}\}$, where $n_1 > k_1, k_2 - k_1 = n_2 - 1 > 1$ (these conditions must be met otherwise the game will be decomposable). If g is any player from level 2 of Γ_1, then Γ is not weighted by Lemma 6.2.8 applied to the following trading transform

$$(\{1^{k_1}, 2^{k_2-k_1}\}, \{1^{k_1}, 2^{k_2-k_1-1}\}; \{1^{k_1-1}, 2^{k_2-k_1+1}\}, \{1^{k_1+1}, 2^{k_2-k_1-2}\}).$$

This is because the second coalition is 2-winning, since $\{1^{k_1}, 2^{k_2-k_1}\}$ meets the two threshold requirements as stated on page 107. Also, since $k_2 - k_1 + 1 = n_2$ (also on page 107), then the coalition $\{1^{k_1-1}, 2^{k_2-k_1+1}\}$ is legitimate. $\square$

Proposition 6.2.10. *Let* $\Gamma = \Gamma_1 \circ_g \Gamma_2$*. If* Γ_1 *is of type* $\boldsymbol{B}_2$*, then* Γ *is not weighted.*

Proof. Let $\Gamma_1 = (P, W)$ be of type $\mathbf{B}_2$, so $\Gamma_1 = H_\exists(\mathbf{n}, \mathbf{k}), \mathbf{k} = (k_1, k_1+1)$. So the shift-minimal winning coalitions of Γ_1 are $\{1^{k_1}\}, \{2^{k_1+1}\}$. If g is any player from level 2 of Γ_1, then Γ is not weighted by Lemma 6.2.8 applied to the following trading transform

$$(\{1^{k_1}\}, \{2^{k_1}\}; \{1^{\lceil \frac{k_1}{2} \rceil}, 2^{\lfloor \frac{k_1}{2} \rfloor}\}, \{1^{\lfloor \frac{k_1}{2} \rfloor}, 2^{\lceil \frac{k_1}{2} \rceil}\}).$$

This is because the second coalition is 2-winning. □

Proposition 6.2.11. *Let* $\Gamma = \Gamma_1 \circ_g \Gamma_2$*. If* Γ_1 *is of type* $\boldsymbol{B}_3$*, then* Γ *is not weighted.*

Proof. Let $\Gamma_1 = (P, W)$ be of type $\mathbf{B}_3$, so $\Gamma_1 = H_\exists(\mathbf{n}, \mathbf{k}), \mathbf{k} = (k_1, k_1 + 1)$. Since Γ_1 is of type $\mathbf{B}_3$, then $n_2 < k_2$. Also, we have $n_2 > k_2 - k_1 + 1$ (otherwise it is decomposable). So let $n_2 = k_2 - k_1 + x, x \geq 2$. Then the two shift-minimal winning coalitions are $\{1^{k_1}\}$ and $\{1^{k_1-x}, 2^{k_2-k_1+x}\}$, and if g is any player from level 2 of Γ_1, then Γ is not weighted by Lemma 6.2.8 applied to the following trading transform

$$(\{1^{k_1}\}, \{1^{k_1-x}, 2^{k_2-k_1+x-1}\}; \{1^{k_1-1}, 2\}, \{1^{k_1-x+1}, 2^{k_2-k_1-x-2}\}).$$

Since the second coalition is 2-winning, then the above trading transform is legitimate. □

Now we look at the cases for Γ_1 being a weighted TSG and also indecomposable.

Proposition 6.2.12. *Let* $\Gamma = \Gamma_1 \circ_g \Gamma_2$*. If* Γ_1 *is of type* $\boldsymbol{T}_1$*, then* Γ *is not weighted.*

Proof. A weighted indecomposable tripartite simple game of type $\mathbf{T}_1$ has $m-t = 1, t > 1$, and $d > 1$. So the two shift-minimal winning coalitions of Γ_1 are $\{1^{m-1}\}$ and $\{2^{m-d}, 3^d\}$. If g is any player from level 3 of Γ_1, then Γ is not weighted by Lemma 6.2.8 applied to the following trading transform

$$(\{1^{m-1}\}, \{2^{m-d}, 3^{d-1}\}; \{1^{m-2}, 2\}, \{1, 2^{m-d-1}, 3^{d-1}\}).$$

This is because the second coalition is 3-winning, since $\{2^{m-d}, 3^d\}$ is a shift-minimal winning coalition as stated on page 107. $\square$

And finally we turn to the two types $\mathbf{T}_{3a}$ and $\mathbf{T}_{3b}$.

Proposition 6.2.13. *Let $\Gamma = \Gamma_1 \circ_g \Gamma_2$. If Γ_1 is of either type $\boldsymbol{T}_{3a}$ or $\boldsymbol{T}_{3b}$, then Γ is not weighted.*

Proof. Firstly, recall that in $\mathbf{T}_{3a}$ we have $n_3 = d$, we also have $m - t = 1$ and $t - n_2 > m - d$ and $n_2 = 1$. So its two shift-minimal winning coalitions are $\{1^{t-1}, 2\}$ and $\{1^{m-d}, 3^d\}$. Since we have $t-1 > m-d$, then $t-1 = m-d+x, x \geq 1$. If g is any player from level 3 of Γ_1, then Γ is not weighted by Lemma 6.2.8 applied to the following trading transform

$$(\{1^{t-1}, 2\}, \{1^{t-1-x}, 3^{d-1}\}; \{1^{t-1}, 3\}, \{1^{t-1-x}, 2, 3^{d-2}\}).$$

The second coalition is 3-winning. Also, we can derive from the winning coalition $\{1^{m-d}, 3^d\}$ and the fact $t - 1 = m - d + x$ that $m - d + d = m = t + d - x - 1$, meaning the number of players in $\{1^{t-1-x}, 2, 3^{d-2}\}$ is $m - 1$, so it is losing.

Secondly, in $\mathbf{T}_{3b}$ we have $n_3 < d$, and also $m - t = 1$ and $t - n_2 > m - d$ and $n_2 = d - n_3 + 1$. So its two shift-minimal winning coalitions are $\{1^{t-n_2}, 2^{n_2}\}$ and $\{1^{m-d}, 2^{d-n_3}, 3^{n_3}\}$. Since we have $t-n_2 > m-d$, then $t-n_2 = m-d+x, x \geq 1$. If g is any player from level 3 of Γ_1, then Γ is not weighted by Lemma 6.2.8 applied to the following trading transform

$$(\{1^{t-n_2}, 2^{n_2}\}, \{1^{t-n_2-x}, 2^{n_2-1}, 3^{n_3-1}\}; \{1^{t-n_2}, 2^{n_2-1}, 3\}, \{1^{t-n_2-x}, 2^{n_2}, 3^{n_3-2}\}).$$

The second coalition is 3-winning. Also, we can derive from the winning coalition $\{1^{m-d}, 2^{d-n_3}, 3^{n_3}\}$ and the two facts $t - n_2 = m - d + x$ and $n_2 = d - n_3 + 1$ that $m - d + d - n_3 + n_3 = t - n_2 - x + n_2 + n_3 - 1 = t + n_3 - x - 1 = m$, meaning the number of players in $\{1^{t-n_2-x}, 2^{n_2}, 3^{n_3-2}\}$ is $m - 1$, so it is losing. $\square$

This concludes our analysis of all the cases where the compositions are over a player of the least desirable level of Γ_1.

Now, recall that in order to achieve a weighted composed game $\Gamma = \Gamma_1 \circ \Gamma_2$, we need to assume that the composition is over a player from the least desirable level of Γ_1 by Corollary 5.2.4. However, Corollary 5.2.4 assumes that Γ_2 is neither a unanimity nor an anti-unanimity game. So it remains to investigate these two possibilities, and it turns out that none of them produce a weighted Γ, which is what we show in the next section.

6.2.3 The remaining cases of compositions that are ideal and nonweighted

Let us firstly consider the case where Γ_2 is A_n.

Theorem 6.2.14. *Let $\Gamma = (P, W)$ be a simple game of one of the types $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{T}_1$, $\mathbf{T}_{3a}$, and $\mathbf{T}_{3b}$, and let $g \in P$ be a player of Γ such that g is strictly more desirable than some nondummy player $g' \in P$. Then the composition $\Gamma \circ_g A_n$ is not weighted.*

Proof. Denote players of A_n by P_{A_n}, and let $a_1, a_2 \in P_{A_n}$. Let us first consider the case where g is from the most desirable level of Γ. We will apply Lemma 5.2.3 to show that $\Gamma \circ_g A_n$ is not complete. So in what follows we show that for each case there exists $g, g' \in P$ and coalitions X_1 and X_2 of Γ which satisfy (5.2.1).

In the following three cases, g is a player of level 1 and g' is a player of level 2.

(i) $\mathbf{B}_1$: X_1 is of type $\{1^{k_1-1}, 2^{k_2-k_1}\}$, and X_2 is of type $\{1^{k_1-1}, 2^{k_2-k_1}\}$;

(ii) $\mathbf{B}_2$: X_1 is of type $\{1^{k_1-1}\}$, and X_2 is of type $\{2^{k_1}\}$;

(iii) $\mathbf{B}_3$: X_1 is of type $\{1^{k_1-1}\}$, and X_2 is of type $\{1^{k_2-n_2}, 2^{n_2-1}\}$.

And for the following three cases, g is a player of level 1 and g' is a player of level 3.

(iv) $\mathbf{T}_1$: X_1 is of type $\{1^{m-2}\}$, and X_2 is of type $\{2^{m-d}, 3^{d-1}\}$;

(v) $\mathbf{T}_{3a}$: X_1 is of type $\{1^{t-2}, 2\}$, and X_2 is of type $\{1^{m-d-1}, 3^d\}$;

(vi) $\mathbf{T}_{3b}$: X_1 is of type $\{1^{t-n_2-1}, 2^{n_2}\}$, and X_2 is of type $\{1^{m-d-1}, 3^d\}$.

With regards to the $\mathbf{T}$ types, the result follows when the composition is over the most desirable level of Γ_1 by (i)-(vi). But let us now consider composing games of the $\mathbf{T}$ types over a player of level 2. We start with $\mathbf{T}_1$. Recall from the analysis regarding $\mathbf{T}_1$ on page 117, that a game of type $\mathbf{B}_1$ is a subgame of a game of type $\mathbf{T}_1$, and since a game of type $\mathbf{B}_1$ when composed with A_n was shown to be nonweighted by (i) above, then the same applies to a game of type $\mathbf{T}_1$.

Now we look at $\mathbf{T}_{3a}$. Recall that $d = n_3$ in $\mathbf{T}_{3a}$ (see page 107), where $n_3 \geq 2$ by Proposition 4.2.3 (C′8). So in this case X_1 is of type $\{1^{t-n_2}, 2^{n_2-1}\}$, X_2 is of type $\{1^{m-d}, 3^{d-1}\}$, where g is a player of level 2 and g' is a player of level 3.

Finally we look at $\mathbf{T}_{3b}$. Here X_1 is of type $\{1^{t-n_2}, 2^{n_2-1}\}, X_2$ is of type $\{1^{m-d}, 2^{d-n_3}, 3^{n_3-1}\}$, where g is a player of level 2 and g' is a player of level 3. $\square$

This completes the study of compositions where Γ_2 is the anti-unanimity game A_n, such that the compositions are not over the least desirable level of Γ_1.

Finally, we consider compositions where Γ_2 is the unanimity game U_n. It turns out that none of these compositions give a weighted composed game either, which is what we show next.

Theorem 6.2.15. *Let $\Gamma_1 = (P, W)$ be a simple game of one of the types $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{T}_1$, $\mathbf{T}_{3a}$, and $\mathbf{T}_{3b}$ and let $g \in P$ be a player not from the least desirable level of Γ_1. Then the composition $\Gamma = \Gamma_1 \circ_g U_n$ is not weighted.*

Proof. Let U_n be defined on P_{U_n}, and let $Z = P_{U_n}$. We start with Γ_1 being of type $\mathbf{B}_1$. A shift-minimal winning coalition of Γ_1 has the only form $\{1^{k_1}, 2^{k_2-k_1}\}$, where $k_1 < n_1$. We compose over level 1 of Γ_1. Then Γ is nonweighted by Lemma 6.2.8 applied to the following trading transform (where $Z = P_{U_n}$):

$$(\{1^{k_1}, 2^{k_2-k_1}\}, \{1^{k_1-1}, 2^{k_2-k_1}\}; \{1^{k_1}, 2^{k_2-k_1-1}\}, \{1^{k_1-1}, 2^{k_2-k_1+1}\}).$$

This is because the second coalition is 1-winning.

Note that $k_2 - k_1 + 1 = n_2 \geq 2$ in a game of type $\mathbf{B}_1$ (see page 107), so the coalition $\{1^{k_1-1}, 2^{k_2-k_1+1}\}$ is allowed.

Now let Γ_1 be of type $\mathbf{B}_2$. The shift-minimal winning coalitions of Γ_1 here are $\{1^{k_1}\}, \{2^{k_1+1}\}$, and if we compose with U_n over level 1 of Γ_1, then Γ is nonweighted by Lemma 6.2.8 applied to the following trading transform:

$$(\{1^{k_1-1}\}, \{2^{k_1+1}\}; \{1^{k_1-1}, 2\}, \{2^{k_1}\}).$$

This is because the first coalition is 1-winning.

Now let Γ_1 be of type $\mathbf{B}_3$. Recall that in a game of type $\mathbf{B}_3$ we have $k_1 \leq n_1$ (page 107), and also $k_2 - n_2 < k_1$ (see page 58). So the shift-minimal winning coalitions of Γ_1 are $\{1^{k_1}\}, \{1^{k_2-n_2}, 2^{n_2}\}$. If we compose with U_n over level 1 of Γ_1, then Γ is nonweighted by Lemma 6.2.8 applied to the following trading transform:

$$(\{1^{k_2-n_2}, 2^{n_2}\}, \{1^{k_1-1}\}; \{1^{k_2-n_2}, 2^{n_2-1}\}, \{1^{k_1-1}, 2\}).$$

This is because the second coalition is 1-winning.

Next we look at the games $\mathbf{T}_1$, $\mathbf{T}_{3a}$, and $\mathbf{T}_{3b}$, also let $Z = P_{U_n}$. Since they have three levels each, then we need to consider what happens when composing over level 1 and when composing over level 2 separately. Let us start with $\mathbf{T}_1$.

