Homogeneity and Monotonicity of Distance-Rationalizable Voting Rules

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Abstract

Distance rationalizability is a framework for classifying existing voting rules by interpreting them in terms of distances and consensus classes. It also allows to design new voting rules with desired properties. A particularly natural and versatile class of distances that can be used for this purpose is that of *votewise* distances [EFS10b], which "lift" distances over individual votes to distances over entire elections using a suitable norm. In this paper, we continue the investigation of the properties of votewise distance-rationalizable rules that was initiated by Elkind, Faliszewski, and Slinko [EFS10b]. We describe a number of general conditions on distances and consensus classes that ensure that the resulting voting rule is homogeneous or monotone. This complements the results of [EFS10b], where the authors focus on anonymity, neutrality and consistency. We also provide a detailed study of the class of rules that are votewise distance-rationalizable with respect to the simple majority consensus, which received little attention so far.

1 Introduction

In collaborative environments, agents often need to make joint decisions based on their preferences over possible outcomes. Thus, social choice theory emerges as an important tool in the design and analysis of multiagent systems [ER97]. However, voting procedures that have been developed for human societies are not necessarily optimal for artificial agents and vice versa. For instance, there are voting rules (such as, e.g., single transferable vote) that allow for polynomial-time winner determination, yet have been deemed too complicated to be comprehended by an average voter in many countries; in contrast, for an autonomous agent the perceived complexity of a rule is not an issue. Further, unlike an electoral committee in a human society, the designer of a multi-agent voting system is usually unencumbered by legacy issues or the need to appeal to uneducated public, and can choose a voting rule that is most suitable for the application at hand, or, indeed, design a brand-new voting rule that satisfies the axioms that he deems important.

A recently proposed *distance rationalizability* framework [MN08, EFS09, EFS10b, EFS10a] is ideally suited for such settings. Under this framework, one can define a voting rule by a class of consensus elections and a distance over elections; the winners of an election are defined as the winners in the nearest consensus election. In other words, for any election this rule seeks the most similar election with an obvious winner (where the similarity is measured by the given distance), and outputs its winner. Examples of natural consensus classes include *strong unanimity consensus*, where all voters agree on the ranking of all candidates, and *Condorcet consensus*, where there is a candidate that is preferred by a majority of voters to every other candidate. Combined with the *swap distance* (defined as the number of swaps of adjacent candidates that transforms one election into the other), these consensus classes produce, respectively, the Kemeny rule and the Dodgson rule.

The examples above illustrate that the distance rationalizability framework can be used to interpret (rationalize) existing voting rules in terms of a search for consensus (see [MN08] for a comprehensive list of results in this vein). It can also be applied to design new voting rules: for instance, in [EFS09] the authors investigate the rule obtained by combining the Condorcet consensus with the Hamming distance. Further, by decomposing a voting rule into a consensus class and a distance we can hope to gain further insights into the structure of the rule. This is especially true for the so-called *votewise* distances introduced in [EFS10b]. These are distances over elections that are obtained by aggregating distances between individual votes using a suitable norm, such as ℓ_1 or ℓ_∞. Indeed, paper [EFS10b] shows that one can derive conclusions about anonymity, neutrality and consistency of votewise rules (i.e., rules rationalized via votewise distances) from the basic properties of the underlying distances on votes, norms, and consensus classes.

In this paper we pick up this thread of research and study two important properties of voting rules not considered in [EFS10b], namely, monotonicity and homogeneity. Briefly put, monotonicity ensures that a voting rule is in some way responsive to voters' preferences (specifically, providing more support to a winning candidate cannot turn him into a loser) and homogeneity ensures that the result of an election depends on the proportions of particular votes and not on their absolute counts. Both properties are considered highly desirable for reasonable voting rules (although, for example, single transferable vote and plurality run-off used in political elections in, respectively, Australia, and France, are known not to be monotone). We focus on the four standard consensus classes considered in the previous work (strong unanimity $\mathcal{S}$, unanimity $\mathcal{U}$, majority $\mathcal{M}$ and Condorcet $\mathcal{C}$) and ℓ_1- and ℓ_∞-norms, Our aim is to identify distances on votes that, combined with these norms and consensus classes, produce homogeneous and/or monotone rules.

Of the four consensus classes considered in this paper, the majority consensus $\mathcal{M}$ received relatively little attention in the existing literature. Thus, in order to study the homogeneity and monotonicity of the rules that are distance-rationalizable with respect to $\mathcal{M}$, we need to develop a better understanding of such rules. Our main result here is a characterizaton of all voting rules that are rationalizable with respect to $\mathcal{M}$ via a neutral distance on votes and the ℓ_1-norm. It turns out that such rules have a very natural interpretation: they are "majority variants" of ordinary scoring rules. This characterization enables us to analyze the homogeneity of the rules in this class, leading to a dichotomy result.

As argued above, a votewise distance-rationalizable rule can be characterized by three parameters: a distance on votes, a norm, and a consensus class. From this perspective, it is interesting to ask how much the voting rule changes if we vary one or two of these parameters. We provide two results that contribute to this agenda. First, we show that essentially any rule that is votewise-rationalizable with respect to $\mathcal{M}$ can also be rationalized with respect to $\mathcal{U}$, by modifying the norm accordingly. This enables us to answer a question left open in [EFS10a]. Second, we show that, for any consensus class and any distance on votes, replacing the ℓ_1-norm with the ℓ_∞-norm produces a voting rule that is an n-approximation of the original rule, where n is the number of voters. For the Dodgson rule, this transformation produces a rule that is polynomial-time computable and homogeneous. This line of work also emphasises the constructive aspect of the distance rationalizability framework: we are able to derive new voting rules with attractive properties by combining a known consensus class with a known distance measure in a novel way.

Related work. The formal theory of distance rationalizability was initiated by Meskanen and Nurmi [MN08], though the idea, in one shape or another, appeared in earlier papers as well (see, e.g., [Nit81, Bai87, Kla05b, Kla05a]). The goal of Meskanen and Nurmi was to seek best possible distance-rationalizations of classical voting rules. This research program was advanced by Elkind, Faliszewski, and Slinko [EFS09, EFS10b, EFS10a], who, in addition to further classification work, also suggested studying general properties of distance-rationalizable voting rules. In particular, in [EFS10a] they identified an interesting and versatile class of distances—which they called votewise distances—that lead to rules whose properties can be meaningfully studied.

The study of distance rationalizability is naturally related to the study of another—much older—framework of interpreting voting rules as maximum likelihood estimators, which could be dated back to Condorcet and which has been pursued, among others, by Young [You77], Conitzer and Sandholm [CS05], Conitzer, Rognlie, and Xia [CRX09], and—in the context of combinatrial domains—by Xia, Conitzer and Lang [XCL10]. To date, most of the research on the MLE framework regards classifying existing voting rules as maximum likelihood estimators; however, paper [XCL10] also shows that the MLE approach can be used to deduce new useful voting rules.

This paper is loosely related to the work of Caragiannis et al. [CKKP10], where the authors give a monotone, homogeneous voting rule that calculates scores which approximate candidates' Dodgson scores up to an $O(m \log m)$ multiplicative factor, where m is the number of candidates. The relation to our work is twofold. First, we also focus on monotonicity and homogeneity, although our goal is to come up with a general method of constructing monotone and homogeneous rules and not to approximate particular rules. Second, in the course of our study we discover a homogeneous and poynomial-time computable voting rule that approximates the scores of candidates in Dodgson elections up to a multiplicative factor of n, where n is the number of voters. While the number of voters is usually much bigger than the number of candidates, and thus our algorithm is usually inferior to that of [CKKP10], it illustrates the power of the distance rationalizability framework.

Organization of the paper. The paper is organized as follows. Section 2 contains preliminary definitions regarding voting rules in general and the distance-rationalizability framework specifically. In Section 3 we provide a detailed study of votewise rules with respect to the majority consensus. Section 4 presents our results on homogeneity of votewise rules. In Section 5 we briefly depart from our path of studying homogeneity and monotonicity and show that ℓ_∞-votewise rules form weak approximations of ℓ_1-votewise rules. In particular, we provide an interesting polynomial-time approximation algorithm for Dodgson's rule. Finally, in Section 6 we present our results on monotonicity of votewise rules and argue that our monotonicity conditions supplement distance-rationalizability framework in general. We conclude in Section 7.

2 Preliminaries

2.1. Basic notation. An *election* is a pair $E = (C, V)$, where $C = \{c_1, \ldots, c_m\}$ is the set of *candidates* and $V = (v_1, \ldots, v_n)$ is the set of *voters*. Voter v_i is identified with a total order $\succ_i$ over C, which we will refer to as v_i's *preference order*, or *ranking*. We write $c_j \succ_i c_\ell$ to denote that voter v_i prefers c_j to c_ℓ. We denote by $\mathcal{P}(C)$ the set of all preference orders over C. For a voter v, we denote by $\mathrm{top}(v)$ the candidate ranked first by v, and set $\mathcal{P}(C, c) = \{v \in \mathcal{P}(C) \mid \mathrm{top}(v) = c\}$. For any voter $v_i \in V$ and a candidate $c \in C$, we denote by $\mathrm{rank}(v_i, c)$ the position of c in v_i's ranking. For example, if $\mathrm{top}(v_i) = c$ then $\mathrm{rank}(v_i, c) = 1$. A *voting rule* is a mapping $\mathcal{R}$ that for any election (C, V) outputs a non-empty subset of candidates $W \subseteq C$ called the *election winners*. Given an election $E = (C, V)$ and $s \in \mathbb{N}$, we denote by sE the election (C, sV), where sV is obtained by concatenating s copies of V. Assuming anonymity, we may view sV as a society where every voter of V is cloned s times.