The shift-minimal winning coalitions of Γ_1 are $\{1^{m-1}\}$ and $\{2^{m-d}, 3^d\}$. Here we need to consider two compositions, one over level 1, and one over level 2.
Case (i). If we compose with U_n over level 1 of Γ_1 then Γ is nonweighted by Lemma 6.2.8 applied to the following trading transform:

$$(\{1^{m-2}\}, \{2^{m-d}, 3^d\}; \{1^{m-2}, 2\}, \{2^{m-d-1}, 3^d\}).$$

This is because the first coalition is 1-winning.

Case (ii). If we compose with U_n over level 2 of Γ_1, then Γ is nonweighted by Lemma 6.2.8 applied to the following trading transform:

$$(\{1^{m-1}\}, \{2^{m-d-1}, 3^d\}; \{1^{m-2}, 2\}, \{1, 2^{m-d-2}, 3^d\}).$$

This is because the second coalition is 2-winning.

Now let Γ_1 be of type $\mathbf{T}_{3a}$. Recall from page 109 that a game of type $\mathbf{T}_{3a}$ has $n_2 = 1$, so the shift-minimal winning coalitions of Γ_1 here are $\{1^{t-1}, 2\}$ and $\{1^{m-d}, 3^d\}$. Here we need to consider two compositions, one over level 1, one over level 2.

Case (i). If we compose with U_n over level 1 of Γ_1, then since $n_1 \geq t-1 > m-d$ by Proposition 4.2.3 (C′2), then Γ is nonweighted by Lemma 6.2.8 applied to the following trading transform:

$$(\{1^{t-2}, 2\}, \{1^{m-d}, 3^d\}; \{1^{t-1}\}, \{1^{m-d-1}, 2, 3^d\}).$$

This is because the first coalition is 1-winning.

Case (ii). If we compose with U_n over level 2 of Γ_1, then Γ is nonweighted by Lemma 6.2.8 applied to the following trading transform:

$$(\{1^{t-1}\}, \{1^{m-d}, 3^d\}; \{1^{t-1}, 3\}, \{1^{m-d}, 3^{d-1}\}).$$

This is because the first coalition is 2-winning.

Finally, let Γ_1 be of type $\mathbf{T}_{3b}$. The shift-minimal winning coalitions of Γ_1 are $\{1^{t-n_2}, 2^{n_2}\}$ and $\{1^{m-d}, 2^{d-n_3}, 3^{n_3}\}$. Here we need to consider two compositions, one over level 1, one over level 2.

Case (i). If we compose Γ_1 with U_n over level 1 of Γ_1, then since $n_1 \geq t-1 > m-d$, then Γ is nonweighted by Lemma 6.2.8 applied to the following trading transform:

$$(\{1^{m-d}, 2^{d-n_3}, 3^{n_3}\}, \{1^{m-d-1}, 2^{d-n_3}, 3^{n_3}\}; \{1^{m-d}, 2^{d-n_3-1}, 3^{n_3}\},$$

$$\{1^{m-d-1}, 2^{d-n_3+1}, 3^{n_3}\}).$$

This is because the second coalition is 1-winning.

Note that $d-n_3+1 = n_2$ in a game of type $\mathbf{T}_{3b}$ (see page 107), so the coalition $\{1^{m-d-1}, 2^{d-n_3+1}, 3^{n_3}\}$ is allowed.

Case (ii). If we compose Γ_1 with U_n over level 2 of Γ_1, then Γ is nonweighted by Lemma 6.2.8 applied to the following trading transform:

$$(\{1^{m-d}, 2^{d-n_3}, 3^{n_3}\}, \{1^{m-d}, 2^{d-n_3-1}, 3^{n_3}\}; \{1^{m-d+1}, 2^{d-n_3-2}, 3^{n_3}\},$$

$$\{1^{m-d-1}, 2^{d-n_3+1}, 3^{n_3}\}).$$

This is because the second coalition is 2-winning. □

In conclusion we see that none of the six games above produce a weighted game when composed with U_n over a player not from the least desirable level of the first game.

Proof of Theorem 6.2.1. Follows from combining results in Sections 6.2.1-6.2.3.

Now the characterisation theorem of all ideal weighted simple games can be derived easily.

6.3 The proof of the characterisation theorem

The following proposition will be useful to show the uniqueness of the decomposition of an ideal weighted game.

Proposition 6.3.1. *Let H' be a game of type **H**, $G' = (P', W')$ be an ideal weighted simple game, B_2 be a game of type $\mathbf{B}_2$ such that g is a player from its level 2, and A_n is an anti-unanimity game. Then $H' \circ G' \not\cong B_2 \circ_g A_n$.*

Proof. Recall that isomorphisms preserve Isbell's desirability relation (Carreras & Freixas, 1996). Therefore, if we show that the two compositions have different winning requirements, then it shows that Isbell's desirability relation is used differently in the two compositions, meaning they cannot be isomorphic to each other. Consider first $H' \circ G'$. Suppose the minimal winning coalition of H' is of the form $\{1^k\}$, then a minimal winning coalition in the composition will have either k or $k-1$ players from its most desirable level.

Now consider $B_2 \circ_g A_n$. Let the two minimal winning coalitions of B_2 be of the forms $\{1^l\}$ and $\{2^{l+1}\}$, then a minimal winning coalition in $B_2 \circ_g A_n$ has either l players from the most desirable level, or l players from the second most desirable level (since the composition is over the least desirable level of B_2). By comparing the minimal winning coalitions in the two compositions, we see that the game $H' \circ G'$ requires at least $k-1$ players from its most desirable level to be present in a coalition in order to make it winning. But since no such requirement exists in $B_2 \circ_g A_n$, then the two games have different winning requirements, and therefore cannot be isomorphic to each other. $\square$

Proof of Theorem 6.2.1. Either G is decomposable or not. If it is not, then it is either A_2, U_2 or one of the indecomposable types $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{T}_1$, $\mathbf{T}_{3a}$ and $\mathbf{T}_{3b}$ by Theorem 6.1.2. Suppose now that G is decomposable, so $G = G_1 \circ G_2$. Then:

(i) If G_1 is not of type $\mathbf{H}$, then by Theorem 6.2.1 it must be of type $\mathbf{B}_2$, and also $G_2 = A_n$ such that the composition is over level 2 of G_1.

(ii) If G_1 is of type $\mathbf{H}$, then since G_2 is ideal weighted by Corollary 5.3.2, the result follows by induction.

Moreover, Proposition 6.3.1 shows that G cannot have two equal decompositions where in one decomposition G_1 is of type $\mathbf{H}$, and in the other G_1 is of type $\mathbf{B}_2$. It is also not difficult to see that there cannot be two decompositions where in both of them G_1 is of type $\mathbf{B}_2$, and G_2 is not the same game in the two decompositions. So if G has more than one decomposition, then G_1 is of type $\mathbf{H}$ in each decomposition, this has two possibilities.

Firstly, if in this case G_2 has no passers or vetoers, then the decomposition is unique by Lemma 5.1.8.

Secondly, suppose G_2 has passers or vetoers, this also has two possibilities.

(i). Consider first the case where G has passers. The fact that G has passers implies that G_1 has passers. But G_1 is indecomposable, so it must be A_2. If there is a second decomposition for G, then G_1 in the second decomposition is also

A_2 (since $A_n, n > 2$ will produce more passers in G). It follows that G_2 in the two decompositions is also the same, therefore the decomposition of G is unique. The case where G has vetoers is similar, it implies that $G_1 = U_2$, and a similar argument to the above can be used.

(ii). Finally, if G has neither passer nor vetoers, then $G_1 = H_{n,k}$. Suppose $G_2 = G_3 \circ \ldots \circ G_m$, where $G_3 \ldots G_m$ are all indecomposable ideal weighted games. If G_2 has passers, then G_3 has passers, but since G_3 is indecomposable ideal weighted, then it is A_2. Since A_2 has two passers, then $G_3 \circ \ldots \circ G_m$ has exactly one passer (the other passer in A_2 was composed over). It follows that if there is a second equal decomposition for G, then it must have $G_1 = H_{n,k}$ and also $G_3 = A_2$, and it is easy now to see that the two decompositions are the same. The case where G_2 has vetoers is similar. Therefore we conclude that the decomposition of G is unique.

This concludes the proof of the main result of this chapter and one of the main results of this thesis, where a complete characterisation of all ideal weighted simple games was given. In the next chapter, we consider the problem of characterising ideal games in the larger class of roughly weighted simple games.

Chapter 7

Ideal Roughly Weighted Simple Games

All the chapters so far have focused on weightedness of simple games. We undertook a thorough study of hierarchical simple games, tripartite simple games and the games that are composed of smaller games. We then applied our understanding of them to achieve a complete characterisation of ideal weighted simple games. Now, in this chapter, we extend our study of simple games to a bigger class, namely the class of roughly weighted simple games. We first recap some of the concepts related to roughly weighted simple games in general. We start by re-visiting an example of a nonweighted roughly weighted game from Chapter 2. Recall from Chapter 1 that our strategy of characterising ideal roughly weighted games consisted of three steps (RW1 - RW3 on page 23), of which the first one (RW1) poses the question: Is any roughly weighted game a composition of indecomposable roughly weighted games? In the first section of this chapter we show that any roughly weighted game, with few exceptions, is indeed a composition of indecomposable roughly weighted games. Then the second section will look at the two types of roughly weighted games, namely complete and incomplete, and demonstrate with examples that we have ideal complete roughly weighted games and ideal incomplete roughly weighted games. In the third section, regarding (RW2), we characterise nonweighted roughly weighted hierarchical games,

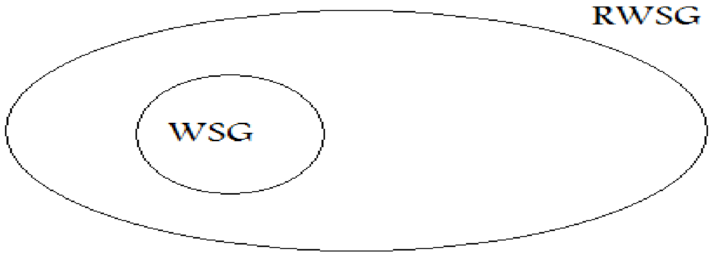

Figure 7.1: Rough weightedness includes weightedness and more.

which shows, among other things, that the class of ideal roughly weighted games is larger than the class of ideal weighted games, since hierarchical games are ideal. Section four then gives an example of a nonweighted roughly weighted tripartite game (the class of tripartite games was also shown to be ideal in Chapter 4), this is also related to (RW2). Finally in the last section, we answer the question posed in (RW3) in the positive, where we illustrate with an example that the class of ideal complete roughly weighted games which are neither hierarchical nor tripartite is nonempty.

7.1 On the Decomposition of Roughly Weighted Games

In this section we show that any roughly weighted game, with few exceptions, is a composition of indecomposable roughly weighted games. Recall that subgames and reduced games are called minors.

The operation of taking minors will be a very useful tool later on. Minors preserve rough weightedness provided they meet one condition as shown in the next lemma, whose proof is straightforward.

Lemma 7.1.1. *Let $G = (P, W)$ be a roughly weighted game with rough voting representation $[q; w_1, \ldots, w_n]$. Suppose that $A \subseteq P$ such that $w(b) > 0$ for some $b \in A^c$. Then the subgame G_A and the reduced game G^A are roughly weighted.*

In rough voting representations of G_A and G^A the weights of players are the same as in G and quotas are q and $\max(0, q - w(A))$, respectively.

In the lemma above, the condition '$w(b) > 0$ for some $b \in A^c$' is needed, if this condition does not hold, then all players of the reduced game will have weights equal to zero as well as the quota q' being zero, and hence Definition 2.2.15 of rough weightedness will not apply. But we will restrict our attention to the class of reduced games which have at least one player with nonzero weight.

Now suppose we are given a roughly weighted game G such that $G = G_1 \circ G_2$, where both G_1 and G_2 are indecomposable. We want to investigate whether both G_1 and G_2 are roughly weighted. Recall that if a game has a potent certificate of nonweightedness, then it is not roughly weighted (see page 42). Then we have the following.

Lemma 7.1.2. *Suppose a composition $G = G_1 \circ_g G_2$ is roughly weighted with $G_1 = (P_1, W_1)$, $G_2 = (P_2, W_2)$ and $g \in P_1$. Suppose there is a player $b \in P_2$ such that $w(b) > 0$. Then both G_1 and G_2 are roughly weighted.*

Proof. Let $P = P_1 \setminus \{g\} \cup P_2$ be the set of players of G. The fact that G_2 is roughly weighted follows directly from Lemma 7.1.1. Indeed, G_2 is a reduced game of G, i.e., $G_2 \cong G^A$, where $A = \{P_1\} \setminus \{g\}$, and there is a player $b \in P_2$ such that $w(b) > 0$.

So it remains to show that G_1 is roughly weighted. Suppose it is not. Then G_1 has a potent certificate of nonweightedness

$$(X_1, \ldots, X_j, P_1; Y_1, \ldots, Y_j, \emptyset), \tag{7.1.1}$$

where $X_1, \ldots, X_j$ are winning in G_1 and $Y_1, \ldots, Y_j$ are losing. Since (7.1.1) is a trading transform, then, without loss of generality, for some k, player g belongs to the first k winning coalitions $X_1, \ldots, X_k$ and the first $k + 1$ losing coalitions $Y_1, \ldots, Y_{k+1}$ (we must have in mind that P_1 also contains g which balances the transform). Define

$$X'_i = \begin{cases} X_i \setminus \{g\} \cup P_2 & i \in \{1, \ldots, k\}, \\ X_i & i \in \{k+1, \ldots, j\} \end{cases}$$

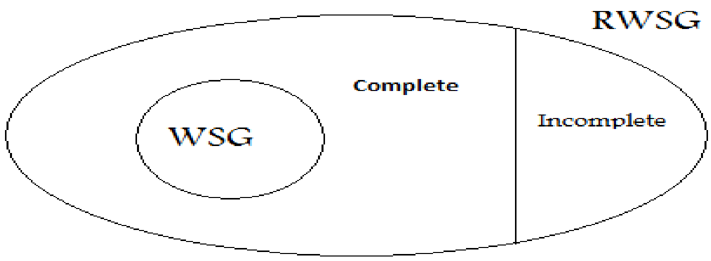

Figure 7.2: Roughly weighted games can be complete or incomplete.

and

$$Y_i' = \begin{cases} Y_i \setminus \{g\} \cup P_2 & i \in \{1, \ldots, k+1\}, \\ Y_i & i \in \{k+2, \ldots, j\}. \end{cases}$$

Then

$$(X_1', \ldots, X_j', P;\ Y_1', \ldots, Y_j', \emptyset), \tag{7.1.2}$$

is a potent certificate of nonweightedness for G since $X_1', \ldots, X_j'$ are winning in G and $Y_1', \ldots, Y_j'$ are losing in G. □

7.2 Ideal Complete and Ideal Incomplete Roughly Weighted Games

In the early chapters of this thesis, we have discussed the fact that all weighted games are complete. As far as roughly weighted games are concerned, however, due to the fact that coalitions whose total weight is equal to the threshold can be winning or losing, it is not necessary that all roughly weighted games are complete. We can have either complete or incomplete roughly weighted games. Example 2.2.11 was a demonstration of an ideal complete roughly weighted game, since it was a hierarchical game, which is known to be both complete and ideal.

The following is an example of an incomplete roughly weighted game.

Example 7.2.1. In the game $G = (P, W)$, let $P = \{a, b, c, d, e\}$ be the set of players. And let W consist of all coalitions with four or more players, together with $\{a, b, c\}$ and $\{a, d, e\}$. This game is roughly weighted according to this system of weights: $w(a) = w(b) = w(c) = w(d) = w(e) = 1$ and the threshold being $q = 3$. Since all coalitions with total weight greater than 3 are winning, and all those with total weight less than 3 are losing, whereas among those whose total weight is 3, only two are winning. But since, for instance, the two coalitions $\{a, b, d\}$ and $\{a, e, c\}$ are losing, then this game is not complete, due to the following trading transform

$$(\{a, b, c\}, \{a, d, e\}; \{a, b, d\}\{a, e, c\}).$$

Since the above trading transform required the swap of only one player from the first winning coalition with one from the second winning coalition, then it is a certificate of incompleteness by Theorem 2.2.14.

We saw that roughly weighted simple games (RWSGs) can be either complete or incomplete, and we already saw an example of an ideal complete RWSG. It turns out, that there are also ideal incomplete RWSGs, as we shall demonstrate shortly. So the task of characterising ideal RWSGs can be divided into two parts: (i) The characterisation of ideal complete RWSGs, and (ii) the characterisation of ideal incomplete RWSGs. The study of ideal incomplete roughly weighted games is a separate question beyond the scope of this thesis, and can be considered as an open problem (see Chapter 8). We will, therefore, for the remaining part of this thesis consider only ideal complete roughly weighted games. However, just to demonstrate that the class of ideal incomplete RWSGs is nonempty, we give an example of such a game, and prove it to be ideal using Matroid Theory. First we need to introduce some basics of Matroid Theory.

7.2.1 The connection between matroids and secret sharing schemes

A connection between ideal secret sharing schemes and matroids has been established in (Brickell & Davenport, 1991), and this connection was the main tool used in the two papers (Beimel, Tassa, & Weinreb, 2008) and (Farràs & Padró, 2010), where a substantial progress towards characterising ideal WSGs was made. For a more exhaustive and interesting study of matroids, the reader is encouraged to look at (Oxley, 1992).

Definition 7.2.2. A *matroid* M is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E, known as the *ground set*, and a collection $\mathcal{I}$ of subsets of E, known as the *independent sets* of M, satisfying the following three conditions:

(I1) $\emptyset \in \mathcal{I}$.

(I2) *If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.*

(I3) *If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 \setminus I_1$ such that $I_1 \cup e \in \mathcal{I}$.*
Condition (I3) is called the *independence augmentation axiom.*

If a set is not independent, then it is called *dependent*. A *circuit* in a matroid is a minimal dependent set. Meaning, removing any one element from a circuit results in an independent set.

It has been shown in (Brickell & Davenport, 1991) that an ideal secret sharing scheme has a corresponding matroid associated with it, denoted herein $\mathcal{M}$, but not necessarily the other way around. The relation between ideal secret sharing schemes and matroids is as follows. Let Γ be an ideal access structure on a set of users $U = \{u_1, \ldots, u_n\}$. The elements of $\mathcal{M}$ are the users in U together with an additional element, denoted u_0, that could be thought of as representing the dealer. We denote hereinafter by

$$\mathcal{C}_0(\Gamma) = \{X \cup \{u_0\} : X \text{ is a minterm of } \Gamma\}$$

the set of all Γ-minterms, supplemented by u_0.

Theorem 7.2.3. *(Brickell & Davenport (1991), Theorem 1) Let Γ be a connected ideal access structure on U. Then there exists a matroid $\mathcal{M}$ on $U \cup \{u_0\}$, such that $\mathcal{C}_0(\Gamma)$ is exactly its set of circuits that contain u_0.*

The matroid whose existence is guaranteed by Theorem 7.2.3 is unique up to an isomorphism, and it is referred to as the *matroid corresponding to* Γ. The next definition will enable us to explicitly define the matroid corresponding to Γ using the authorised sets in Γ.

Definition 7.2.4. (Critical User) Let M_1 and M_2 be distinct minterms of Γ. A user $x \in M_1 \cup M_2$ is critical for $M_1 \cup M_2$ if the set $M_1 \cup M_2 \setminus \{x\}$ is unauthorised. In addition we define

$$D(M_1, M_2) = (M_1 \cup M_2) \setminus \{x \in M_1 \cup M_2 : x \text{ is critical for } M_1 \cup M_2\}.$$

The following result is very important.

Lemma 7.2.5. *(Beimel, Tassa, & Weinreb (2008), Corollary 4.4) Let Γ be a connected ideal access structure on U. Then there exists a unique connected matroid $\mathcal{M}$ on $U \cup \{u_0\}$, such that $\mathcal{C}_0(\Gamma)$ is exactly its set of circuits that contain u_0. Furthermore, the minimal sets in $\{D(M_1, M_2) : M_1, M_2$ are distinct minterms of $\Gamma\}$ are the circuits that do not contain u_0.*

We can also try to obtain an ideal access structure from a matroid, but this will not work for every matroid. A *representation* of a matroid $\mathcal{M}$ is a matrix A with entries from some field $\mathbb{F}$, such that each element in the ground set of $\mathcal{M}$ represents some column in A. Moreover, there is a one-to-one correspondence between the columns of A and the ground set of $\mathcal{M}$, meaning a set of columns in A is linearly independent (as vectors) if and only if the corresponding set is independent in $\mathcal{M}$. In such a case the matroid $\mathcal{M}$ is called *representable*. With this definition in mind, we have the following.

Theorem 7.2.6. *(Brickell & Davenport (1991), Theorem 2) Let $\mathcal{M}$ be a representable matroid over a field. Then there exists an ideal secret sharing scheme S with access structure Γ, such that the set of circuits of $\mathcal{M}$ containing u_0 is $\mathcal{C}_0(\Gamma)$.*

Now we are ready for the example.

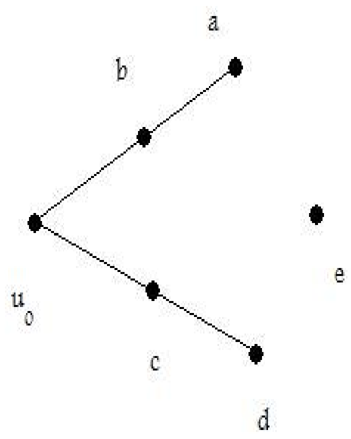

Figure 7.3: The matroid Q_6

7.2.2 An example of an ideal incomplete roughly weighted game

Example 7.2.7. Let $G = (P_G, W_G)$ be a game such that $P_G = \{a, b, c, d, e\}$ is the set of players. The set of minimal winning coalitions is $W_G^{\min} = \{\{a, b\}, \{c, d\}, \{a, e, d\}, \{a, e, c\}, \{b, e, d\}, \{b, e, c\}\}$. The rough weights are $rw(a) = rw(b) = rw(c) = rw(d) = rw(d) = \frac{1}{2}$ and the quota $q = 1$. We can see that the game is roughly weighted, but it is not complete by the following trading transform (see Theorem 2.2.14):

$$(\{a, b\}, \{c, d\}; \{d, b\}, \{c, a\}).$$

Finally, let us think of $W_G^{\min}$ as an access structure for some secret sharing scheme realising it. We can form the following set $\mathcal{C}_0 = \{X \cup \{u_0\}\}$, where X is a minterm from $W_G^{\min}$ above, and $u_0 \notin P_G$. The set $\mathcal{C}_0$ above happens to be the set of all circuits containing the point u_0 of a matroid known in matroid theory as Q_6 (see Oxley (1992), p.503). The geometric representation of Q_6 is shown in Figure 7.3. Now, Q_6 is representable over a field $\mathbb{F}$ if and only if $|\mathbb{F}| \geq 4$ (also in Oxley (1992), p.503). It follows from Theorem 7.2.6 that $W_G^{\min}$ above carries an ideal secret sharing scheme. This completes our example for the existence of a roughly weighted game that is also ideal and incomplete.

In the remaining part of the thesis, building upon the characterisation of ideal

WSGs of Chapter 6, we shall solve some problems regarding the characterisation of ideal complete roughly weighted games. So the rest of the thesis is as follows. In the next Section 7.3, we consider the class of HSGs and identify six categories of nonweighted roughly weighted disjunctive HSGs, and then obtain the corresponding results for conjunctive HSGs by duality. In Section 7.4, as a demonstration of the fact that the class of roughly weighted TSGs is larger than the class of weighted TSGs, we give an example for such a nonweighted roughly weighted TSG, which also happens to be an indecomposable game. Finally, in Section 7.5, we discover an ideal complete roughly weighted game, which is neither hierarchical nor tripartite. We also prove its indecomposability and ideality.

7.3 Roughly Weighted Hierarchical Simple Games

Let us now generalise Theorem 3.6.2 and classify roughly weighted disjunctive hierarchical games. As the classification of weighted hierarchical games is already given in Theorem 3.6.2, we will characterize only nonweighted ones. Firstly, we formulate the two main theorems. The first one is about disjunctive hierarchical games.

Theorem 7.3.1. *Let $H = H_\exists(\boldsymbol{n}, \boldsymbol{k})$ be an m-level nonweighted hierarchical game without passers. Then it is roughly weighted if and only if one of the following is true:*

(i) $\mathbf{k} = (2, 4)$ *with* $n_1 \geq 2$ *and* $n_2 \geq 4$*;*

(ii) $\mathbf{k} = (k, k+2)$*, with* $n_1 \geq k > 2$ *and* $n_2 = 4$*;*

(iii) $\mathbf{k} = (2, 3, 4)$ *and* $\mathbf{n} = (n_1, 2, n_3)$ *with* $n_1 \geq 2, n_3 \geq 3$*;*

(iv) $\mathbf{k} = (k, k+1, k+2)$*, and* $\mathbf{n} = (n_1, 2, 2)$*, where* $2 \leq k \leq n_1$ *or* $\mathbf{n} = (n_1, n_2, 2)$ *with* $2 \leq k \leq n_1$ *and* $n_2 \geq 3$*;*

(v) $\mathbf{k} = (k, k+1, k_3)$*,* $\mathbf{n} = (n_1, n_2, n_3)$ *such that* $2 \leq k \leq n_1$*, and* $n_3 = k_3 - k \geq 3$.

(vi) $k_m = k_{m-1} + n_m$ *and the subgame* $H_\exists(\mathbf{n}, \mathbf{k})_{\{m^{n_m}\}} = H_\exists(\mathbf{n}', \mathbf{k}')$*, where* $\mathbf{n}' = (n_1, \ldots, n_{m-1})$ *and* $\mathbf{k}' = (k_1, \ldots, k_{m-1})$*, falls under one of the types (i)–(v).*

This will imply the following classification of conjunctive hierarchical games.

Theorem 7.3.2. *Let* $H = H_\forall(\boldsymbol{n}, \boldsymbol{k})$ *be an* m*-level nonweighted conjunctive hierarchical game without vetoers. Then it is roughly weighted and nonweighted if and only if one of the following is true:*

(i) $\mathbf{k} = (n_1 - 1, n_1 + n_2 - 3)$*, where* $n_1 \geq 2$*,* $n_2 \geq 4$*;*

(ii) $\mathbf{k} = (k, k + 2)$*, with* $1 \leq k < n_1 - 1$*, and* $n_2 = 4$*;*

(iii) $\mathbf{k} = (n_1 - 1, n_1, n_1 + n_3 - 1)$ *and* $\mathbf{n} = (n_1, 2, n_3)$ *such that* $n_1 \geq 2, n_3 \geq 3$*;*

(iva) $\mathbf{k} = (k, k + 1, k + 2)$*, where* $1 \leq k \leq n_1 - 1$ *and* $\mathbf{n} = (n_1, 2, 2)$*;*

(ivb) $\mathbf{k} = (k_1, k_2, k_2 + 1)$ *with* $\mathbf{n} = (n_1, n_2, 2)$*, where* $k_2 - k_1 = n_2 - 1$*,* $1 \leq k_1 \leq n_1 - 1$ *and* $n_3 \geq 3$*;*

(v) $\mathbf{k} = (k_1, k_2, k_2 + 1)$*, with* $1 \leq k \leq n_1 - 1$ *and* $n_2 \geq 3$*;*

(vi) $k_{m-1} = k_m$*, and the reduced game* $H_\forall(\mathbf{n}, \mathbf{k})^{\{m^{n_m}\}} = H_\forall(\mathbf{n}', \mathbf{k}')$*, where* $\mathbf{n}' = (n_1, \ldots, n_{m-1})$ *and* $\mathbf{k}' = (k_1, \ldots, k_{m-1})$*, falls under one of the types (i)–(v).*

So let us outline the methodology of the proof of Theorem 7.3.1. In the next section we prove that a disjunctive hierarchical game with no passers may not have more than four levels (Lemma 7.3.12). This gives us the opportunity to consider cases $m = 2, 3, 4$ levels separately. This is what we do in Section 7.3.3.

Section 7.3.3 will start with describing the strategy that we shall follow in our proofs. Then we consider two-level roughly weighted disjunctive hierarchical games and fully classify them in Lemma 7.3.13. What follows after that is our investigation of the three-level games. This is the main part of the characterisation.

Our first goal here will be a full classification of disjunctive hierarchical games with $\mathbf{k} = (2, 3, 4)$ (Lemma 7.3.15). This is the most important type of disjunctive hierarchical games. This is because they are either subgames or reduced games of virtually any other disjunctive hierarchical game. The next goal after that will be a classification of games with $\mathbf{k} = (2, 3, k)$ for $k \geq 5$. This will be achieved in Lemma 7.3.16.

We then briefly discuss how the weights of a roughly weighted disjunctive hierarchical game G change if we pass on to a reduced game of G. With this tool we consider disjunctive hierarchical games with arbitrary vector $\mathbf{k} = (k_1, k_2, k_3)$. The next milestone is Lemma 7.3.20 which states that rough weightedness implies $k_2 - k_1 = 1$. The method that we intensively use is to find a reduced game to which we know what the weights should be and then make conclusions about the possible weights of the original game.