Two important properties of voting rules that will be studied in this paper are homogeneity and monotonicity.

Homogeneity. A voting rule $\mathcal{R}$ is *homogeneous* if for every election $E = (C, V)$ and every positive integer s it holds that $\mathcal{R}(E) = \mathcal{R}(sE)$.

Monotonicity. A voting rule $\mathcal{R}$ is *monotone* if for every election $E = (C, V)$, every $c \in \mathcal{R}(E)$ and every $E' = (C, V')$ obtained from E by moving c up in some voters' rankings (but not changing their rankings in any other way) we have $c \in \mathcal{R}(E')$.

2.2. Voting rules. We will now define the classic voting rules discussed in this paper, namely, scoring rules, (simplified) Bucklin, and Dodgson.

Scoring rules Throughout this paper, we will use a somewhat nonstandard definition of a scoring rule. Any vector $\alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{R}_+ \cup \{0\})^n$ can be used as a scoring vector and it defines a partial voting rule $\mathcal{R}_\alpha$ that can be used in elections for a fixed number m of candidates. For each preference order u on the set of m candidates a candidate c gets $\alpha_{\mathrm{rank}(u,c)}$ points (as is standard) and these values are summed up together to obtain the score of c. However, we define the winners to be the candidates with the lowest score (rather than with the highest; as is typical when discussing scoring rules). A sequence of score vectors $(\alpha^{(m)})_{m\in\mathbb{N}}$ defines a voting rule $\mathcal{R}_{(\alpha^{(m)})}$ which is applicable for any number of alternatives.

For example, in this notation the Borda rule is defined by a family of scoring vectors $\alpha^{(m)} = (0, 1, \dots, m-1)$ and the k-approval is the family of scoring vectors given by $\alpha_i^{(m)} = 0$ for $i \leq k$, $\alpha_i^{(m)} = 1$ for $i > k$. The 1-approval rule is also known as Plurality. The traditional model, where the winners are the candidates with the highest score, it can be converted to our notation by setting $\alpha_i' = \alpha_{\max} - \alpha_i$, where $\alpha_{\max} = \max_{i=1}^m \alpha_i$. The reason for this deviation is that in the context of this paper it will be much more convenient to speak of minimizing one's score. Note that, in general, we do not require $\alpha_1 \leq \dots \leq \alpha_m$, although this assumption is obviously required for monotonicity.

Note also that scoring vectors $(\alpha_1, \dots, \alpha_m)$ and $(\beta\alpha_1, \dots, \beta\alpha_m)$ define same voting rules for any $\beta > 0$; the same is true for $(\alpha_1, \dots, \alpha_m)$ and $(\alpha_1 + \gamma, \dots, \alpha_m + \gamma)$ for any $\gamma \geq 0$. Thus, in what follows, we normalize all scoring rules by requiring that the smallest entry in the scoring vector is 0, and the smallest non-zero entry is 1.

Bucklin The *Bucklin rule*[1] $\mathcal{R}_B$ can be thought of as an adaptive version of k-approval. Under this rule, we first determine the smallest value of k such that some candidate is ranked in top k positions by more than half of the voters. The winner(s) are the candidates that are ranked in the top k positions the maximum number of times. Under the *simplified Bucklin rule* $\mathcal{R}_{sB}$, the winners are all candidates ranked in top k positions by a majority of voters. For any election E we have $\mathcal{R}_B(E) \subseteq \mathcal{R}_{sB}(E)$.

Dodgson To define the Dodgson rule, we need to introduce the concept of a *Condorcet winner*. A Condorcet winner is a candidate that is preferred to any other candidate by a majority of voters. The *Dodgson score* of a candidate c is the smallest number of swaps of adjacent candidates that have to be performed on the votes to make c the Condorcet winner. The winner(s) under the Dodgson rule are the candidates with the lowest Dodgson score.

2.3. Norms and Metrics. A *norm* on $\mathbb{R}^n$ is a mapping $N : \mathbb{R}^n \to \mathbb{R}$ that has the following properties for all $x, y \in \mathbb{R}^n$:

(1) $N(\alpha x) = |\alpha| N(x)$ for all $\alpha \in \mathbb{R}$;

(2) $N(x) \geq 0$ and $N(x) = 0$ if and only if $x = (0, \dots, 0)$;

(3) $N(x + y) \leq N(x) + N(y)$.

Two important properties of norms that will be of interest to us are symmetry and monotonicity. We say that a norm N is *symmetric* if for each permutation $\sigma : [1, n] \to [1, n]$ it holds that $N(x_1, \dots, x_n) = N(x_{\sigma(1)}, \dots, x_{\sigma(n)})$. For monotonicity, we make use of the definition proposed in [BSW61]. Specifically, we say that a norm N is *monotone in the*

[1] known also as majoritarian compromise.

positive orthant, or $\mathbb{R}_+^n$*-monotone*, if for any two vectors $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{R}_+^n$ such that $x_i \le y_i$ for all $i = 1, \dots, n$ we have $N(x_1, \dots, x_n) \le N(y_1, \dots, y_n)$.

A well-studied class of norms are the ℓ_p*-norms* given by

$$\ell_p(x_1, \dots, x_n) = (|x_1|^p + \dots + |x_n|^p)^{\frac{1}{p}}$$

for $p \in \mathbb{N}$. This definition extends to $p = +\infty$ by setting $\ell_\infty(x_1, \dots, x_n) = \max\{x_1, \dots, x_n\}$. Observe that for any $p \in \mathbb{N} \cup \{+\infty\}$ the ℓ_p norm is, in fact, a family of norms, i.e., it is well-defined on $\mathbb{R}^i$ for any $i \in \mathbb{N}$. Also, any such norm is clearly symmetric and monotone in the positive orthant.

A *metric*, or *distance*, on a set X is a mapping $d : X^2 \to \mathbb{R}$ that satisfies the following conditions for all $x, y, z \in X$:

(1) $d(x, y) \ge 0$;

(2) $d(x, y) = 0$ if and only if $x = y$;

(3) $d(x, y) = d(y, x)$;

(4) $d(x, z) \le d(x, y) + d(y, z)$.

A function that satisfies conditions (1), (3) and (4), but not (2), is called a *pseudodistance*.

Given a distance d on X and a norm N on $\mathbb{R}^n$, we can define a distance $N \circ d$ on X^n by setting $(N \circ d)((x_1, \dots, x_n), (y_1, \dots, y_n)) = N(d(x_1, y_1), \dots, d(x_n, y_n))$ for all vectors $(x_1, \dots, x_n), (y_1, \dots, y_n) \in X^n$. A distance defined in this manner is called a *product metric*.

In this paper, we will study distances over votes and their extensions to distances over elections via product metrics (see below). Some examples of distances over votes are given by the *discrete distance* d_{discr}, the *swap distance* d_{swap}, and the *Sertel distance* d_{ser}, defined as follows. For any set of candidates C and any $u, v \in \mathcal{P}(C)$, we set $d_{\mathrm{discr}}(u, v) = 0$ if $u = v$ and $d_{\mathrm{discr}}(u, v) = 1$ otherwise. The swap distance d_{swap} is given by $d_{\mathrm{swap}}(u, v) = \frac{1}{2}|\{(c, c') \in C^2 \mid c \succ_u c', c' \succ_v c\}|$, where $\succ_u$ and $\succ_v$ are the preference orders associated with u and v, respectively. The Sertel distance between u and v is defined as the smallest value of i such that for all $j > i$ voters u and v rank the same candidate in position j.

A distance d on $\mathcal{P}(C)$ is called *neutral* if for any $u, v \in \mathcal{P}(C)$ and any permutation $\pi : C \to C$ we have $d(u, v) = d(\pi(u), \pi(v))$, where $\pi(x)$ denotes the vote obtained from x by moving candidate c_i into position $\mathrm{rank}(x, \pi(c_i))$, for $i = 1, \dots, |C|$. Clearly, all distances listed above are neutral.

2.4. Distance Rationalizability. Intuitively, a consensus class is a collection of elections with an obvious winner. Formally, a *consensus class* is a pair $(\mathcal{E}, \mathcal{W})$ where $\mathcal{E}$ is a set of elections and $\mathcal{W} \colon \mathcal{E} \to C$ is a function. This function, for each election $E \in \mathcal{E}$, outputs the alternative called the *consensus winner*. The following four consensus classes have been considered in the previous work on distance rationalizability:

Strong unanimity. Denoted $\mathcal{S}$, contains elections $E = (C, V)$ where all voters report the same preference order. The consensus winner is the candidate ranked first by all voters.

Unanimity. Denoted $\mathcal{U}$, contains all elections $E = (C, V)$ where all voters rank the same candidate first. The consensus winner is the candidate ranked first by all voters.

Majority. Denoted $\mathcal{M}$, contains all elections $E = (C, V)$ where more than half of the voters rank the same candidate first. The consensus winner is the candidate ranked first by the majority of voters.

Condorcet. Denoted $\mathcal{C}$, contains all elections $E = (C, V)$ with a Condorcet winner. The consensus winner is the Condorcet winner.

We say that a voting rule $\mathcal{R}$ is *compatible* with a consensus class $\mathcal{K}$ if for any consensus election $E \in \mathcal{K}$ it holds that $\mathcal{W}(E) = \mathcal{R}(E)$. Similarly, $\mathcal{R}$ is said to be *weakly compatible* with $\mathcal{K}$ if for any $E \in \mathcal{K}$ we have $\mathcal{W}(E) \in \mathcal{R}(E)$. Essentially all well-known voting rules are weakly compatible with $\mathcal{S}$, $\mathcal{U}$ and $\mathcal{M}$, but there are rules that are not compatible with any of these consensus classes (e.g., k-approval for $k > 1$). The rules that are compatible with $\mathcal{C}$ are also known as *Condorcet-consistent* rules; we use the term "compatibility" rather than "consistency" to avoid confusion with the consistency property of voting rules.

We are now ready to define the concept of distance rationalizability. Our definition below is taken from [EFS10b], which itself was inspired by [MN08, EFS09].

Definition 2.1. *Let d be a distance over elections and let $\mathcal{K} = (\mathcal{E}, \mathcal{W})$ be a consensus class. The $(\mathcal{K}, d)$-*score *of a candidate c in an election E is the distance (according to d) between E and a closest election $E' \in \mathcal{E}$ such that $c \in \mathcal{W}(E')$. A voting rule $\mathcal{R}$ is* distance-rationalizable *via a consensus class $\mathcal{K}$ and a distance d over elections (is $(\mathcal{K}, d)$-*rationalizable*) if for each election E the set $\mathcal{R}(E)$ consists of all candidates with the smallest $(\mathcal{K}, d)$-score.*

A particularly useful class of distances to be used in distance rationalizability constructions is that of *votewise* distances, which are obtained by combining a distance over votes with a suitable norm. Formally, given a set of candidates C, consider a distance d over $\mathcal{P}(C)$ and a family of norms $\mathcal{N} = (N_i)_{i=1}^{\infty}$, where N_i is a norm over $\mathbb{R}^i$. We define a distance $\widehat{d^{\mathcal{N}}}$ over elections with the set of candidates C as follows: for any $E = (C, V)$, $E' = (C, V')$, we set $\widehat{d^{\mathcal{N}}}(E, E') = (N_i \circ d)(V, V')$ if $|V| = |V'| = i$, and $\widehat{d^{\mathcal{N}}}(E, E') = +\infty$ if $|V| \neq |V'|$. A voting rule $\mathcal{R}$ is said to be $\mathcal{N}$*-votewise distance-rationalizable* (or simply $\mathcal{N}$*-votewise*) with respect to a consensus class $\mathcal{K}$ if there exists a distance d over votes such that $\mathcal{R}$ is $(\mathcal{K}, \widehat{d^{\mathcal{N}}})$-rationalizable. When $\mathcal{N}$ is the ℓ_p-norm for some $p \in \mathbb{N} \cup \{+\infty\}$, we write $\widehat{d^p}$ instead of $\widehat{d^{\ell_p}}$, and when $\mathcal{N} = \ell_1$, we omit the index altogether and write $\widehat{d}$. It is known that any voting rule is distance-rationalizable with respect to any consensus class that is compatible with it [EFS10b]. However, there exist voting rules that are not $\mathcal{N}$-votewise distance-rationalizable with respect to standard consensus classes for any reasonable norm $\mathcal{N}$ [EFS10a].

Let us now consider some examples of distance-rationalizations of voting rules. Nitzan [Nit81] was the first to show that Plurality is $(\mathcal{U}, \widehat{d}_{\text{discr}})$-rationalizable and Borda is $(\mathcal{U}, \widehat{d}_{\text{swap}})$-rationalizable. It is easy to see that Dodgson is $(\mathcal{C}, \widehat{d}_{\text{swap}})$-rationalizable and Kemeny is

$(\mathcal{S}, \widehat{d}_{\mathrm{swap}})$-rationalizable. The distance $\widehat{d^{\infty}_{\mathrm{ser}}}$, combined with the majority consensus, yields the Simplified Bucklin rule [EFS10b].

Let $C = \{c_1, \dots, c_m\}$ be a set of candidates and let u and v be two votes over C. For each scoring vector $\alpha = (\alpha_1, \dots, \alpha_m)$, paper [EFS10b] defines a (pseudo)distance $d_\alpha(u, v)$ as $d_\alpha(u, v) = \sum_{j=1}^{m} |\alpha_{\mathrm{rank}(u,c_j)} - \alpha_{\mathrm{rank}(v,c_j)}|$, and shows that if $\alpha_1 = 0$ then $\mathcal{R}_\alpha$ is $(\mathcal{U}, \widehat{d}_\alpha)$-(pseudo)distance-rationalizable. The following lemma, proved as part of Theorem 3 of [EFS09],[2] will be useful for us later on.

Lemma 2.2 ([EFS09])**.** *Let $C = \{c_1, \dots, c_m\}$ be a set of candidates, $\alpha = (\alpha_1, \dots, \alpha_m)$ be a normalized scoring vector, and c be a candidate. For each vote v over C it holds that* $\min\{d_\alpha(v, u) \mid u \in \mathcal{P}(C, c)\} = 2|\alpha_{\mathrm{rank}(v,c)} - \alpha_1|$.

3 $\mathcal{M}$-Counterparts of Classical Scoring Rules

The majority consensus is a very natural notion of agreement in the society; however, it has received little attention in the literature so far. Here we will show how it leads to a series of interesting rules with nice properties.

Definition 3.1. *For any scoring vector $\alpha = (\alpha_1, \dots, \alpha_m)$, let $\mathcal{M}$-$\mathcal{R}_\alpha$ be a partial voting rule defined on the profiles with m alternatives as follows. Given an election $E = (C, V)$ with $C = \{c_1, \dots, c_m\}$ and $V = (v_1, \dots, v_n)$, for each candidate $c \in C$, we define the $\mathcal{M}$-score of c as the sum of $\lfloor \frac{n}{2} \rfloor + 1$ lowest values among $\alpha_{\mathrm{rank}(v_1,c)}, \dots, \alpha_{\mathrm{rank}(v_n,c)}$. The candidates with the lowest $\mathcal{M}$-$\mathcal{R}_\alpha$ scores are the $\mathcal{M}$-$\mathcal{R}_\alpha$-winners. As in the classical case, for a family of scoring vectors $(\alpha^{(i)})_{i\in\mathbb{N}}$ we can define an $\mathcal{M}$-scoring rule $\mathcal{M}$-$\mathcal{R}_{(\alpha^{(i)})}$.*

We will refer to voting rules from Definition 3.1 as *$\mathcal{M}$-scoring rules*. Such rules (or their slight modifications) are often used for score aggregation in real-life settings; for example, it is not unusual for a professor to grade the students on the basis of their five best assignments out of six or in some sport competitions to award winners on the basis of their several best attempts.

It is not hard to see that $\mathcal{M}$-Plurality is equivalent to Plurality: under both rules, the winners are the candidates with the maximum number of first-place votes. However, essentially all other scoring rules differ from their $\mathcal{M}$-counterparts.

Proposition 3.2. *For any normalized scoring vector $\alpha = (\alpha_1, \dots, \alpha_m)$ the rule $\mathcal{M}$-$\mathcal{R}_\alpha$ is equivalent to $\mathcal{R}_\alpha$ (i.e., outputs exactly the same set of winners on any preference profile) if and only if (i) $\alpha_1 = \cdots = \alpha_m$ or (ii) $\alpha_i = 0$, $\alpha_j = 1$ for some $i \in [1, m]$ and all $j \neq i$.*

Proof. Clearly, if all coordinates of the scoring vector are equal, both $\mathcal{M}$-$\mathcal{R}_\alpha$ and $\mathcal{R}_\alpha$ output the set of all candidates on any preference profile. Further, we have already argued that if $\alpha_1 = 0$, $\alpha_j = 1$ for each $j > 1$, then $\mathcal{M}$-$\mathcal{R}_\alpha$ is the Plurality rule. Clearly, this argument also applies if the only 0 appears in a different position of the scoring vector.

[2] The proof in [EFS09] assumes that—in our notation—$\alpha_1 \leq \cdots \leq \alpha_m$, but it is not hard to see that it goes through as long as we require $\alpha_1 = 0 \leq \alpha_k$ for all $k > 1$.