This leaves us with the two remaining cases to consider $\mathbf{k} = (k, k+1, k+2)$, $k \geq 3$ and $\mathbf{k} = (k, k+1, k_3)$, such that $k_3 - k \geq 3$. This cases are dealt with in Lemmata 7.3.22 and 7.3.23.

Finally we prove the main theorem on roughly weighted disjunctive hierarchical games by proving that the fourth nontrivial level is not possible (the two-level and three-level games by that time are already classified). Then by duality we deduce the main result on roughly weighted conjunctive hierarchical games.

It was mentioned in the earlier chapters that it is a well-known fact that any weighted game can be given a voting representation in which players of equal Isbell's desirability have equal weights. However, we need a similar statement that would be also applicable to roughly weighted games. In other words, we want to be able to say that a simple game $G = (P, W)$ is a roughly weighted game if and only if the corresponding simple game $\bar{G} = (\bar{P}, \bar{W})$ is. It is easy to check that if we take a class of equivalent players and assign them the average weight of players in this class, then the resulting system of weights will again give us a rough voting representation for the same game. If we do this with every class of equivalent players we will achieve the result. We formalise it as follows.

Lemma 7.3.3. *A simple game $G = (P, W)$ is a roughly weighted majority game if and only if the corresponding simple game $\bar{G} = (\bar{P}, \bar{W})$ is.*

Proof. Suppose there are m equivalence classes $P_1, \ldots, P_m$ of players and let us denote $[i]$ the equivalence class to which i belongs. The statement is nontrivial only in one direction. The nontrivial part is to prove that if the game G on P is roughly weighted, then the game $\bar{G}$ on $\bar{P}$ is also roughly weighted. So suppose that there exists a system of weights $w_1, \ldots, w_n$ and the quota $q \geq 0$, not all equal to zero, such that $\sum_{i \in X} w_i > q$ implies $X \in W$ and $\sum_{i \in X} w_i < q$ implies $X \in L$. Our statement will be proved if we can find another system of weights $u_1, \ldots, u_n$ for G which satisfy the conditions:

(i) $u_i = u_j$ if $[i] = [j]$,

(ii) $\sum_{i \in X} u_i > q$ implies $X \in W$.

(iii) $\sum_{i \in X} u_i < q$ implies $X \notin W$.

We define this alternative system of weights by setting $u_i = \frac{1}{|[i]|} \sum_{j \in [i]} w_j$, i.e., we replace the weight of ith player with the average weight of players in the equivalence class to which i belongs. It obviously satisfies (i). Let us prove that it satisfies (ii).

Let $X \subseteq P$ and $\sum_{i \in X} u_i > q$. Let $k_i = |X \cap P_i|$. Let X^+ be the subset of P which results in replacing in X, for all $i = 1, 2, \ldots, m$, all k_i players of P_i with the "heaviest" players from the same class. Then the weight of X^+ relative to the old system of weights is greater or equal to $\sum_{i \in X} u_i$ and hence greater than q. So X^+ is winning in G, and so is X, because we replaced all weights of players with equivalent weights (the average weight of players in each equivalence class). (iii) is proved similarly. $\square$

7.3.1 Minors of Disjunctive Hierarchical Simple Games

In proving our classification of nonweighted roughly weighted HSGs, we will first work with disjunctive hierarchical games, and then obtain the result for conjunctive hierarchical games by duality. Hence in this section we restrict ourselves with the disjunctive case only. The following statements are easy to check.

Proposition 7.3.4. *Let* $\mathbf{n}' = (n_2+k_1-1, n_3, \ldots, n_m)$ *and* $\mathbf{k}' = (k_2, \ldots, k_m)$. *Then* $H_\exists(\mathbf{n}', \mathbf{k}')$ *is a subgame of* $G = H_\exists(\mathbf{n}, \mathbf{k})$. *This subgame does not have passers and it has dummies if and only if G had.*

Proof. Indeed, $H_\exists(\mathbf{n}', \mathbf{k}') = G_A$ for $A = \{1^{n_1-k_1+1}\}$. If we make $n_1 - k_1 + 1$ elements of level one unavailable the first constraint loses any bite and the first level collapses. □

Lemma 7.3.5. *For any* $i = 1, 2, \ldots, m-1$ *there exists* n'_i, *such that for* $\mathbf{n}' = (n'_i, n_{i+1})$ *and* $\mathbf{k}' = (k_i, k_{i+1})$ *the game* $G' = H_\exists(\mathbf{n}', \mathbf{k}')$ *is a subgame of* $G = H_\exists(\mathbf{n}, \mathbf{k})$.

Proof. Follows directly from Propositions 3.4.1 and 7.3.4. □

Proposition 7.3.6. *Let* $G = H_\exists(\mathbf{n}, \mathbf{k})$, *where* $\mathbf{n} = (n_1, \ldots, n_m)$, $\mathbf{k} = (k_1, \ldots, k_m)$. *Suppose that* $k_i > k_{i-1} + 1$ *for some* $i \in \{1, \ldots, m\}$. *Then for*

$$\mathbf{n}' = (n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_m),$$
$$\mathbf{k}' = (k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1} - 1, \ldots, k_m - 1)$$

$G' = H_\exists(\mathbf{n}', \mathbf{k}')$ *is a reduced game of G. Moreover, if G did not have dummies, then* G' *would not have them.*

Proof. Since all representations are canonical, the condition $k_i > k_{i-1}+1$ implies that $n_i > k_i - k_{i-1} \geq 2$, so $n_i \geq 3$. We note now that $G' = G^A$ for $A = \{i\}$. It is easy to check that all conditions (a) and (b) are satisfied for the new values of parameters $\mathbf{n}'$ and $\mathbf{k}'$. □

7.3.2 Roughly Weighted Disjunctive Hierarchical Simple Games have at most four levels

We will consider a roughly weighted hierarchical m-level game, call it H. It will be convenient to have the quota equal to 1. Also by Lemma 7.3.3 we may consider that all players of level i have weight w_i, so that any rough voting representation has the form $[1; w_1, \ldots, w_m]$. If X is a coalition of H, by $w(X)$ we will denote the total weight of X.

Our approach will be based on Theorem 3.5.1, and need the following observation stated in Proposition 7.3.7 below. But first, we note the following which will help us make sense of the next proposition.

We note that if $[q; w_1, \ldots, w_n]$ is a rough voting representation of a game G, then $w_i > w_j$ implies that player i is at least as desirable as player j. This, in particular, implies that if player k is strictly less desirable than player ℓ, then $w_k \leq w_\ell$. As we will see later it is possible that player k is strictly less desirable than player ℓ but $w_k = w_\ell$ and, in particular, a nondummy player can have zero weight.

Proposition 7.3.7. *Let H be a disjunctive hierarchical game, and let M be its unique shift-maximal losing coalition. Suppose H is roughly weighted with rough voting representation $[1; w_1, \ldots, w_m]$. Then $w(M) \geq w(L)$ for any losing coalition L.*

Proof. Since any shift replaces a player with a less influential one, the weight of the latter must be not greater than the weight of the former. This secures that if a coalition S is obtained from a coalition T by a shift, then $w(T) \geq w(S)$. If S is a subset of T, then also $w(T) \geq w(S)$. Since M is a unique shift-maximal losing coalition we will have $w(M) \geq w(L)$ for any losing coalition L. □

This simple proposition has a useful corollary.

Corollary 7.3.8. *Let H be a disjunctive hierarchical game, and let M be its unique shift-maximal losing coalition. Suppose H is roughly weighted with rough voting representation $[1; w_1, \ldots, w_n]$ but not weighted. Then $w(M) = 1$.*

Proof. If $w(M) < 1$, then by Proposition 7.3.7 there is no losing coalitions on the threshold. In this case the game is weighted. □

The following will also be very useful.

Proposition 7.3.9. *Let $H_\exists(\mathbf{n},\mathbf{k})$ be the m-level disjunctive hierarchical game with no passers and no dummies. Suppose it is roughly weighted with rough voting representation $[1; w_1, \ldots, w_n]$. Then*

(i) $w_1 \geq w_2 \geq \ldots \geq w_m$.

(ii) $w_i > 0$ *for* $i = 1, 2, \ldots, m-1$.

Proof. As there are no dummies, $k_m - k_{m-1} < n_m$ is satisfied. We also have $k_1 > 1$ as no passers are present. By Theorem 3.5.1 we know that $H_\exists(\mathbf{n},\mathbf{k})$ has a unique shift-maximal losing coalition. This coalition then would be

$$M = \{1^{k_1-1}, 2^{k_2-k_1}, \ldots, m^{k_m-k_{m-1}}\}. \tag{7.3.1}$$

By Corollary 7.3.8 we have

$$w(M) = (k_1-1)w_1 + (k_2-k_1)w_2 + \ldots + (k_m-k_{m-1})w_m = 1.$$

If only $w_{i+1} > w_i$, then

$$\begin{aligned} w(M) \geq &(k_1-1)w_1 + \ldots + (k_i-k_{i-1})w_i + (k_{i+1}-k_i)w_{i+1} > \\ &\quad (k_1-1)w_1 + \ldots + (k_{i-1}-k_{i-2})w_{i-1} + (k_{i+1}-k_{i-1})w_i \geq 1, \end{aligned}$$

since the latter is the weight of a winning coalition $\{1^{k_1-1}, 2^{k_2-k_1}, \ldots, i^{k_{i+1}-k_{i-1}}\}$ (indeed the cardinality of this multiset is $k_{i+1}-1 \geq k_i$). This contradiction proves (i).

To prove (ii) we note that by Theorem 3.2.1 we have $k_i - k_{i-1} < n_i$, and hence every level in multiset M is not completely filled and has some capacity. Suppose first that $k_m - k_{m-1} \leq n_m - 2$. Then the multiset

$$M' = \{1^{k_1-2}, 2^{k_2-k_1}, \ldots, m^{k_m-k_{m-1}+2}\}$$

is winning from which we see that $w_m > 0$. If $k_i - k_{i-1} = n_i - 1$, then the multiset

$$M'' = \{1^{k_1-2}, 2^{k_2-k_1}, \ldots, (m-1)^{k_{m-1}-k_{m-2}+1}, m^{k_m-k_{m-1}+1}\}$$

is winning whence $w_{m-1} > 0$. This proves (ii). $\square$

The two following results will be very useful later on in characterising roughly weighted hierarchical simple games with three levels and more.

Lemma 7.3.10. *Suppose that an m-level disjunctive hierarchical game $H = H_\exists(\mathbf{n}, \mathbf{k})$ without passers and without dummies is roughly weighted with rough voting representation $[1; w_1, \ldots, w_m]$ but not weighted. If the subgame $H' = H_\exists(\mathbf{n}', \mathbf{k}')$, where $\mathbf{n}' = (n_1, \ldots, n_{m-1})$ and $\mathbf{k}' = (k_1, \ldots, k_{m-1})$, is also not weighted, then $w_m = 0$.*

Proof. By Proposition 3.4.1 H' is a subgame of H, hence it is roughly weighted with rough voting representation $[1; w_1, \ldots, w_{m-1}]$. By Proposition 7.3.9 all the weights are nonzero. The shift-maximal losing coalition M for H will be (7.3.1) and for H' it will be

$$M' = \{1^{k_1-1}, 2^{k_2-k_1}, \ldots, (m-1)^{k_{m-1}-k_{m-2}}\}.$$

If the game H' is not weighted, then by Corollary 7.3.8 we have $w(M') = 1$. As $1 = w(M) = w(M') + (k_m - k_{m-1})w_m$ and $k_m > k_{m-1}$, this implies $w_m = 0$. □

The reader might expect that $w_m = 0$ implies that the mth level must consist of dummies. In a roughly weighted game this may not be the case. Here is an example illustrating this.

Example 7.3.11. Let us consider disjunctive hierarchical game $H = H_\exists(\mathbf{n}, \mathbf{k})$ with $\mathbf{n} = (3, 3, 3)$ and $\mathbf{k} = (2, 3, 5)$. It is roughly weighted relative to the weights $[1; \frac{1}{2}, \frac{1}{2}, 0]$. Indeed, the shift-minimal winning coalitions of H are $\{1^2\}$, $\{2^3\}$, $\{2^2, 3^3\}$. They all have weight at least 1. The unique shift-maximal losing coalition $\{1, 2, 3^2\}$ also has weight 1 but this is allowed. The players of the third level are not dummies despite having weight 0. Moreover in any other system of weights consistent with the game H, players of level three will have weight 0.

Proof. Let us prove the last statement about the game in this example. If $[1; w_1, w_2, w_3]$ is any rough voting representation for H, then the following system of inequalities must hold:

$$w_1 \geq \frac{1}{2} \tag{7.3.2}$$

$$w_2 \geq \frac{1}{3} \tag{7.3.3}$$

$$2w_2 + 3w_3 \geq 1 \tag{7.3.4}$$

$$w_1 + w_2 + 2w_3 = 1. \tag{7.3.5}$$

However (7.3.2) and (7.3.5) imply $w_2 + 2w_3 \leq \frac{1}{2}$, which implies $2w_2 + 4w_3 \leq 1$, which together with (7.3.4) implies $w_3 = 0$. $\square$

Now we can restrict the number of nontrivial levels to four.

Lemma 7.3.12. *A roughly weighted m-level disjunctive hierarchical game $H = H_\exists(\mathbf{n}, \mathbf{k})$ without passers and without dummies may have no more than four levels, i.e., $m \leq 4$.*

Proof. Suppose $m \geq 5$. Consider the game $H' = H_\exists(\mathbf{n}', \mathbf{k}')$, where $\mathbf{n}' = (n_1, \ldots, n_{m-1})$ and $\mathbf{k}' = (k_1, \ldots, k_{m-1})$. By Proposition 3.4.1 H' is a subgame of H and has no passers or dummies. By Lemma 7.1.1 it is roughly weighted. As it has four or more levels, by Theorem 3.2.1 it is not weighted. By Proposition 7.3.9 we have $w_{m-1} > 0$, but we also have $w_{m-1} = 0$ by Lemma 7.3.10 applied to H'. This contradiction proves the lemma. $\square$

We will see in the following section that four nontrivial levels are also not achievable.

7.3.3 The characterization of Roughly Weighted Disjunctive Hierarchical Simple Games

Now we can start our full characterisation of all roughly weighted hierarchical games. Due to the results of the previous section our main focus will be on 2-level ones, then 3-level ones, and then showing that the fourth level may not be added unless we allow dummies or passers.