We will now show that the converse direction is also true. Note that we can assume without loss of generality that $\alpha_1 \le \cdots \le \alpha_m$: it is not hard to see that for any permutation $\sigma : [1, m] \to [1, m]$ it holds that $\mathcal{M}$-$\mathcal{R}_\alpha$ is equivalent to $\mathcal{R}_\alpha$ if and only if $\mathcal{M}$-$\mathcal{R}_{\sigma(\alpha)}$ is equivalent to $\mathcal{R}_{\sigma(\alpha)}$, where $\sigma(\alpha)$ is the scoring vector given by $(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(m)})$. Thus for any scoring rule that satisfies neither (i) nor (ii) we can assume that either (a) $\alpha_1 = \alpha_2 = 0$, $\alpha_m > 0$ or (b) $\alpha_1 = 0$, $\alpha_2 = 1$, and $\alpha_m > 1$. We will argue that in both of these cases $\mathcal{M}$-$\mathcal{R}_\alpha$ is not equivalent to $\mathcal{R}_\alpha$.

Indeed, consider two candidates c and w and an election E with n voters where $\lfloor \frac{n}{2} \rfloor + 1$ voters rank c first and w second, and the remaining voters rank w first and c last.

In case (a), the $\mathcal{M}$-$\mathcal{R}_\alpha$-score of both c and w is 0, so both of them are among the winners under $\mathcal{M}$-$\mathcal{R}_\alpha$. On the other hand, c's $\mathcal{R}_\alpha$-score is at least $\lceil \frac{n}{2} \rceil - 1$, while w's $\mathcal{R}_\alpha$-score is zero, so w is among the winners under $\mathcal{R}_\alpha$ and c is not. Thus, we have $\mathcal{M}$-$\mathcal{R}_\alpha(E) \neq \mathcal{R}_\alpha(E)$.

In case (b), c is the unique winner under $\mathcal{M}$-$\mathcal{R}_\alpha$. On the other hand, under $\mathcal{R}_\alpha$ candidate c gets $\alpha_m(\lceil \frac{n}{2} \rceil - 1)$ points, and candidate w gets $\lfloor \frac{n}{2} \rfloor + 1$ points. Since $\alpha_m > 1$, for large enough values of n (it suffices to pick $n > \frac{\alpha_m + 1}{\alpha_m - 1}$) candidate w has a lower score under $\mathcal{R}_\alpha$, i.e., c cannot be the winner of E. □

Indeed, $\mathcal{M}$-scoring rule $\mathcal{M}$-$\mathcal{R}_\alpha$ for a particular scoring vector $\alpha = (\alpha_1, \dots, \alpha_m)$ may be in some respect better than $\mathcal{R}_\alpha$. For example, it is known that Borda is highly manipulable. $\mathcal{M}$-Borda is also manipulable but the scope of possible manipulations seems to be much more limited. In particular, moving an alternative to the bottom of your ranking may not have an effect on the score of that alternative.

In this section we will show that these rules are also very interesting from the distance rationalizability point of view: it turns out that they essentially coincide with the class of rules that are ℓ_1-votewise rationalizable with respect to $\mathcal{M}$.

We will first need to generalize a result from [EFS10b] to pseudodistances and weak compatibility.

Proposition 3.3. *If a voting rule is pseudodistance-rationalizable with respect to a consensus class $\mathcal{K}$, it is weakly compatible with $\mathcal{K}$.*

Proof. Consider a $\mathcal{K}$-consensus $E = (C, V)$ with winner c and a $(\mathcal{K}, d)$-rationalizable voting rule $\mathcal{R}$, where d is a pseudodistance. We have $d(E, E) = 0$, so $d(E, E) \le d(E, E')$ for any election E'. Therefore, $c \in \mathcal{R}(E)$. □

As a side remark, recall that in [EFS10b] the authors show that if we replace 'pseudodistance rationalizability' and 'weak compatibility' with 'distance rationalizability' and 'compatibility' in the statement of Proposition 3.3, then the converse is also true: any $\mathcal{K}$-compatible rule is distance-rationalizable with respect to $\mathcal{K}$. It is not clear if the same is true in our case, as the proof given in [EFS10b] does not generalize immediately to our setting.

We start by characterizing $\mathcal{M}$-scoring rules that are (pseudo)distance-rationalizable with respect to $\mathcal{M}$.

Proposition 3.4. *Consider a normalized scoring vector $\alpha = (\alpha_1, \dots, \alpha_m)$. The rule $\mathcal{M}$-$\mathcal{R}_\alpha$ is ℓ_1-votewise distance-rationalizable with respect to $\mathcal{M}$ if and only if $\alpha_1 = 0$, $\alpha_j > 0$ for all $j \neq 1$. Further, $\mathcal{M}$-$\mathcal{R}_\alpha$ is ℓ_1-votewise pseudodistance-rationalizable with respect to $\mathcal{M}$ if and only if $\alpha_1 = 0$.*

Proof. Suppose first that $\alpha_1 \neq 0$. Since α is normalized, there exists a $j \neq 1$ such that $\alpha_j = 0$. Consider a preference profile in which some candidate c is ranked first by everyone, and some other candidate w is ranked in the j-th position by everyone. Clearly, c is the majority winner, but under $\mathcal{M}$-$\mathcal{R}_\alpha$ w is a winner, and c is not. Thus, by Proposition 3.3 no such rule can be pseudodistance-rationalizable with respect to $\mathcal{M}$.

Now, suppose that $\alpha_1 = 0$. Consider the pseudodistance d_α, an election $E = (C, V)$, a candidate $c \in C$, and a voter $v \in V$ that ranks c in the j-th position. By Lemma 2.2, $\min\{d_\alpha(v, u) \mid u \in \mathcal{P}(C, c)\} = 2\alpha_j$ This implies that in E for any candidate $c \in C$ his $\mathcal{M}$-$\mathcal{R}_\alpha$-score is twice the distance to the nearest $\mathcal{M}$-consensus with winner c. Hence, the rule $\mathcal{M}$-$\mathcal{R}_\alpha$ is $(\mathcal{M}, \widehat{d_\alpha})$-rationalizable.

Clearly, d_α is not necessarily a distance. Indeed, if we have $\alpha_j = 0 = \alpha_1$ for some $j \neq 1$, the distance between a vote v and the vote obtained from v by swapping the candidates in the first and the j-th position is 0. This argument also shows that in this case $\mathcal{M}$-$\mathcal{R}_\alpha$ is not distance-rationalizable. Indeed, if all voters rank c first and rank w in the j-th position, then both c and w are winners under $\mathcal{M}$-$\mathcal{R}_\alpha$, even though c is the unique majority winner. Now, suppose that $\alpha_j \neq 0$ for all $j \neq 1$. It may still happen that $\alpha_j = \alpha_k$ for some $j, k \in \{2, \dots, m\}$, in which case d_α is still a pseudodistance. However, in this case we can set $\varepsilon = \min\{|\alpha_j - \alpha_k| \mid \alpha_j \neq \alpha_k\}$ and let $d'_\alpha(u, v) = 0$ if $u = v$ and $d'_\alpha(u, v) = \min\{d_\alpha(u, v), \varepsilon\}$ otherwise. It is not hard to see that d'_α is a distance; in particular, we have $d'_\alpha(u, v) \neq 0$ for $u \neq v$ by construction, and the triangle inequality is satisfied by our choice of ε. Further, consider a vote v that ranks c in the j-th position, $j > 1$, and the nearest (with respect to d_α) vote u that ranks c first. We have $d_\alpha(v, u) = 2\alpha_j > 0$, so $d'_\alpha(u, v) = d_\alpha(u, v)$. Therefore, the argument showing that $\mathcal{M}$-$\mathcal{R}_\alpha$ is $(\mathcal{M}, \widehat{d_\alpha})$-rationalizable applies to $\widehat{d'_\alpha}$ as well, and hence $\mathcal{M}$-$\mathcal{R}_\alpha$ is ℓ_1-votewise distance-rationalizable. □

We remark that our characterization also applies to scoring rules and $\mathcal{U}$, thus answering a question left open in [EFS09], where the authors ask whether scoring rules with $\alpha_i = \alpha_j$ for $i, j > 1$ can be distance-rationalized (rather than pseudodistance-rationalized). Further, in [EFS09] the authors consider only monotone scoring rules, i.e., rules that satisfy—in our notation—$\alpha_1 \leq \dots \leq \alpha_m$, while our result holds for all scoring vectors.

The following lemma (proof omitted) explains how to find an $\mathcal{M}$-consensus that is nearest to a given election with respect to a given ℓ_1-votewise distance.

Lemma 3.5. *Let $\mathcal{R}$ be a voting rule that is $(\mathcal{M}, \widehat{d})$-rationalized. Let $E = (C, V)$ be an arbitrary election where $V = (v_1, \dots, v_n)$ and let $E' = (C, U)$ be an $\mathcal{M}$-consensus such that $\widehat{d}(E, E')$ is minimal among all $\mathcal{M}$-consensuses in $\mathcal{P}^n(C)$. Let $c \in C$ be the consensus winner of (C, U). Then, for each $i = 1, \dots, n$, either $u_i \in \arg\min_{x \in \mathcal{P}(C,c)} d(x, v_i)$ or $u_i = v_i$.*

Combining Lemma 3.5 with the argument in the proof of Theorem 4.9 in [EFS10b], we can show that the converse of Proposition 3.4 is also true: any voting rule that can be pseudodistance-rationalized via $\mathcal{M}$ and a neutral ℓ_1-votewise pseudodistance is, in fact, an $\mathcal{M}$-scoring rule. Also, any $\mathcal{M}$-scoring rule is obviously neutral. We can summarize these observations in the following theorem.