The strategy

Here is our general strategy to analyse if a particular disjunctive hierarchical game $G = H_\exists(\mathbf{n}, \mathbf{k})$ without passers is roughly weighted or not. Firstly, if a system of rough weights for G exists, then, due to the absence of passers the quota is strictly positive so, by normalising, we may assume that in a voting representation we look for the quota equal to 1, i.e., our rough voting representation must be $[1; w_1, \ldots, w_n]$, where $w_1 \geq w_2 \geq \ldots \geq w_n$. We then list all shift-minimal winning coalitions and write a system of inequalities S in $w_1, \ldots, w_n$ that is equivalent to the fact that in the game with rough voting representation $[1; w_1, \ldots, w_n]$ these coalitions are above or on the threshold. For example if $\{1^2, 3, 4^3\}$ is a shift-minimal winning coalition, we add the inequality $2w_1 + w_3 + 3w_4 \geq 1$ to S. Requiring that those shift-minimal winning coalitions are on or above the threshold is sufficient for ensuring that all winning coalitions are on or above the threshold. This is due to the fact that every shift reduces the weight or leaves it fixed and adding players does not decrease the weight either. By Theorem 3.5.1 there is a unique shift-maximal losing coalition M. So then (assuming no dummies) by Corollary 7.3.8 we have the following equation

$$(k_1 - 1)w_1 + (k_2 - k_1)w_2 + \ldots + (k_n - k_{n-1})w_n = 1, \tag{7.3.6}$$

which expresses the fact that the weight of M is exactly on the threshold. We add equation (7.3.6) to S. This system has a solution if and only if the game is roughly weighted. However, the existence of a solution may lead occasionally to a weighted game. We need to check these solutions against Theorem 3.6.2 to make sure that the game is not weighted.

The possible shift-minimal winning coalitions in a two- or a three-level disjunctive hierarchical game $G = H_\exists(\mathbf{n}, \mathbf{k})$ and the inequalities corresponding to them are as follows:

- When $k_i \leq n_i$, we have shift-minimal winning coalition $\{i^{k_i}\}$ and the corresponding inequality
$$k_i w_i \geq 1. \tag{7.3.7}$$

- In the case when $k_2 > n_2$ the coalition $\{1^{k_2-n_2}, 2^{n_2}\}$ is a shift-minimal winning coalition, so we have

$$(k_2 - n_2)w_1 + n_2 w_2 \geq 1. \tag{7.3.8}$$

- In the case when $k_3 > n_3$ there are two possibilities, either $k_2 \leq n_2$, or $k_2 > n_2$. Suppose $k_2 \leq n_2$. Since $\{2^{k_3-n_3}, 3^{n_3}\}$ is a shift-winning coalition, then we have

$$(k_3 - n_3)w_2 + n_3 w_3 \geq 1. \tag{7.3.9}$$

  (We note that $k_3 - n_3 < k_2 \leq n_2$ in this case.) And if $k_2 > n_2$, then since $k_3 - n_3 < k_2 < k_1 + n_2$, and $k_1 \leq n_1$, the coalition $\{1^{k_3-n_2-n_3}, 2^{n_2}, 3^{n_3}\}$ is a shift-minimal winning coalition, and we have

$$(k_3 - n_2 - n_3)w_1 + n_2 w_2 + n_3 w_3 \geq 1. \tag{7.3.10}$$

Two-level games

Lemma 7.3.13. *Let $H = H_\exists(\mathbf{n}, \mathbf{k})$ be a two-level disjunctive hierarchical game with no passers and no dummies. Then H is roughly weighted but not weighted if and only if one of the following conditions is satisfied:*

(i) $\mathbf{k} = (2, 4)$ *with $n_1 \geq 2$ and $n_2 \geq 4$;*

(ii) $\mathbf{k} = (k, k + 2)$*, where $k > 2$, with $n_1 \geq k$ and $n_2 = 4$.*

If $[1; w_1, w_2]$ is a rough voting representation for H, then $w_2 = w_1/2$. Moreover, in case (i) we have $(w_1, w_2) = (\frac{1}{2}, \frac{1}{4})$.

Proof. Let $[1; w_1, w_2]$ be a rough voting representation for H and M be its unique shift-maximal losing coalition. As we do not have passers we have $k_1 > 1$. We need to consider two cases: (i) $k_2 \leq n_2$ and (ii) $k_2 > n_2$. In the first case, due to (7.3.7), we have $k_1 w_1 \geq 1$, $k_2 w_2 \geq 1$ and by Corollary 7.3.8 $w(M) = (k_1 - 1)w_1 + (k_2 - k_1)w_2 = 1$. If only we had $k_1 w_1 > 1$ or $k_2 w_2 > 1$ we could decrease w_1 or w_2 and make $w(M) < 1$ in which case the game would be weighted. Hence $k_1w_1 = k_2w_2 = (k_1 - 1)w_1 + (k_2 - k_1)w_2 = 1$. This implies $1/k_1 + k_1/k_2 = 1$. Let $k_2 - k_1 = d$. Then $\frac{1}{k_1} + \frac{k_1}{k_1+d} = 1$, which is equivalent to $k_1 + d = k_1d$ or $d = \frac{k_1}{k_1-1}$. It implies $1 < d \leq 2$ whence $d = 2$ and $k_1 = 2$. Thus we have only one solution: $k_1 = 2$ and $k_2 = 4$. This implies $\mathbf{w} = (w_1, w_2) = (\frac{1}{2}, \frac{1}{4})$.

Let us consider the second case. Due to (7.3.6), (7.3.7) and (7.3.8) $\mathbf{w}$ satisfy the inequalities $k_1w_1 \geq 1$, $(k_2 - n_2)w_1 + n_2w_2 \geq 1$ and the equality $(k_1 - 1)w_1 + (k_2 - k_1)w_2 = 1$. The latter line must be a supporting line of the polyhedron area given by

$$k_1w_1 \geq 1, \quad (k_2 - n_2)w_1 + n_2w_2 \geq 1, \quad w_2 \geq 0.$$

Indeed, if it cuts across this area, then we will be able to find a point (w_1, w_2) in this area with $(k_1 - 1)w_1 + (k_2 - k_1)w_2 < 1$. The game then will be weighted relative to $[1; w_1, w_2]$. This area has only two extreme points and the line must pass through at least one of them. This is either when $w_2 = 0$ or when $k_1w_1 = 1$ and $(k_2 - n_2)w_1 + n_2w_2 = 1$. Firstly, let us consider the case when $w_2 = 0$. In such a case $(k_2 - n_2)w_1 \geq 1$ and $(k_1 - 1)w_1 = 1$. This can only happen when $k_2 - n_2 \geq k_1 - 1$ or $n_2 \leq k_2 - k_1 + 1$, but by Theorem 3.2.1 we cannot have $n_2 < k_2 - k_1 + 1$, so it must be that $n_2 = k_2 - k_1 + 1$. But in this case H is weighted by Theorem 1.1.1.

Suppose now $k_1w_1 = (k_2 - n_2)w_1 + n_2w_2 = 1$ and $(k_1 - 1)w_1 + (k_2 - k_1)w_2 = 1$. Expressing w_1 and w_2 from the first two equations and substituting into the third we obtain $(k_2 - k_1)(n_2 - (k_2 - k_1)) = n_2$. Denoting $d = k_2 - k_1$ we can rewrite this as $n_2 = d + 1 + \frac{1}{d-1}$. As n_2 must be an integer we get $d = 2$ and $n_2 = 4$. Now $w_2 = \frac{k_1 - k_2 + n_2}{n_2k_1} = \frac{1}{2k_1} = \frac{w_1}{2}$. It is easy to check that these weights indeed make H roughly weighted and $w(M) = (k - 1)w_1 + 2 \cdot \frac{w_1}{2} = kw_1 = 1$. $\square$

Three-level games

Following our strategy outlined earlier, we now focus our investigation on three-level disjunctive HSGs.

Lemma 7.3.14. *Let $H = H_{\exists}(\mathbf{n}, \mathbf{k})$ with no passers and no dummies, where $\mathbf{n} = (n_1, n_2, n_3)$ and $\mathbf{k} = (k, k+1, k+a)$, where $a \geq 2$ is a positive integer, with $k_i \leq n_i$ for $i = 1, 2, 3$. Then H is not roughly weighted.*

Proof. The shift-minimal winning coalitions are $\{1^k\}$, $\{2^{k+1}\}$ and $\{3^{k+a}\}$ and the inequalities in this case will be $kw_1 \geq 1$, $(k+1)w_2 \geq 1$ and $(k+a)w_3 \geq 1$, respectively. The shift-maximal equation in this case will be $(k-1)w_1 + w_2 + (a-1)w_3 = 1$. As in the proof of Lemma 7.3.13, we may assume that all three aforementioned inequalities are in fact equalities, that is $w_1 = \frac{1}{k}$, $w_2 = \frac{1}{k+1}$, and that $w_3 = \frac{1}{k+a}$. Substituting these weights in the shift-maximal equation we get a contradiction as $(k-1)\frac{1}{k} + \frac{1}{k+1} + \frac{a-1}{k+a} = 1$ is equivalent to $\frac{1}{k+1} + \frac{a-1}{k+a} = \frac{1}{k}$ which never happens for $k \geq 2$ and $a \geq 2$ as this is equivalent to $(a-1)k^2 + (a-2)k - a = 0$ from which $a = \frac{k^2+2k}{k^2+k-1} < 2$, a contradiction. $\square$

As mentioned at the start of this section, the most basic type of disjunctive hierarchical games that we will be referring to constantly is the one with $\mathbf{k} = (2, 3, 4)$. We will characterise these in Lemmata 7.3.15 and 7.3.16.

Lemma 7.3.15. *Let $\mathbf{k} = (2, 3, 4)$. The 3-level game $G = H_{\exists}(\mathbf{n}, \mathbf{k})$ with no dummies is roughly weighted if and only if $n_1 \geq 2$ and one of the following is true:*

(i) $\mathbf{n} = (n_1, 2, n_3)$*, where* $n_3 \geq 3$ *and* $\mathbf{w} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$*;*

(ii) $\mathbf{n} = (n_1, n_2, 2)$*, where* $n_2 \geq 3$ *and* $\mathbf{w} = (\frac{1}{2}, \frac{1}{2} - \alpha, \alpha)$ *with* $\alpha \in [0, \frac{1}{6}]$*;*

(iii) $\mathbf{n} = (n_1, 2, 2)$ *and* $\mathbf{w} = (\frac{1}{2}, \frac{1}{2} - \alpha, \alpha)$ *with* $\alpha \in [0, \frac{1}{4}]$.

Proof. Firstly, we note that by Theorem 3.2.1 we have $2 = k_1 \leq n_1$. We also note that the shift-maximal equation in this case by (7.3.6) is in this case

$$w_1 + w_2 + w_3 = 1. \tag{7.3.11}$$

Case (i). $k_i \leq n_i$ for $i = 1, 2, 3$ is considered in Lemma 7.3.14. There are no solutions in this case.

Case (ii). Suppose $k_2 > n_2, k_3 \leq n_3$, then $n_3 \geq 4$. As $k_2 = 3$, by Corollary 3.2.4 it follows that n_2 must be 2. The shift-minimal winning coalitions are $\{1^2\}$, $\{1, 2^2\}$, $\{3^4\}$. So the corresponding inequalities are $w_1 \geq \frac{1}{2}$, $w_1 + 2w_2 \geq 1$, and $w_3 \geq \frac{1}{4}$. As in the proof of Lemma 7.3.14, we may assume $w_3 = \frac{1}{4}$. From (7.3.11) we get $w_1 + w_2 = \frac{3}{4}$. It follows that $w_1 + 2(\frac{3}{4} - w_1) \geq 1$, whence $w_1 \leq \frac{1}{2}$, forcing $w_1 = \frac{1}{2}, w_2 = \frac{1}{4}, w_3 = \frac{1}{4}$. So it is roughly weighted only when $\mathbf{n} = (n_1, 2, n_3)$, and $\mathbf{w} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, as required.

Case (iii). Suppose $k_2 \leq n_2$, $k_3 > n_3$. Then $n_3 \leq 3$ and $n_2 \geq 3$. Then the shift-minimal winning coalitions are $\{1^2\}$, $\{2^3\}$, $\{2^{4-n_3}, 3^{n_3}\}$. To justify this we have to note that by Theorem 3.2.1 $k_3 - n_3 < k_2 \leq n_2$ whence $4 - n_3 \leq n_2$ and the last coalition is legitimate. The inequalities then will be $w_1 \geq \frac{1}{2}, w_2 \geq \frac{1}{3}$ and $(4-n_3)w_2 + n_3 w_3 \geq 1$. As above we may assume $w_1 = \frac{1}{2}$. Substituting this value of w_1 into (7.3.11) in this case we get $w_2 + w_3 = \frac{1}{2}$. So $(4-n_3)w_2 + n_3(\frac{1}{2} - w_2) \geq 1$. Now by Corollary 3.2.4 n_3 is either 2 or 3. If it is 3, then we get $w_2 + 3(\frac{1}{2} - w_2) \geq 1$ giving $w_2 \leq \frac{1}{4}$, but we know that $w_2 \geq \frac{1}{3}$, contradiction. If it is 2, then the system has solutions for any $w_2 \geq \frac{1}{3}$ and the game is roughly weighted with $\mathbf{w} = (\frac{1}{2}, \frac{1}{2} - \alpha, \alpha)$, where $\alpha \in [0, \frac{1}{6}]$. In this case, $\mathbf{n} = (n_1, n_2, 2)$.

Case (iv). Suppose $k_2 > n_2$ and $k_3 > n_3$. Then $n_3 \leq 3$ and $n_2 = 2$. Since by Theorem 3.2.1 and Corollary 3.2.4 we have $4 = k_3 < k_2 + n_3$, then $4 - n_3 \leq 2 = n_2$ and the shift-minimal winning coalitions are $\{1^2\}$, $\{1, 2^2\}$, $\{2^{4-n_3}, 3^{n_3}\}$. Then the inequalities will be: $w_1 \geq \frac{1}{2}$, $w_1 + 2w_2 \geq 1$, $(4-n_3)w_2 + n_3 w_3 \geq 1$. If $n_3 = 2$, then the latter becomes $2w_2 + 2w_3 \geq 1$ or $w_2 + w_3 \geq \frac{1}{2}$ which, in particular, imply $w_1 = 1 - w_2 - w_3 \leq \frac{1}{2}$, whence $w_1 = \frac{1}{2}$ and $w_2 + w_3 = \frac{1}{2}$. Now from $w_1 + 2w_2 \geq 1$ we get $w_2 \geq \frac{1}{4}$. So this gives the solution $n = (n_1, 2, 2)$ with $w = (\frac{1}{2}, \frac{1}{2} - \alpha, \alpha)$, where $\alpha \in [0, \frac{1}{4}]$, as required. Now if $n_3 = 3$, then the inequalities will be $w_1 \geq \frac{1}{2}$, $w_1 + 2w_2 \geq 1$, $w_2 + 3w_3 \geq 1$. Again substituting $w_3 = 1 - w_1 - w_2$ into the latter inequality gives $3w_1 + 2w_2 \leq 2$. As $2w_2 \geq 1 - w_1$ we get $1 + 2w_1 \leq 2$ and $w_1 \leq \frac{1}{2}$. Hence $w_1 = \frac{1}{2}$ and $w_2 + w_3 = \frac{1}{2}$. This together with $w_2 + 3w_3 \geq 1$ gives $w_3 \geq \frac{1}{4}$. But since $w_1 = \frac{1}{2}$, then $w_1 + 2w_2 \geq 1$ gives $w_2 \geq \frac{1}{4}$, and so $w_3 \leq \frac{1}{4}$. Thus we have the weights $w = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, with $n = (n_1, 2, 3)$. This works. $\square$

Now we turn our attention to the games with $\mathbf{k} = (2, 3, k)$ for $k \geq 5$ and show that they are seldom roughly weighted.

Lemma 7.3.16. *Let $H = H_\exists(\mathbf{n}, \mathbf{k})$, where $\mathbf{k} = (2, 3, k)$ and $k \geq 5$, be a 3-level disjunctive hierarchical game with no dummies. Then $n_1 \geq 2$ and it is roughly weighted if and only if $n_3 = k - 2$, and $\mathbf{w} = (\frac{1}{2}, \frac{1}{2}, 0)$.*

Proof. We note that the absence of dummies means that $k_2 + n_3 > k_3$ or $n_3 > k - 3$. The shift-maximal equation is now

$$w_1 + w_2 + (k - 3)w_3 = 1. \tag{7.3.12}$$

Case (i). The case when $k_i \leq n_i$ for all i is treated in Lemma 7.3.14. There are no solutions in this case.

Case (ii). As $k_1 \leq n_1$, suppose $k_2 > n_2$ and $k_3 \leq n_3$. It follows that n_2 must be 2. The shift-minimal winning coalitions then are $\{1^2\}$, $\{1, 2^2\}$, $\{3^k\}$. So the corresponding inequalities are $w_1 \geq \frac{1}{2}$, $w_1 + 2w_2 \geq 1$ and $w_3 \geq \frac{1}{k}$. By the usual trick we may assume that $w_3 = \frac{1}{k}$. Then from the shift-maximal equation (7.3.12) we get $w_1 + w_2 = \frac{3}{k}$. It follows that $w_1 + 2(\frac{3}{k} - w_1) \geq 1$, so $w_1 \leq \frac{6-k}{k} \leq \frac{1}{k}$, but $w_1 \geq \frac{1}{2}$, contradiction.

Case (iii). Suppose $k_2 \leq n_2$, and $n_3 < k_3 = k$. Then $k_3 - k_2 + 1 = k - 2 \leq n_3 \leq k - 1$ and, in particular, by Corollary 3.2.4 $k - n_3 \leq 2 \leq n_2$. Then the shift-minimal winning coalitions are $\{1^2\}$, $\{2^3\}$, $\{2^{k-n_3}, 3^{n_3}\}$, giving the inequalities $w_1 \geq \frac{1}{2}$, $w_2 \geq \frac{1}{3}$ and $(k - n_3)w_2 + n_3 w_3 \geq 1$. We may set $w_1 = \frac{1}{2}$ which implies $w_2 + (k-3)w_3 = \frac{1}{2}$. Let us consider two cases: (a) $n_3 = k-1$ and (b) $n_3 = k-2$.

(a) In this case the two inequalities become $w_2 + (k - 3)w_3 = \frac{1}{2}$ and $w_2 + (k - 1)w_3 \geq 1$. These imply $w_3 \geq \frac{1}{4}$. But then $w_2 + (k - 3)w_3 \geq \frac{1}{3} + \frac{k-3}{4} > \frac{1}{2}$, contradiction.

(b) In this case the two inequalities become $w_2 + (k - 3)w_3 = \frac{1}{2}$ and $2w_2 + (k - 2)w_3 \geq 1$. This implies that either $w_3 = 0$ or $2(k - 3) \leq k - 2$. The latter implies $k \leq 4$, hence the only solution in this case is $\mathbf{w} = (\frac{1}{2}, \frac{1}{2}, 0)$.

Case (iv). Suppose $k_2 > n_2$, $n_3 < k_3 = k$, so $n_2 = 2$ and, as in case (iii), $k-2 \leq n_3 \leq k-1$. Then the shift-minimal winning coalitions are $\{1^2\}$, $\{1, 2^2\}$, $\{2^{k-n_3}, 3^{n_3}\}$, giving the inequalities $w_1 \geq \frac{1}{2}$, $w_1 + 2w_2 \geq 1$ and $(k-n_3)w_2 + n_3w_3 \geq 1$. We have either (a) $n_3 = k-2$ or $n_3 = k-1$.

(a) Suppose $n_3 = k-2$. Then the last inequality becomes $2w_2+(k-2)w_3 \geq 1$. From the shift-maximal equation (7.3.12) we get $w_2 + (k-3)w_3 \leq \frac{1}{2}$, which together with the previous inequality implies either $w_3 = 0$ or $2(k-3) \leq k-2$. As the latter implies $k \leq 4$, we again have the solution $\mathbf{w} = (\frac{1}{2}, \frac{1}{2}, 0)$.

(b) Suppose $n_3 = k-1$. Then the last inequality becomes $w_2+(k-1)w_3 \geq 1$. From the shift-maximal equation (7.3.12) we get $w_2 + (k-3)w_3 \leq \frac{1}{2}$ from which $w_3 \geq \frac{1}{4}$. But this contradicts to (7.3.12) since $w_1+w_2+(k-3)w_3 \geq \frac{1}{2}+w_2+\frac{k-3}{4} > 1$ for any $k \geq 5$. $\square$

Let us now make some useful observations that we will use in the remaining proofs.

Proposition 7.3.17. *Let $H = H_\exists(\mathbf{n}, \mathbf{k})$ be a disjunctive hierarchical game on $P = \{1^{n_1}, 2^{n_2}, 3^{n_3}\}$ with no passers and dummies, and $A = \{1^{s_1}, 2^{s_2}, 3^{s_3}\}$ is a submultiset of P such that either $s_1 \neq n_1$ or $s_2 \neq n_2$. Suppose H is roughly weighted with rough voting representation $[1; w_1, w_2, w_3]$. Then the reduced game H^A is also roughly weighted with rough voting representation*

$$\left[1; \frac{w_1}{1-w(A)}, \frac{w_2}{1-w(A)}, \frac{w_3}{1-w(A)}\right]. \tag{7.3.13}$$

Proof. H^A will be also roughly weighted by Lemmata 7.1.1 and 7.3.9. Since A is a submultiset of P with the total weight $w(A) = s_1w_1 + s_2w_2 + s_3w_3$, then by Lemma 7.1.1, the reduced game H^A has rough voting representation $[1 - w(A); w_1, w_2, w_3]$ or (7.3.13) after normalisation. $\square$

Lemma 7.3.18. *Suppose that a 3-level hierarchical game $H = H_\exists(\mathbf{n}, \mathbf{k})$, where $\mathbf{k} = (2, k_2, k_3)$, has no dummies and is roughly weighted with rough voting representation $[1; w_1, w_2, w_3]$. Then either $w_3 = 0$ or $\mathbf{k} = (2, 3, 4)$.*

Proof. Suppose $w_3 > 0$. If $k_2 > 3$, then $n_2 \geq k_2 - k_1 + 1 = k_2 - 1$ so the second level contains at least $k_2 - 1$ elements and, in particular, $A = \{2^{k_2-3}\}$ is a submultiset of the multiset of players. Let us consider the reduced game H^A. Then by Proposition 7.3.6, $H^A = H(\mathbf{n}', \mathbf{k}')$ with $\mathbf{n}' = (n_1, n_2 - k_2 + 3, n_3)$ and $\mathbf{k}' = (2, 3, k_3 - k_2 + 3)$. Since $n_2 - k_2 + 3 \geq 2$ the reduced game still has three levels. By Proposition 7.3.17 the game H^A is also roughly weighted and, due to (7.3.13) the last weight of it will still be nonzero. Having $k_3 - k_2 + 3 > 4$ would imply by Lemma 7.3.16 that the last weight is zero. Since that is not the case, then we have $k_3 - k_2 + 3 = 4$ so $\mathbf{k}' = (2, 3, 4)$.

Let $w = w(A) = (k_2 - 3)w_2$. Then by Lemma 7.3.15 and (7.3.13) we have $\frac{w_1}{1-w} = \frac{1}{2}$. This means that $2w_1 = 1 - w < 1$ which contradicts the fact that $\{1^2\}$ is a winning coalition in H. Hence $k_2 = 3$ and $\mathbf{k} = (2, 3, 4)$. □

Corollary 7.3.19. *There does not exist a roughly weighted* 3*-level disjunctive hierarchical game* $H = H_\exists(\mathbf{n}, \mathbf{k})$ *with* $\mathbf{k} = (2, 4, k_3)$ *and no dummies.*

Proof. Suppose on the contrary that H is roughly weighted with rough voting representation $[1; w_1, w_2, w_3]$. Then by Lemma 7.3.18 we must have $w_3 = 0$. Consider $H' = H_\exists(\mathbf{n}', \mathbf{k}')$, where $\mathbf{n}' = (n_1, n_2)$ and $\mathbf{k}' = (k_1, k_2)$. If it is weighted, then by Theorem 3.6.2 $n_2 = k_2 - k_1 + 1 = 3$. And if it is not, then $n_2 \geq 4$ by Lemma 7.3.13. In either case we have shift-minimal winning coalitions $\{1^2\}$ and $\{1, 2^3\}$, hence $w_1 \geq \frac{1}{2}$ and $w_1 + 3w_2 \geq 1$. By Theorem 3.2.1 we have $k_3 - n_3 \leq 3$ so the third shift-minimal winning coalition is of the type $\{2^{k_3-n_3}, 3^{n_3}\}$. The weight of such coalition is not greater than $3w_2$. So we must have $w_2 \geq \frac{1}{3}$. At the same time from the shift-maximal equation $w_1 + 2w_2 = 1$ and $w_1 \geq \frac{1}{2}$ we have $w_2 \leq \frac{1}{4}$. This is a contradiction. □

Lemma 7.3.20. *Suppose that a* 3*-level disjunctive hierarchical game* $H = H_\exists(\mathbf{n}, \mathbf{k})$ *with no passers and no dummies is roughly weighted. Then* $k_2 - k_1 = 1$.

Proof. Let $[1; w_1, w_2, w_3]$ be a rough voting representation of H. As there is no passers, $k_1 \geq 2$. Suppose $k_2 - k_1 \geq 2$. Observe that all 3-level games with $k_2 - k_1 \geq 2$ can be reduced to a 3-level game where $k_1 = 2$, $k_2 = 4$ as follows.

First, take the reduced game $H_1 = H^A$ with $A = \{1^{k_1-2}\}$, which will result in a game $H_1 = H_\exists(\mathbf{n}', \mathbf{k}')$, with $\mathbf{n}' = (n_1', n_2, n_3)$ and $\mathbf{k}' = (2, k_2', k_3')$, where $k_2' = k_2 - k_1 + 2$, $k_3' = k_3 - k_1 + 2$ and $n_1' = n_1 - k_1 + 2$. By Proposition 7.3.17 H_1 is roughly weighted. Hence by Corollary 7.3.19 we have $k_2' \geq 5$. By Theorem 3.2.1 $n_2 \geq k_2 - k_1 = k_2' - 2$ and $n_2 - (k_2' - 4) \geq 2$. This shows that n_2 has enough players for a further reduction to $H_1^{A'}$, where $A' = \{2^{k_2'-4}\}$, without collapsing the second level. The resulting game with $\mathbf{k}'' = (2, 4, k_3' - (k_2' - 4))$ is not roughly weighted by Corollary 7.3.19 which contradicts Proposition 7.3.17 and proves the lemma. $\square$

By combining Lemmata 7.3.16 and 7.3.20 we get the following

Corollary 7.3.21. *If a* 3*-level disjunctive hierarchical game* $H = H_\exists(\mathbf{n}, \mathbf{k})$ *does not have passers and dummies and is roughly weighted, then it belongs to one of the following two categories:*

(i) $\mathbf{k} = (k, k+1, k+2)$*;*

(ii) $\mathbf{k} = (k, k+1, k_3)$*, such that* $n_3 = k_3 - k \geq 3$.

Proof. By Lemma 7.3.20 we have $k_2 = k_1 + 1$. To prove the other claims we make a reduction of H and consider $H' = G^A$ with $A = \{1^{k_1-2}\}$. Then H' has parameters $\mathbf{n}' = (n_1 - k_1 + 2, n_2, n_3)$ and $\mathbf{k}' = (2, k_2 - k_1 + 2, k_3 - k_1 + 2) = (2, 3, k_3 - k_1 + 2)$. Now either $k_3' = 4$, and in this case $k_3 = k + 2$, or by Lemma 7.3.16 $n_3 = k_3' - 2 = k_3 - k_1$. Since in the latter case we have $k_3' \geq 5$, then we get $k_3 - k \geq 3$. $\square$

So it is these two categories of games that we need to analyse. They will be analysed in the following two lemmas. We refer in their study to Lemmas 7.3.15 and 7.3.16.

Lemma 7.3.22. *A* 3*-level game* $H = H_\exists(\mathbf{n}, \mathbf{k})$ *with* $\mathbf{k} = (k, k+1, k+2)$ *and* $k \geq 3$ *is roughly weighted if and only if* $n_1 \geq k$ *and one of the following conditions is satisfied:*

(a) $\mathbf{n} = (n_1, 2, 2)$, $\mathbf{w} = \left(\frac{1}{k}, \frac{1-2\alpha}{k}, \frac{2\alpha}{k}\right)$, *where* $\alpha \in \left[0, \frac{1}{4}\right]$*;*

(b) $\mathbf{n} = (n_1, n_2, 2)$*,* $n_2 \geq 3$*, and* $\mathbf{w} = \left(\frac{1}{k}, \frac{1}{k}, 0\right)$.

Proof. Firstly, it is easy to check that the games in (a) and (b) are indeed, roughly weighted with the specified set of weights.