Theorem 3.6. *Let $\mathcal{R}$ be a voting rule. There exists a neutral ℓ_1-votewise pseudodistance $\widehat{d}$ such that $\mathcal{R}$ is $(\mathcal{M}, \widehat{d})$-rationalizable if and only if $\mathcal{R}$ can be defined as an $\mathcal{M}$-scoring rule $\mathcal{M}\text{-}\mathcal{R}_{(\alpha^{(i)})}$ such that $\alpha_1^{(i)} \leq \alpha_j^{(i)}$ for all $j > 1$ and all $i \in \mathbb{N}$.*

We stress that the above characterization applies to neutral ℓ_1-votewise rules and not, for example, ℓ_∞-votewise rules, which can be substantially different and, as we will see in the next section, can have quite different properties.

The discussion above suggests that using the majority consensus to rationalize a voting rule is similar to using the unanimity consensus, except that we take the best "half-plus-one" votes into account only. In fact, it turns out that under very weak assumptions we can translate a votewise rationalization of a rule with respect to $\mathcal{M}$ to a votewise rationalization of that rule with respect to $\mathcal{U}$.

Definition 3.7. *Let $\mathcal{N} = (N_i)_{i=1}^{\infty}$ be a family of functions where for each i, $i \geq 1$, N_i is a mapping from $\mathbb{R}^i$ to $\mathbb{R}$. We define a family $\mathcal{N}^{\mathcal{M}} = (N_i^{\mathcal{M}})_{i=1}^{\infty}$ as follows. For each $i \geq 1$, $N_i^{\mathcal{M}}$ is a mapping from $\mathbb{R}^i$ to $\mathbb{R}$ given by*

$$N_i^{\mathcal{M}}(x_1, \ldots, x_i) = N(|x_{\pi(1)}|, \ldots, |x_{\pi(\lfloor \frac{i}{2} \rfloor + 1)}|),$$

where π is a permutation of $[1, i]$ such that $|x_{\pi(1)}| \geq |x_{\pi(2)}| \geq \cdots \geq |x_{\pi(i)}|$.

For a family of symmetric norms $\mathcal{N} = (N_i)_{i=1}^{\infty}$ that are monotone in the positive orthant, the family $\mathcal{N}^{\mathcal{M}}$ is also a family of norms, which we will call the *majority variant* of $\mathcal{N}$.

Proposition 3.8. *Let $\mathcal{N} = (N_i)_{i=1}^{\infty}$ be a family of norms, where each N_i is a norm on $\mathbb{R}^i$ that is symmetric and monotone in the positive orthant. Then the family $\mathcal{N}^{\mathcal{M}} = (N^{\mathcal{M}})_{i=1}^{\infty}$ is also a family of norms that are symmetric and monotone in the positive orthant.*

Proof. Let us fix a positive integer n. We will first show that $N_n^{\mathcal{M}}$ is a norm. It is easy to see that since $N_{\lfloor \frac{n}{2} \rfloor}$ is a norm, for every $(x_1, \ldots, x_n) \in \mathbb{R}^n$ it holds that (a) $N_n^{\mathcal{M}}(x_1, \ldots, x_n) \geq 0$, (b) $N_n^{\mathcal{M}}(x_1, \ldots, x_n) = 0$ if and only if $x_1 = \cdots = x_n = 0$, and (c) for each $\alpha \in \mathbb{R}$ it holds that $N_n^{\mathcal{M}}(\alpha x_1, \ldots, \alpha x_n) = |\alpha| N_n^{\mathcal{M}}(x_1, \ldots, x_n)$.

Let us now show that $N_n^{\mathcal{M}}$ satisfies the triangle inequality. Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be two vectors in $\mathbb{R}^n$. We need to show that $N_n^{\mathcal{M}}(x_1 + y_1, \ldots, x_n + y_n) \leq N_n^{\mathcal{M}}(x_1, \ldots, x_n) + N_n^{\mathcal{M}}(y_1, \ldots, y_n)$. Let π, σ_x and σ_y be permutations of $[1, n]$ such that $|x_{\pi(1)} + y_{\pi(1)}| \geq \cdots \geq |x_{\pi(n)} + y_{\pi(n)}|$, $|x_{\sigma_x(1)}| \geq \ldots \geq |x_{\sigma_x(n)}|$, $|y_{\sigma_y(1)}| \geq \ldots \geq |y_{\sigma_y(n)}|$. Let $h = \lfloor \frac{n}{2} \rfloor + 1$. We have

$$\begin{aligned}
N_n^{\mathcal{M}}(x_1 + y_1, \ldots, x_n + y_n) &= N_h(|x_{\pi(1)} + y_{\pi(1)}|, \ldots, |x_{\pi(h)} + y_{\pi(h)}|) \\
&\leq N_h(|x_{\pi(1)}| + |y_{\pi(1)}|, \ldots, |x_{\pi(h)}| + |y_{\pi(h)}|) \\
&\leq N_h(|x_{\pi(1)}|, \ldots, |x_{\pi(h)}|) + N_h(|y_{\pi(1)}|, \ldots, |y_{\pi(h)}|) \\
&\leq N_h(|x_{\sigma_x(1)}|, \ldots, |x_{\sigma_x(h)}|) + N_h(|y_{\sigma_y(1)}|, \ldots, |y_{\sigma_y(h)}|) \\
&= N_n^{\mathcal{M}}(x_1, \ldots, x_n) + N_n^{\mathcal{M}}(y_1, \ldots, y_n),
\end{aligned}$$

where the second inequality follows by triangle inequality for N_h, and the third one follows by N_h's symmetry and monotonicity in the positive orthant. As a result, $N_n^{\mathcal{M}}$ is a norm.

By construction, $\mathcal{M}$-N_n is both symmetric and monotone in the positive orthant. This completes the proof. $\square$

As an immediate corollary we get the following result.

Corollary 3.9. *Let $\mathcal{N}$ be a family of symmetric norms that are monotone in the positive orthant and let d be a distance over votes. Let $\mathcal{R}$ be a voting rule that is $(\mathcal{M}, \widehat{d^{\mathcal{N}}})$-rationalized. Then $\mathcal{R}$ is $(\mathcal{U}, \widehat{d^{\mathcal{N}^{\mathcal{M}}}})$-rationalized.*

This discussion illustrates that when a rule can be rationalized in several different ways, the right choice of a consensus class plays an important role, as it may greatly simplify the underlying norm and hence the distance. This is why it pays to keep a variety of consensus classes available and search for best distance rationalizations possible. Corollary 3.9 also has a useful application: Paper [EFS10a] shows that STV[3] cannot be rationalized with respect to $\mathcal{U}$ by any neutral $\mathcal{N}$-votewise distance, where $\mathcal{N}$ is a family of symmetric norms monotone in the positive orthant. Corollary 3.9 allows us to extend their result to $\mathcal{M}$.

Theorem 3.10. *For three candidates, STV (together with any intermediate tie-breaking rule) is not distance-rationalizable with respect to the majority consensus and any anonymous neutral $\mathcal{N}$-votewise distance, where $\mathcal{N}$ is monotone in the positive orthant.*

4 Homogeneity

Homogeneity is a very natural property of voting rules. It can be interpreted as a weaker form of another appealing property, namely, consistency. Recall that a voting rule $\mathcal{R}$ is said to be *consistent* if for any two elections $E_1 = (C, V_1)$ and $E_2 = (C, V_2)$ with $\mathcal{R}(E_1) \cap \mathcal{R}(E_2) \neq \emptyset$ it holds that $\mathcal{R}(C, V_1 + V_2) = \mathcal{R}(E_1) \cap \mathcal{R}(E_2)$, where $V_1 + V_2$ denotes the concatenation of V_1 and V_2. Thus, homogeneity effectively imposes the same requirement as consistency, but only for the restricted case $V_1 = V_2$. Now, consistency is known to be hard to achieve: by Young's famous theorem [You75], the only voting rules that are simultaneously anonymous, neutral and consistent are the scoring rules (or their compositions). In contrast, we will now argue that for many consensus classes and many values of $p \in \mathbb{N} \cup \{+\infty\}$, the rules that are ℓ_p-votewise rationalizable with respect to these classes are homogeneous.

We start by showing that this is the case for ℓ_p, $p \in \mathbb{N}$, and consensus classes $\mathcal{S}$ and $\mathcal{U}$. We then provide a complete characterization of all homogeneous rules that are ℓ_1-votewise distance rationalizable with respect to $\mathcal{M}$, assuming that the underlying distance on votes is neutral. Next, we show that combining ℓ_∞ with $\mathcal{S}$, $\mathcal{U}$ or $\mathcal{M}$ results in homogeneous rules, too. However, for $\mathcal{C}$ this is not the case, and we conclude the section by discussing the homogeneity (or lack thereof) of the rules that are votewise rationalizable with respect to $\mathcal{C}$.