If H is roughly weighted, then upon reducing it to $H^A = H_\exists(\mathbf{n}', \mathbf{k}')$, where $A = \{1^{k-2}\}$, we get $\mathbf{k}' = (2, 3, 4)$. Also, $\mathbf{n}'$ must fall into one of the three cases given in Lemma 7.3.15. Let us analyze them one by one.

Case (i). $\mathbf{n}' = (n_1', 2, n_3)$, where $n_1' \geq 2$, $n_3 \geq 3$, and $\mathbf{w} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. By Proposition 7.3.17 the rough voting representation of H^A will then be

$$\left[1; \frac{w_1}{1 - w_1(k-2)}, \frac{w_2}{1 - w_1(k-2)}, \frac{w_3}{1 - w_1(k-2)}\right]$$

and it has to match the voting representation of the reduced game at hand, namely $[1; \frac{1}{2}, \frac{1}{4}, \frac{1}{4}]$. It follows that $\frac{w_1}{1-w_1(k-2)} = \frac{1}{2}$, so $w_1 = \frac{1}{k}$. Also, $\frac{w_2}{1-w_1(k-2)} = \frac{w_3}{1-w_1(k-2)} = \frac{1}{4}$, giving $w_2 = w_3 = \frac{1}{2k}$. Now we test to see if these weights indeed define the original hierarchical game H. Now since $(k+2)w_3 = \frac{k+2}{2k}$ is never greater than or equal to 1 for $k \geq 3$ coalition $\{3^{k_3}\}$ does not exist in H (otherwise it would be winning), that is, $n_3 \leq k+1$. Also $w_2 + (k+1)w_3 = 2w_2 + kw_3 = \frac{k+2}{2k}$, and a shift-minimal winning coalition $\{2^i, 3^{k+2-i}\}$, for $i = 1, 2$, also does not exist in H for $k \geq 3$.

So the coalition $\{1^{(k+2)-n_2-n_3}, 2^{n_2}, 3^{n_3}\}$ must be a shift-minimal winning coalition in H. So its weight should be at least the threshold, which is 1, i.e., $\frac{k+2-2-n_3}{k} + \frac{2}{2k} + \frac{n_3}{2k} = \frac{2k-n_3+2}{2k} \geq 1$. But this is never true for $n_3 \geq 3$. Therefore H is not roughly weighted in this case.

Case (ii). $\mathbf{n}' = (n_1', n_2, 2)$, where $n_1' \geq 2$, $n_2 \geq 3$, and $\mathbf{w} = (\frac{1}{2}, \frac{1}{2} - \alpha, \alpha)$ for some $\alpha \in [0, \frac{1}{6}]$. Here w_1 is still $\frac{1}{k}$. But $\frac{w_2}{1-w_1(k-2)} = \frac{1}{2} - \alpha$. It follows that $\frac{kw_2}{2} = \frac{1}{2} - \alpha$, so that $w_2 = \frac{1-2\alpha}{k}$. Also, $\frac{w_3}{1-w_1(k-2)} = \alpha$, implying $w_3 = \frac{2\alpha}{k}$. As we do not have dummies there must be a winning coalition consisting of $k+2$ players. This would be either $\{2^k, 3^2\}$ or $\{1^{k-n_2}, 2^{n_2}, 3^2\}$ depending on what is greater n_2 or k. But $w(\{2^k, 3^2\}) = k\frac{1-2\alpha}{k} + 2\frac{2\alpha}{k} = 1 - 2\alpha + \frac{4\alpha}{k}$. For this to be winning we need $1-2\alpha+\frac{4\alpha}{k}\geq 1$, giving $\frac{4\alpha}{k}\geq 2\alpha$. So either we have $\frac{2}{k}\geq 1$ or $\alpha=0$. As $k>2$ we have the latter. Thus $\{2^k,3^2\}$ can be winning only for $\alpha=0$ in which case $\mathbf{w}=(\frac{1}{k},\frac{1}{k},0)$.

Suppose now that the winning coalition consisting of $k+2$ players is $\{1^{k-n_2},2^{n_2},3^2\}$. Its weight then is $\frac{k-n_2}{k}+\frac{n_2(1-2\alpha)}{k}+\frac{4\alpha}{k}\geq 1$. It follows that $2\alpha\geq\alpha n_2$, whence $\alpha=0$. In both cases we have (b).

Case (iii). $\mathbf{n}'=(n_1',2,2)$, where $\mathbf{w}=(\frac{1}{2},\frac{1}{2}-\alpha,\alpha)$, and $\alpha\in[0,\frac{1}{4}]$. This gives us case (a). □

Now to the remaining case.

Lemma 7.3.23. *Any 3-level game $G=H(\mathbf{n},\mathbf{k})$, where $\mathbf{k}=(k,k+1,k_3)$ such that $k_3-(k+1)\geq 2$ and G has no passers and no dummies, is roughly weighted if and only if the following is true.*

(i) $\mathbf{n}=(n_1,n_2,n_3)$, *where* $n_3=k_3-k\geq 3$, *and* $\mathbf{w}=(\frac{1}{k},\frac{1}{k},0)$.

Proof. Upon reducing G to $G^A=H'(\mathbf{n}',\mathbf{k}')$, where $A=\{1^{k-2}\}$, we get $\mathbf{k}'=(2,3,k_3')$, where $k_3'\geq 5$. Since the game G^A is roughly weighted by Proposition 7.3.17, then by Lemma 7.3.16 it has to have $\mathbf{n}=(n_1,n_2,k_3'-2)$, where $n_2\geq 2$, and $n_1\geq 2$, and the weights consistent with $\mathbf{w}=(\frac{1}{2},\frac{1}{2},0)$. So we get $\frac{w_1}{1-w_1(k-2)}=\frac{1}{2}$, so $w_1=\frac{1}{k}$. Also, $\frac{w_2}{1-w_1(k-2)}=\frac{1}{2}$, meaning $\frac{w_2}{1-w_1k+2w_1}=\frac{1}{2}$, so $\frac{w_2}{1-1+2\frac{1}{k}}=\frac{1}{2}$. Therefore $w_2=\frac{1}{k}$, and $w_3=0$. It can be easily checked that these weights give a valid hierarchical game where $\mathbf{n}=(n_1,n_2,k_3-k), w=(\frac{1}{k},\frac{1}{k},0)$. □

Proofs of Theorems 7.3.1 and 7.3.2

Finally we are ready to collect all facts together and prove the two theorems that characterise roughly weighted disjunctive and conjunctive hierarchical games.

Proof of Theorem 7.3.1. Firstly, we note that $k_1>1$ since H does not have passers. Secondly, we note that if $k_m\geq k_{m-1}+n_m$, then the mth level of players consists of dummies. This game will be roughly weighted if and only if the game $H_\exists(\mathbf{n}', \mathbf{k}')$ is. This situation is described in (vi). So we consider that $k_m < k_{m-1} + n_m$ and assume that H does not have passers and dummies. Lemma 7.3.12 now gives us that $m \le 4$.

If $m = 2$ the result follows from Lemma 7.3.13. If $m = 3$ the result follows from Lemmata of the previous section. We will now show that the fourth level cannot be added without introducing dummies. Suppose that H has the fourth level whose players are not dummies. The game $H_\exists(\mathbf{n}', \mathbf{k}')$ is roughly weighted and, since it has more than two levels, it is not weighted by Theorem 1.1.1. By Lemma 7.3.10 we get then $w_4 = 0$. By Lemma 7.3.9 each of the weights w_1, w_2, w_3 is nonzero.

By letting $A = \{1^{k_1-2}\}$ and considering the reduced game $H' = H^A$ we obtain a 4-level game $H' = H_\exists(\mathbf{n}', \mathbf{k}')$ which is roughly weighted by Proposition 7.3.17. We can now consider the subgame $H'' = H_\exists(\mathbf{n}'', \mathbf{k}'')$ of H', where $\mathbf{n}'' = (n'_1, n'_2, n'_3)$ and $\mathbf{k}'' = (k'_1, k'_2, k'_3)$. It will be again roughly weighted by Lemma 7.1.1 and we may apply Lemma 7.3.18 to this 3-level game. By Lemma 7.1.1 all weights of this game will be nonzero. By Lemma 7.3.18 we will then have $\mathbf{k}'' = (2, 3, 4)$ and thus $\mathbf{k}' = (2, 3, 4, k_4)$, $\mathbf{n}' = (n'_1, n_2, n_3, n_4)$, where $n'_1 = n_1 - k_1 + 2$. As there are no dummies in H, by Theorem 3.2.1 we have $n_4 \ge k_4 - 4 + 1 = k_4 - 3$. Thus either $\{3^3, 4^{k_4-3}\}$ or $\{2, 3^2, 4^{k_4-3}\}$ is a winning coalition as total number of players in each is k_4. By Lemma 7.3.15, in all cases when H' is roughly weighted we have $w_2 + w_3 = \frac{1}{2}$ and $w_3 \le \frac{1}{4}$. However, the weight of the coalition $\{2, 3^2, 4^{k_4-3}\}$ is $w_2 + 2w_3 = \frac{1}{2} + w_3 \le \frac{3}{4}$. The same is true for the coalition $\{3^3, 4^{k_4-3}\}$, giving a contradiction.

The characterisation of conjunctive hierarchical games will now be deduced using duality between disjunctive and conjunctive games.

Proof of Theorem 7.3.2. The proof will consists of calculating the duals for the games listed in Theorem 7.3.1.

Firstly, we note that we have $k_1 = 1$ if and only if in the dual game we have

$n_1 - k_1 + 1 = k_1^*$, meaning $k_1^* = n_1$. Hence no passers translates in the dual game into no vetoers.

(i) Here we have $k_1^* = n_1 - k_1 + 1$ where $k_1 = 2$, so $k_1^* = n_1 - 1$. Also $k_2^* = n_1 + n_2 - k_2 + 1 = n_1 + n_2 - 3$.

(ii) In the dual game $k_1^* = n_1 - k + 1$ and $k_2^* = n_1 + n_2 - (k+2) + 1 = n_1 - k + 3 = k_1^* + 2$. As $n_1 \geq k > 2$, we have $n_1 - 1 > k_1^* \geq 1$.

(iii) Since $\mathbf{k} = (2, 3, 4)$ in the dual game, then this gives $n_1 - k_1 + 1 = k_1^*$, so that $k_1^* = n_1 - 1$. Also, $n_1 + n_2 - k_2 + 1 = k_2^*$, giving $k_2^* = n_1 + n_2 - 2 = n_1$ since $n_2 = 2$. We also get $k_3^* = n_1 + n_2 + n_3 - k_3 + 1 = n_1 + n_3 - 1$.

(iv) We have two cases here so we treat them separately. Calculating the parameters of the dual game we get:

(a) $k_1^* = n_1 - k + 1$, $k_2^* = n_1 + n_2 - (k+1) + 1 = n_1 - k + 2 = k_1^* + 1$, and $k_3^* = n_1 + n_2 + n_3 - (k+2) + 1 = n_1 - k + 3 = k_1^* + 2$.

(b) $k_1^* = n_1 - k + 1$, $k_2^* = n_1 + n_2 - (k+1) + 1 = n_1 + n_2 - k$, and $k_3^* = n_1 + n_2 + n_3 - (k+2) + 1 = n_1 + n_2 - k + 1 = k_2^* + 1$.

(v) Calculating the parameters of the dual game we get $k_1^* = n_1 - k_1 + 1$, $k_2^* = n_1 + n_2 - (k_1 + 1) + 1 = n_1 + n_2 - k_1 = k_1^* + n_2 - 1$, $k_3^* = n_1 + n_2 + n_3 - k_3 + 1 = n_1 + n_2 - k_1 + 1 = k_2^* + 1$.

(vi) Finally, in the last case we have $n_m - k_m = k_{m-1}$ and this implies $k_m^* = k_{m-1}^*$.

This brings us to the end of characterising the first class of ideal complete roughly weighted games. Now we look at the second one, the class of tripartite games.

7.4 Example of a Roughly Weighted Tripartite Simple Game

The second class of ideal complete roughly weighted games, is the class of tripartite games discussed in Chapter 4. We don't have a full characterisation of roughly weighted tripartite games yet, but below is an example confirming that this class is nonempty, in fact it is an example of an indecomposable game. First recall the definition of Δ_1.

Definition 7.4.1. Let $P = \{1^{n_1}, 2^{n_2}, 3^{n_3}\}$ and m, d, t be positive integers such that $m \geq t$. If $n_2 > d + t - m$, then we define the simple game Δ_1 as follows: a submultiset $X = \{1^{l_1}, 2^{l_2}, 3^{l_3}\}$ is winning in Δ_1 if and only if

$$l_1 \geq t \text{ or } (l_1 + l_2 + l_3 \geq m \text{ and } l_1 + l_2 \geq m - d).$$

Example 7.4.2. Consider $\Delta_1 = (P_C, W_C)$, where $P_C = \{1^2, 2^2, 3^4\}$, with $m = 4, d = 3, t = 2$. The set of shift-minimal winning coalitions is $\{\{1^2\}, \{2, 3^3\}\}$. We assign the following rough weights to the players: $rw(1) = \frac{1}{2}, rw(2) = rw(3) = \frac{1}{4}$, and we let the quota to be 1. Then we see that Δ_1 is roughly weighted since all coalitions with total weight greater than 1 are winning, all coalitions with total weight less than 1 are losing. Also, among the coalitions with total weight equaling 1, $\{3^4\}$ is the only losing one. This game, however, is not weighted since we have the following certificate of nonweightedness:

$$(\{1^2\}, \{2, 3^3\}; \{1, 2, 3\}, \{1, 3^2\}).$$

We also know from item (5) on page 107 that this is actually an example of an indecomposable tripartite game of type T_1[1].

In the next section, we introduce an ideal complete simple game that is neither hierarchical nor tripartite. We then give an example for this game, and show that it is indecomposable and ideal, thus discovering a new game in the class of ideal complete roughly weighted games.

[1]The only condition from (5) on page 107 that is violated here is $m - 1 = t$, which is actually needed for the weightedness of T_1, and not for its indecomposability.

7.5 Example of an Ideal Roughly Weighted Game, which is neither Hierarchical nor Tripartite

The example we give in this section is of a 4-partite simple game, which is a complete game of four desirability levels that we will define shortly. This type of game will be referred to as **NHNT**. Let each of the four levels $1 \dots 4$, be denoted $L_1 \dots L_4$ respectively, where $|L_i| = n_i$ for $1 \leq i \leq 4$, so its multiset representation is on the multiset of players $\{1^{n_1}, 2^{n_2}, 3^{n_3}, 4^{n_4}\}$. The construction of this game is based on the idea that this game should be neither a hierarchical nor a tripartite game, but if we were to change the size of its least desirable level L_4 from n_4 to $n_4 - 1$, then the resulting subgame with multiset representation on $\{1^{n_1}, 2^{n_2}, 3^{n_3}, 4^{n_4-1}\}$ is a hierarchical game. We start by giving a formal definition.