[3] We skip the description of STV due to space constraints, but we mention that STV is one of the very few nontrivial election systems that are in practical use in real-life political systems.

Theorem 4.1. *Suppose that a voting rule $\mathcal{R}$ is $(\mathcal{K}, \widehat{d^p})$-rationalizable, where $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}\}$ and $p \in \mathbb{N}$. Then $\mathcal{R}$ is homogeneous.*

Proof. Let us consider the case of $\mathcal{U}$ and some ℓ_p-votewise distance $\widehat{d^p}$ first. Let $\mathcal{R}$ be $(\mathcal{U}, \widehat{d^p})$-rationalizable and let $E = (C, V)$ be an election with $C = \{c_1, \dots, c_m\}$ and $V = (v_1, \dots, v_n)$. Let s be an arbitrary positive integer. We will show that $\mathcal{R}(E) = \mathcal{R}(sE)$.

Let c be a candidate in $\mathcal{R}(E)$ and let (C, U), where $U = (u_1, \dots, u_n)$, be a $\mathcal{U}$-consensus witnessing this fact. For the sake of contradiction assume that $c \notin \mathcal{R}(sE)$. Let d be some $\mathcal{R}$-winner of sE and let (C, W) be a $\mathcal{U}$-consensus witnessing this fact. It is easy to see that we can pick W so that it is of the form sW', where $W' = (w_1, \dots, w_n)$. Since c is not a winner of sE, it holds that

$$\left(\sum_{i=1}^{n} s\,(d(v_i, u_i))^p \right)^{\frac{1}{p}} > \left(\sum_{i=1}^{n} s\,(d(v_i, w_i))^p \right)^{\frac{1}{p}}.$$

Since c is a winner of E, we also have

$$\left(\sum_{i=1}^{n} (d(v_i, u_i))^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} (d(v_i, w_i))^p \right)^{\frac{1}{p}}.$$

However, it is easy to see that these two inequalities are contradictory, and hence $c \in \mathcal{R}(sE)$. Using the same reasoning we can show that any winner of sE must be a winner of E.

For the consensus class $\mathcal{S}$ we can use essentially the same argument as for $\mathcal{U}$. Indeed, in the case of $\mathcal{S}$ we simply have $u_1 = u_2 = \cdots = u_n$ and $w_1 = w_2 = \cdots = w_n$, and the rest of the argument goes through without change. □

In contrast, $\mathcal{M}$-Borda, i.e., the rule obtained by combining $\mathcal{M}$ with $\widehat{d_{\text{swap}}}$, is not homogeneous.

Example 4.2. Consider the following election.

v_1	v_2	v_3	v_4	v_5	v_6
b	a	c	c	c	d
a	b	b	b	d	a
d	d	a	a	a	b
e	e	e	e	e	e
c	c	d	d	b	c

A simple calculation shows that to become a majority winner a needs four swaps, b needs three swaps, c needs four swaps, and d needs five swaps. Thus, b is a winner according to $\mathcal{M}$-Borda. However, if we replace each voter by two identical ones, it turns out that b needs five swaps to become a majority winner, but c requires only four (and, in fact, is the $\mathcal{M}$-Borda winner of the election).

More generally, we can fully characterize homogeneous rules that can be rationalized via $\mathcal{M}$ and a neutral ℓ_1-votewise pseudodistance (recall that by Theorem 3.6 all such rules are necessarily $\mathcal{M}$-scoring rules). For convenience, we state the following theorem for scoring vectors that satisfy $\alpha_1 \leq \cdots \leq \alpha_m$; obviously this can be done without loss of generality.

Theorem 4.3. *A voting rule $\mathcal{M}$-$\mathcal{R}_\alpha$ with a normalized scoring vector $\alpha = (\alpha_1, \ldots, \alpha_m)$ that satisfies $\alpha_1 \leq \cdots \leq \alpha_m$ is homogeneous if and only if $\alpha_m = 1$ (i.e., the rule is k-approval for some $k \geq 1$) or $\alpha_{\lceil \frac{m}{2} \rceil} = 0$.*

Proof. Suppose first that $\alpha_m = 1$. Then there exists some k, $1 \leq k < m$, such that $\alpha_i = 0$ for $i \leq k$, $\alpha_i = 1$ for $i > k$. Consider an election $E = (C, V)$ with $V = (v_1, \ldots, v_n)$ and fix an integer $s > 1$. If there are candidates ranked in top k positions by a majority of voters, these candidates form the set of winners both in E and in sE. Otherwise, each candidate has a strictly positive score under $\mathcal{M}$-$\mathcal{R}_\alpha$. Moreover, in this case the $\mathcal{M}$-$\mathcal{R}_\alpha$-score of each $c \in C$ is simply the difference between $\lfloor \frac{m}{2} \rfloor + 1$ and the number of voters that rank c in top k positions. Hence the winners in both E and sE are the candidates that are ranked in top k positions by the maximum number of voters.

Now, set $h = \lceil \frac{m}{2} \rceil$ and suppose that $\alpha_h = 0$. Again, consider an election $E = (C, V)$ with $V = (v_1, \ldots, v_n)$ and an integer $s > 1$. If m is odd or $\alpha_{h+1} = 0$, then by the pigeonhole principle there is at least one candidate $c \in C$ that is ranked in top h positions by a majority of voters. In this case, the sets of winners in both E and sE consist of all such candidates. It remains to consider the case $m = 2h$, $\alpha_{h+1} = 1$. If there exists a candidate $c \in C$ that is ranked among the top h positions by more than half of the voters, then the same argument as in the previous case shows that $\mathcal{M}$-$\mathcal{R}_\alpha(E) = \mathcal{M}$-$\mathcal{R}_\alpha(sE)$. On the other hand, if no candidate is ranked among the top h positions by more than half of the voters, then we see—again by the pigeonhole principle—that each candidate is ranked among the top h positions by exactly $\frac{n}{2}$ voters (note that this case is possible only if n is even). Thus, the $\mathcal{M}$-score of each candidate is of the form α_j, $j > h$. Further, each candidate's score remains the same in E and in sE. Thus, E and sE have the same sets of winners under $\mathcal{M}$-$\mathcal{R}_\alpha$.

It remains to argue that in all other cases, i.e., if $\alpha_m > 1$ and $\alpha_h > 0$, the rule $\mathcal{M}$-$\mathcal{R}_\alpha$ is not homogeneous. For readability, we will first consider the case $\alpha_3 > 1$ (note that this implies $\alpha_2 = 1$). This will be done in the following lemma. Later, we will show how to use ideas from this proof for the general case.

Lemma 4.4. *If $\alpha_3 > 1$ then the rule $\mathcal{M}$-$\mathcal{R}_\alpha$ is not homogeneous.*

Proof. Recall that we have $\alpha_1 = 0$, $\alpha_2 = 1$. Set $\alpha = \alpha_3$. We start by considering the case $m = 3$; later, we will generalize our construction to arbitrary values of m. Suppose first that α is a rational number, i.e., $\alpha = \frac{p}{q}$, where p and q are relatively prime. We construct an election $E = (C, V)$, where $C = \{a, b, c\}$ and V consists of the following votes:

1. $2p + q + 1$ votes $a \succ b \succ c$,

2. $2q + p + 1$ votes $b \succ c \succ a$, and

3. $p+q-2$ votes $c \succ b \succ a$.

We observe that $|V| = 4(p+q)$, each of the candidates a and b gets p points as their $\mathcal{M}$-scores, and c gets $p+q+3$ points (we use the fact that $q \geq 2$). Thus, the $\mathcal{M}$-score of candidate c is higher than that of a and b, and hence both a and b are winners of E.

The reader can verify that if we consider the election $2E = (C, 2V)$, then the $\mathcal{M}$-scores of candidates a and b are, respectively, $(2q-1)\alpha = 2p - \alpha$ and $2p-1$. Since $\alpha > 1$, it cannot be the case that both a and b are winners of $2E$. Thus, in this case $\mathcal{M}$-$\mathcal{R}_\alpha$ is not homogeneous.

Now, if α is irrational, consider its continued fraction expansion $\alpha = (a_0, a_1, \dots)$, and the successive convergents $\frac{h_i}{k_i}$, $i = 0, 1, \dots$, where $h_0 = a_0$, $k_0 = 1$, $h_1 = a_1 h_0 + 1$, $k_1 = a_1$, and $h_i = a_i h_{i-1} + h_{i-2}$, $k_i = a_i k_{i-1} + k_{i-2}$ for $i \geq 2$. We know that for even values of i we have $\frac{h_i}{k_i} < \alpha$ and $|\alpha - \frac{h_i}{k_i}| < \frac{1}{k_i k_{i+1}}$. Also, it is not hard to show that for any $N > 0$ there exists an even value of i such that $k_{i+1} > N$. Thus, we pick an even i such that $k_{i+1} > \frac{2}{\alpha - 1}$ (recall that $\alpha > 1$). We obtain

$$0 < \alpha - \frac{h_i}{k_i} < \frac{1}{k_i k_{i+1}} < \frac{\alpha - 1}{2k_i}.$$

Now, set $p = h_i$, $q = k_i$, let $\varepsilon = \alpha - \frac{p}{q}$, and use the same construction as above. In E, the $\mathcal{M}$-score of a is $q\alpha$, the $\mathcal{M}$-score of b is $p < q\alpha$, and the $\mathcal{M}$-score of c exceeds that of a and b, so b is the unique winner. On the other hand, in $2E$ the $\mathcal{M}$-score of a is $(2q-1)\alpha = 2p + 2q\varepsilon - \alpha$, while the $\mathcal{M}$-score of b is $2p-1$. We have $\varepsilon < \frac{\alpha-1}{2q}$, so a has a lower $\mathcal{M}$- score than b, and hence b cannot be the winner of $2E$. Thus, in this case, too, our rule is not homogeneous.

Finally, it is easy to see that for the case of $m > 3$ it suffices to modify the above construction by adding $m-3$ dummy candidates that each voter ranks last (in some arbitrary order). □

We will now consider the general case. Since we have $\alpha_m > 1$, the scoring vector can be written as

$$(\underbrace{0, \dots, 0}_{x}, \underbrace{1, \dots, 1}_{y}, \underbrace{\alpha, \dots, \alpha}_{z}, \alpha_{x+y+z+1}, \dots, \alpha_m)$$

for some $\alpha > 1$ and $x, y, z \geq 1$. If $x = y = 1$, the condition of Lemma 4.4 is satisfied, so we can assume that this is not the case. Also, since $\alpha_h \neq 0$, we have $x < h$.

We will now modify the construction of the election $E = (C, V)$ from the proof of Lemma 4.4 as follows. We set $C = \{a, b, c\} \cup D$, where $D = \{d_1, \dots, d_{m-3}\}$. If α is a rational number, we set $\alpha = \frac{p}{q}$, where p and q are relatively prime; otherwise, we construct p and q as in the proof of Lemma 4.4.

We replace each voter in V with a voter that grants the same number of points to a, b, and c as the replaced voter. Thus, we construct

1. $2p+q+1$ voters that rank a in the 1-st position, b in the $(x+1)$-st position and c in the $(x+y+1)$-st position,

2. $2q+p+1$ voters that rank b in the 1-st position, c in the $(x+1)$-st position and a in the $(x+y+1)$-st position, and

3. $p+q-2$ voters that rank c in the 1-st position, b in the $(x+1)$-st position and a in the $(x+y+1)$-st position.

The candidates in D are ranked in an arbitrary order among the remaining positions in these votes.

We also construct additonal voters so as to ensure that the candidates in D are not among the winners of E. Set $s = 3p + 2q$. For each candidate $d \in D$ we create s pairs of voters with the following preferences. In each pair, the first voter ranks a in the 1-st position and b in the $(x+y+1)$-st position, and the second voter ranks b in the 1-st position and a in the $(x+y+1)$-st position. Both of these voters rank d in the $(x+1)$-st position. Finally, the voters in each pair rank the candidates in $(D \setminus \{d\}) \cup \{c\}$ in each of the votes in the opposite order in the remaining positions. Since $x < h$, this ensures that no voter in $D \cup \{c\}$ is ranked in the top x positions by both voters in the pair. Altogether, we have $4p + 4q + 2s(m-3)$ voters.

Since both a and b are ranked in the first position by exactly one voter in each newly constructed pair, these new votes do not affect the $\mathcal{M}$-scores of a and b. Indeed, it is easy to see that a has $q\alpha$ points and b has p points. Similarly, the $\mathcal{M}$-score of c is still at least $p+q+3$. Finally, the $\mathcal{M}$-score of any $d \in D$ is at least $s+1-2(p+q) > p$ by our choice of s. Thus, for the resulting election E we have $\mathcal{M}\text{-}\mathcal{R}_\alpha(E) = \{a, b\}$ if α is rational and $\mathcal{M}\text{-}\mathcal{R}_\alpha(E) = \{b\}$ if α is irrational. On the other hand, as in the proof of Lemma 4.4, in $2E$ the $\mathcal{M}$-score of a is $(2q-1)\alpha$, the $\mathcal{M}$-score of b is $2p-1$, and $(2q-1)\alpha < 2p-1$, so $\mathcal{M}\text{-}\mathcal{R}_\alpha(2E) \neq \mathcal{M}\text{-}\mathcal{R}_\alpha(E)$. □

We have demonstrated that many voting rules that are ℓ_1-votewise distance rationalizable with respect to $\mathcal{M}$ are not homogeneous. However, if we use the ℓ_∞-norm instead of ℓ_1, the resulting voting rules are more likely to be homogeneous. For example, Simplified Bucklin has been shown to be distance-rationalizable via $\mathcal{M}$ and an ℓ_∞-votewise distance $\widehat{d^{\infty}_{\rm ser}}$ [EFS10b] and it is not hard to see that Simplified Bucklin is homogeneous. Indeed, this follows from a more general result stating that ℓ_∞-votewise rules are homogeneous as long as they are rationalized via a consensus class that satisfies a fairly weak requirement.

Definition 4.5. *We say that a consensus class $\mathcal{K}$ is* split-homogeneous *if the following two conditions hold:*

(a) If U is a $\mathcal{K}$-consensus then for every positive integer s it holds that sU is a $\mathcal{K}$-consensus with the same winner;

(b) If U and W are two profiles, with n votes each, such that $U+W$ is a $\mathcal{K}$-consensus, then at least one of U and W is a $\mathcal{K}$-consensus with the same winner.

It turns out that combining a split-homogeneous consensus class with an ℓ_∞-votewise distance produces a homogeneous rule.

Theorem 4.6. *Given a set of alternatives C, let d be a pseudodistance on $\mathcal{P}(C)$, let $\mathcal{K}$ be a split-homogeneous consensus class, and let $\mathcal{R}$ be a $(\mathcal{K}, \widehat{d^\infty})$-rationalizable voting rule. Then $\mathcal{R}$ is homogeneous.*

Proof. We will prove that for any election $E = (C, V)$ we have $\mathcal{R}(E) = \mathcal{R}(2E)$; the general case is similar. Let c be a winner of E and let U be the consensus profile that witnesses this. Then for each $U' \in \mathcal{K}$ we have

$$k = \widehat{d^\infty}(V, U) \leq \widehat{d^\infty}(V, U'). \tag{1}$$

Due to the nature of ℓ_∞-metric we have

$$\widehat{d^\infty}(2V, 2U) = \widehat{d^\infty}(V, U) = k, \tag{2}$$

and $2U$ is a consensus profile by condition (a) of Definition 4.5. Suppose that $c \in \mathcal{R}(U) = \mathcal{R}(2U)$ is not a winner of $2E$. Then there exist a profile $X + Y \in \mathcal{K}$, $|X| = |Y| = n$, such that $\widehat{d^\infty}(2V, X + Y) < k$. Since our distance is an ℓ_∞ one, we have

$$\widehat{d^\infty}(V, X) < k \qquad \text{and} \qquad \widehat{d^\infty}(V, Y) < k.$$

However by condition (b) either $X \in \mathcal{K}$ or $Y \in \mathcal{K}$ which contradicts (1) and (2). □

It is not hard to see that the consensus classes $\mathcal{S}$, $\mathcal{U}$ and $\mathcal{M}$ are split-homogeneous. Thus, we obtain the following corollary.

Corollary 4.7. *For any $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}, \mathcal{M}\}$ and any pseudodistance d on votes, the voting rule that is $(\mathcal{K}, \widehat{d^\infty})$-rationalizable is homogeneous.*

In contrast, the Condorcet consensus is not split-homogeneous as the following example demonstrates.

Example 4.8. Consider the following election $E = (C, V)$ with $C = \{a, b, c, d, e\}$ and $V = (v_1, \ldots, v_{12})$.

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
a	b	c	d	e	c	e	a	b	c	d	c
b	c	d	e	a	a	d	e	a	b	c	a
c	d	e	a	b	b	c	d	e	a	b	b
d	e	a	b	c	d	b	c	d	e	a	d
e	a	b	c	d	e	a	b	c	d	e	e

Here, voters $v_1, \ldots, v_5$ form a Condorcet cycle, and voters $v_7, \ldots, v_{11}$ are obtained from voters $v_1, \ldots, v_5$ by reversing their preferences. Voters v_6 and v_{12} are identical and rank candidate c on top. It is not hard to verify that c is the Condorcet winner in E. On the other hand, consider elections $E_1 = (C, V_1)$ and $E_2 = (C, V_2)$, where $V_1 = (v_1, \ldots, v_6)$ and $V_2 = (v_7, \ldots, v_{12})$. In E_1, b is ranked above c in 4 votes, so c is not a Condorcet winner in E_1. Similarly, in E_2, d is ranked above c in 4 votes, so c is not a Condorcet winner in E_2 either.

v_1	v_2	v_3	v_4	v_5	v_6
c	a	c	b	a	c
$\downarrow$	b	a	$\downarrow$	$\downarrow$	$\downarrow$
b	c	b	a	b	a
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
a	$\downarrow$	$\downarrow$	c	c	b
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$

Table 1: Election $E = (C, V)$ from the proof of Proposition 4.9.

There are ℓ_∞-votewise distances that combined with the Condorcet consensus yield nonhomogeneous rules.