Definition 7.5.1. (**NHNT**) Let $C = (P_C, W_C)$ be a simple game on $P_C = \{1^{n_1}, 2^{n_2}, 3^{n_3}, 4^{n_4}\}$. Also let $\mathbf{k} = (k_1, k_2, k_3)$, such that $k_1 < k_2 \leq k_3$ be three integers. Then C is a simple game of type **NHNT** if it satisfies the following: A submultiset $X = \{1^{l_1}, 2^{l_2}, 3^{l_3}, 4^{l_4}\} \subseteq P_C$ is winning in C if and only if

$$(l_1 \geq k_1) \textit{ or } (l_1 + l_2 \geq k_2) \textit{ or } (l_1 + l_2 + l_3 + l_4 \geq k_3 \text{ and } l_1 + l_2 + l_3 \geq k_3 - n_4 + 1).$$

In other words, a coalition X is a minimal winning coalition in the **NHNT** game if it contains either k_1 players from the first level L_1, or it contains k_2 players from L_1 and L_2, or it contains k_3 players from any of the levels $L_1, \dots, L_4$ of which there are at least $k_3 - n_4 + 1$ players from $L_1, \dots, L_3$.

Remark 7.5.2. We will assume that a game of type **NHNT** is canonically represented, meaning it has four distinct nonequivalent desirability levels satisfying the conditions: (i) $n_1 \geq k_1$, (ii) $n_1 + n_2 \geq k_2$, (iii) $n_1 + n_2 + n_3 + n_4 \geq k_3$ such that $n_1 + n_2 + n_3 \geq k_3 - n_4 + 1$, and (iv) in any shift-minimal winning coalition $\{1^{l_1}, 2^{l_2}, 3^{l_3}, 4^{l_4}\}$, we have $l_4 < n_4$. It is not difficult to show that if any of those four conditions fails, then the resulting game will be either a hierarchical or a tripartite game. In fact, in Example 7.5.3, we will show that if in a shift-minimal winning coalition we have $l_4 = n_4$, then the game is a hierarchical one.

Next we show that a game of type **NHNT** is a complete game.

7.5.1 Completeness

Consider the game C which satisfies Definition 7.5.1. We shall refer to $(l_1 \geq k_1)$ as the first winning requirement, $(l_1 + l_2 \geq k_2)$ as the second winning requirement and $(l_1 + l_2 + l_3 + l_4 \geq k_3$ and $l_1 + l_2 + l_3 \geq k_3 - n_4 + 1)$ as the third winning requirement. Consider a winning coalition X meeting the third winning requirement. Since X has at least k_3 players from any of the levels $L_1, \ldots, L_4$, of which $k_3 - n_4 + 1$ players are from $L_1, \ldots, L_3$, then replacing a player from L_i with a player from L_{i-1} will result in a coalition that still meets the third winning requirement, hence winning. Similarly, a winning coalition X meeting the second winning requirement has at least k_2 players from any of the two levels L_1 and L_2, meaning replacing a player from L_2 with a player from L_1 will also result in a winning coalition. These facts, together with the transitivity of Isbell's desirability relation (see Taylor & Zwicker (1999), p.89-90), show that $1 \succeq_C 2 \succeq_C 3 \succeq_C 4$, hence C is a complete simple game.

We don't need to discuss the methodology and proofs by which the **NHNT** simple game was discovered. But we just like to mention that condition (iv) of Remark 7.5.2 is a way of making sure that the **NHNT** game is not a hierarchical game. And since we have four nontrivial levels in the **NHNT** game (assuming the game is canonically represented as discussed earlier), then it is not a tripartite game either. Now, in order to add the **NHNT** game to the list of indecomposable ideal complete roughly weighted nonweighted simple games, we provide an example for one of the smallest cases possible of this new type. This of course is not enough to show that all games of this type are ideal and indecomposable, but it shows that there is at least one ideal and indecomposable game of this new type.

7.5.2 Deriving The Example

The example is derived as follows. We aim for the **NHNT** game C to have the property that its subgame on $\{1^{n_1}, 2^{n_2}, 3^{n_3}, 4^{n_4-1}\}$ is a 3-level hierarchical game (not necessarily the smallest 3-level HSG). So the strategy is as follows. By Theorem 7.3.1, we know that the smallest 3-level HSG is one with $\mathbf{k} = (2,3,4), \mathbf{n} = (2,2,2)$, so we try to build our **NHNT** game upon that. After a number of trials and errors, we find that the next game, though is not the smallest nonweighted roughly weighted and ideal **NHNT** game, is the smallest indecomposable nonweighted and roughly weighted one. We will also show that it has $H_\exists(\mathbf{n}, \mathbf{k})$, $\mathbf{n} = (2,2,4), \mathbf{k} = (2,3,4)$ as its subgame.

Example 7.5.3. Consider the **NHNT** simple game $C = (P_C, W_C), P_C = \{1^2, 2^2, 3, 4^4\}$ and $\mathbf{k} = (2,3,4)$. Also, assign the rough weights as follows: $rw(1) = \frac{1}{2}, rw(2) = rw(3) = rw(4) = \frac{1}{4}$, and the quota $q = 1$. The set of shift-minimal winning coalitions is $\{\{1^2\}, \{1, 2^2\}, \{3, 4^3\}\}$. This game is roughly weighted but not weighted due to the following certificate of nonweightedness:

$$(\{1^2\}, \{2^2, 3, 4\}; \{1, 2, 3\}, \{1, 2, 4\}).$$

Now, let $A = \{4\}$, and consider the subgame C_A on $A^c = \{1^2, 2^2, 3, 4^3\}$. The shift-minimal winning coalitions of C_A are still $\{1^2\}, \{1, 2^2\}$ and $\{3, 4^3\}$. Observe now that $\{3, 4^3\}$ uses all players from level 4 in A^c. So no replacement of a player from level 3 with a player from level 4 is possible, meaning $3 \preceq_C 4$. But $3 \succeq_C 4$ by completeness of C, therefore levels 3 and 4 become equivalent to each other. So the winning requirements of C_A for $X \subseteq A^c$ are now $(|X \cap L_1| \geq k_1)$ *or* $(|X \cap (L_1 \cup L_2)| \geq k_2)$ *or* $(|X| \geq k_3)$, meaning it is in fact a disjunctive hierarchical game $H_\exists(\mathbf{n}, \mathbf{k})$, $\mathbf{n} = (2,2,4), \mathbf{k} = (2,3,4)$.

Now it remains to show that C is indecomposable and ideal.

7.5.3 The Indecomposability

Let us continue our discussion of the game C from Example 7.5.3, here we will prove its indecomposability.

Proof. To prove the indecomposability of C, suppose towards a contradiction that we have $G = (P_G, W_G)$ and $H = (P_H, W_H)$, such that $C = G \circ_g H$ on $P_C = P_G \setminus \{g\} \cup P_H$. Consider the shift-minimal winning coalition from W_C of type $\{3, 4^3\}$. If $\{3, 4^3\} \in W_H$, then it is clear that H is neither a unanimity nor an anti-unanimity game, and it also follows that player g is a passer in G. But since C is a complete game, then it follows from Lemma 5.2.1 that g is the least desirable level of G, meaning all players of G are passers by Lemma 5.2.2, contradicting the fact that C has no passers.

Then $\{3, 4^3\}$ must contain a player from G, which must be 3, hence $\{4^3\} \in W_H$. Again it is clear that H is neither a unanimity (since level 4 in P_C has four players) nor an anti-unanimity game, so it follows from Lemma 5.2.1 that g is the least desirable level of G, and from Lemma 5.2.2 it follows that all levels $1, 2$ and 3 are in $P_G \setminus \{g\}$. But now the coalition $\{1, 2, 3, 4\}$ cannot be a minimal winning in the composition, since $\{4^3\}$ is a minimal winning coalition in H, meaning $\{4\}$ is losing in H, contradiction. Therefore C is indecomposable. $\square$

7.5.4 The Ideality

The only property left to prove for the game in Example 7.5.3 is ideality. To this end, we will use the result from (Farràs & Padró, 2010) that we have already presented and used in Chapter 4, which we restate below for the readers convenience.

Suppose we have a complete simple game of n desirability levels. For a shift-minimal winning coalition $X_j = \{1^{a_1}, 2^{a_2}, \ldots, n^{a_n}\}, a_i \leq n_i$ for all $i \in [n]$, let $\text{supp}(X_j) = \{i \in [n] : a_i \neq 0\}$. Also, let $m_j = \max(\text{supp}(X_j))$. Finally, let X_j^i be the number of players a_i in the shift-minimal winning coalition X_j.

Theorem 7.5.4 (Farràs & Padró (2010), Theorem 16)**.** *Let $\Gamma = (P, W)$ be a complete simple game, where $P = \{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}$. Also, let the set of shift-minimal winning coalitions be $\{X_1, \ldots, X_r\}$. Consider $m_j = max(supp(X_j)), 1 \leq j \leq r$, and suppose that the shift-minimal winning coalitions are listed in such a way that $m_j \leq m_{j+1}$. Then Γ is ideal if and only if* *(i)* $m_j < m_{j+1}$ *and* $|X_j| < |X_{j+1}|$ *for all* $j = 1, \ldots, r-1$*, and*

(ii) $X_j^i \geq X_{j+1}^i$ *if* $1 \leq j \leq r-1$ *and* $1 \leq i \leq m_j$*, and*

(iii) if $X_j^i > X_r^i$ *for some* $1 \leq j < r$ *and* $1 \leq i < m_j$*, then* $n_k = X_j^k$ *for all* $k = i+1, \ldots, m_j$.

Now we can apply the above theorem to show that our **NHNT** game C is ideal.

Proof. Let the shift-minimal winning coalitions be $X_1 = \{1^2\}, X_2 = \{1, 2^2\}$, and $X_3 = \{3, 4^3\}$. Then supp$(X_1) = \{1\}$, meaning $m_1 = 1$. Also, supp$(X_2) = \{1, 2\}, m_2 = 2$ and supp$(X_3) = \{3, 4\}$, so $m_3 = 4$. It follows that $X_1, \ldots, X_3$ are ordered such that $m_j \leq m_{j+1}$. We now check X_1, X_2 and X_3 against the three conditions of Theorem 4.2.2.

(i) This condition applies only to $j = 1, 2$. But $m_1 < m_2$ and $|X_1| < |X_2|$. Also, $m_2 < m_3$ and $|X_2| < |X_3|$. So this condition holds.

(ii) This condition applies first to $j = 1 = i$, and then to $j = 2$ and $i = 1$, and finally to $j = 2$ and $i = 2$. Now $X_1^1 = 2,$ and $X_2^1 = 1$, so $X_1^1 > X_2^1$. Also, $X_2^1 = 1,$ and $X_3^1 = 0$, so $X_2^1 > X_3^1$. Finally, $X_2^2 = 2,$ and $X_3^2 = 0$, so $X_2^2 > X_3^2$. So the second condition also holds.

(iii) Here the two conditions $1 \leq j < r$ and $1 \leq i < m_j$ are both met only when $j = 2$ and $i = 1$ (if $j = 1$ then $m_j = 1$, implying that $i = 0$, contradicting $1 \leq i$). But for $j = 2$ and $i = 1$ we have $X_2^1 = 1, X_3^1 = 0$, meaning $X_2^1 > X_3^1$, and it is true that for $k = 2$ we have $n_2 = 2 = X_2^2$ as required.

Therefore C is ideal. $\square$

This concludes the thesis.

Chapter 8

Conclusion and future research

The problem of characterising all access structures (simple games) that carry an ideal secret sharing scheme is a difficult one, and in this thesis we have considered the approach of characterising all ideal simple games within particular known classes. Moreover, our two main contributions in this direction were characterising all ideal games in the class of weighted simple games, and answering some important questions regarding the characterisation of ideal games in the larger class of roughly weighted simple games. And it is the latter class which requires further study, and a number of open problems have been encountered and formulated in the course of this thesis that can be used as catalysts for future research. They are stated below in a chronological order, see Problems 8.1.1 - 8.1.7.

Also, with regards to future research beyond the characterisation of ideal roughly weighted games, consider the following. We have seen in Chapters 3 and 7 that weighted hierarchical games can have only up to two nontrivial levels, and roughly weighted hierarchical games only up to three nontrivial levels, respectively. So, in general, hierarchical games are rather far from weighted ones. Characterising weighted hierarchical games was the entry point to the problem of characterising ideal weighted games in Beimel, Tassa, & Weinreb (2008), and characterising roughly weighted hierarchical games was also our entry point when considering the problem of characterising ideal roughly weighted games in Chapter 7. Similarly, it can be a starting point for the study of ideal games in even larger classes. Gvozdeva, Hemaspaandra, & Slinko (2013) introduced three hierarchies of simple games, each depends on a single parameter and for each hierarchy the union of all classes is the whole class of simple games. One idea that was suggested is to generalize roughly weighted games as follows. Rough weightedness allows just one value of the threshold $q = 1$ (after normalization), where coalitions of weight 1 can be both losing and winning. Instead of just a single threshold value, we may allow values of thresholds from a certain interval $[1; a]$ to possess this property, that is, coalitions whose weight is between 1 and a can be both winning or losing. They denote this class of games $\mathcal{C}_a$. The question which deserves further study is how big should a be so that all hierarchical n-level games are in $\mathcal{C}_a$.

8.1 Open problems

In Chapter 6 we have answered the question: Under what conditions is the composition of two ideal WSGs also an ideal WSG? This can now be generalised to RWSGs.

Problem 8.1.1. Under what conditions is the composition of two ideal RWSGs also an ideal RWSG?

The problem of characterising ideal complete RWSGs introduced in Chapter 7 can be broken into few parts:

Problem 8.1.2. Characterise all ideal roughly weighted TSGs.

We believe the following is true.

Conjecture 8.1.3. *All* ***NHNT*** *simple games are ideal.*

Problem 8.1.4. Characterise the ideal roughly weighted **NHNT** simple games.

Problem 8.1.5. Are there more ideal complete RWSGs?

Which then leaves us with the study of roughly weighted games that are incomplete:

Problem 8.1.6. Characterise all ideal incomplete RWSGs.

And beyond all the problems posited above, we have the following.

Problem 8.1.7. Characterise ideal complete and incomplete nonroughly weighted simple games.

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