Proposition 4.9. *There exists a set of candidates C and a distance d on $\mathcal{P}(C)$ such that the voting rule rationalized by $(\mathcal{C}, \widehat{d^\infty})$ is not homogeneous.*

Proof. We first define two additional types of swap operations for preference orders. A *forward distance-two swap* of candidate c transforms this preference order as follows: the candidate ranked two positions higher than c, is moved from his current position and placed directly below c. If c were ranked first or second, a forward distance-two swap is not defined. For example, if $C = \{a, b, c, d, e\}$ and the preference order is $a \succ b \succ c \succ d \succ e$, then the result of a forward distance-two swap of candidate c will be $b \succ c \succ a \succ d \succ e$. A *backward distance-two swap* is defined similarly.

It is easy to see that a single forward distance-two swap can be reversed by applying a single backward distance-two swap and the other way round.

We can now define our distance d. Let us fix some candidate set $C = \{c_1, \dots, c_m\}$. For each two preference orders u and v over C we define $d(u, v)$ to be the minimal number of swaps of adjacent candidates and distance-two swaps of candidates needed to transform vote u into vote v. It is easy to see that d indeed is a distance because it counts the number of reversible operations that transform one preference order into the other. As before, $\widehat{d^\infty}$ is the ℓ_∞-votewise extension of d to a distance over elections.

Let $\mathcal{R}$ be a voting rule that is $(\mathcal{C}, \widehat{d^\infty})$-rationalized. We will now build an election $E = (C, V)$ such that $\mathcal{R}(E) \neq \mathcal{R}(2E)$. We set $C = \{a, b, c, x_1, \dots x_t\}$ where t is a sufficiently large integer. (After reading our description of the votes in V it will become clear what we mean by *sufficiently large*.) The set of voters V will contain six voters, $v_1, \dots, v_6$, whose preference orders are presented in Table 1. Note that in this table we only showed how candidates in $\{a, b, c\}$ are ranked. Remaining candidates are ranked in the places of arrows, in such a way that (a) each candidate in $\{a, b, c\}$ is preferred to each candidate x_i, $1 \leq i \leq t$, by a majority of voters, and (b) one needs at least three swaps or distance-two swaps to change the relative order of two candidates from $\{a, b, c\}$ that are separated by an arrow.

We have the following results of head-to-head contests in E: four voters prefer a to b, a and c are tied, and b and c are tied. Thus, a single swap of a and c in vote v_3 makes a a

Condorcet winner of the election. On the other hand, it is easy to see that being allowed one (possibly distance-two) swap per vote, it is impossible to make either b or c the Condorcet winner. Thus, a is the unique $\mathcal{R}$-winner of E.

In $2E$, similarly, a single swap (within one of the copies of v_3) suffices to make a the Condorcet winner. However, now also a single swap per vote suffices to make c a Condorcet winner. Indeed, in one copy of v_2 we transform $a \succ b \succ c$ into $a \succ c \succ b$ and in the other into $b \succ c \succ a$. These two transformations allow c to break a tie with both a and b, and become the Condorcet winner. □

The combination of $\mathcal{C}$ and ℓ_1-votewise distance does not necessarily lead to a homogeneous rule either: it is well known that the Dodgson rule is not homogeneous (see, e.g., [Bra09] for a recent survey of Dodgson voting deficiencies) and yet it is $(\mathcal{C}, \widehat{d}_{\rm swap})$-rationalizable. In fact, we are not aware of any homogeneous voting rule that is ℓ_1-votewise distance-rationalizable with respect to $\mathcal{C}$.

On the other hand, for the case of ℓ_∞-votewise distances there is indeed an example of a homogeneous voting rule that is rationalized via such a distance and the Condorcet consensus. This rule, which we will call Dodgson$^\infty$, is rationalized by $(\mathcal{C}, \widehat{d^\infty}_{\rm swap})$. (The next section will explain better the name of the rule.) We claim that Dodgson$^\infty$ is homogeneous. To prove this, we will first explain how to determine Dodgson$^\infty$'s winners; it turns out that, in constrast to the Dodgson rule itself, Dodgson$^\infty$ admits a polynomial-time winner determination algorithm.

Proposition 4.10. *Given an election $E = (C, V)$, the problem of computing the $(\mathcal{C}, \widehat{d^\infty}_{\rm swap})$-score of a given candidate $c \in C$ is in* P*.*

Proof. Consider the following algorithm:

1. Set $k = 0$.
2. If c is a Condorcet winner of E then return k.
3. For each vote where c is not ranked first, swap c and its predecessor.
4. Increase k by 1.
5. Go to Step 2.

Suppose that c's $(\mathcal{C}, \widehat{d^\infty}_{\rm swap})$-score is k. Since our algorithm does not stop until it finds a Condorcet consensus, it will not stop before step k. On the other hand, there exists a Condorcet consensus U with winner c such that $\widehat{d^\infty}_{\rm swap}(E, U) = k$. Note that we can assume that U has been obtained from E by shifting c upwards, and, moreover, c has been shifted by k positions in at least one vote, and by at most k positions in all remaining votes. Now, consider an election U' in which c has been shifted upwards by exactly k positions in each vote or moved to the top position if its rank is smaller than k. Clearly, U' is also a Condorcet consensus, and $\widehat{d^\infty}_{\rm swap}(E, U') = k$. Moreover, U' will be discovered at the k-th step of our algorithm.

Since the algorithm terminates after at most $|C|$ iterations, it is easy to see that it runs in polynomial time. This completes the proof. $\square$

Proposition 4.11. *Dodgson$^\infty$ is homogeneous.*

Proof. Let $E = (C, V)$ be an election where $V = (v_1, \dots, v_n)$ and let c be a candidate in C. The algorithm in the proof of Proposition 4.10 finds the smallest value of k such that after shifting a given candidate upwards by k positions in each vote, this candidate becomes the Condorcet winner. Therefore, if two votes are identical before running the algorithm, these votes remain identical in the resulting Condorcet consensus. This shows that Dodgson$^\infty$-score of c is the same in E and in kE. $\square$

5 An Interlude: ℓ_1-Votewise Rules versus ℓ_∞-Votewise Rules

Inspired by Proposition 4.11, in this section we take a brief detour from the discussion of homogeneity and monotonicity in votewise rules, and discuss the relationship between ℓ_1-votewise rules and ℓ_∞-votewise rules. It turns out that in a certain weak sense, ℓ_∞-votewise rules are approximations of the corresponding ℓ_1-votewise rules. The next theorem expresses this "weak sense" precisely.

Theorem 5.1. *Consider a voting rule $\mathcal{R}$ that is distance-rationalized via some consensus class $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}, \mathcal{M}, \mathcal{C}\}$ and a distance $\widehat{d}$, where d is a distance over votes. Let $\mathcal{R}^\infty$ be the voting rule rationalized via $\mathcal{K}$ and $\widehat{d^\infty}$. For any candidate $c \in C$, let $\mathrm{score}_E^{\mathcal{R}}(c)$ (respectively, $\mathrm{score}_E^{\mathcal{R}^\infty}(c)$) denote the $(\mathcal{K}, \widehat{d})$-score (respectively, $(\mathcal{K}, \widehat{d^\infty})$-score) of c in E. Then for each election $E = (C, V)$ and each candidate $c \in C$ we have*

$$\mathrm{score}_E^{\mathcal{R}^\infty}(c) \le \mathrm{score}_E^{\mathcal{R}}(c) \le |V| \cdot \mathrm{score}_E^{\mathcal{R}^\infty}(c).$$

Proof. Consider an election $E = (C, V)$ with $C = \{c_1, \dots, c_m\}$ and $V = (v_1, \dots, v_n)$, and fix a candidate $c \in C$.

We first claim that $\mathrm{score}_E^{\mathcal{R}}(c) \le |V| \cdot \mathrm{score}_E^{\mathcal{R}^\infty}(c)$. Let (C, W), where $W = (w_1, \dots, w_n)$, be a $\mathcal{K}$-consensus where i is a winner and such that $\mathrm{score}_E^{\mathcal{R}^\infty}(c) = \widehat{d}^\infty(V, W)$. By definition, we have $\mathrm{score}_E^{\mathcal{R}}(c) \le \widehat{d}(V, W) = \sum_{i=1}^n d(v_i, w_i) \le n \max\{d(v_i, w_i) \mid 1 \le i \le n\} = n\widehat{d^\infty}(V, W) = |V| \cdot \mathrm{score}_E^{\mathcal{R}^\infty}(c)$, which proves the claim.

On the other hand, let (C, U), where $U = (u_1, \dots, u_n)$, be a $\mathcal{K}$-consensus where c is the winner and for which $\mathrm{score}_E^{\mathcal{R}}(c) = \widehat{d}(V, U)$. By definition, it holds that $\mathrm{score}_E^{\mathcal{R}}(c) = \widehat{d}(V, U) = \sum_{i=1}^n d(v_i, u_i) \ge \max\{d(v_i, u_i) \mid 1 \le i \le n\} = \widehat{d}^\infty(V, U) \ge \mathrm{score}_E^{\mathcal{R}^\infty}(c)$, and so $\mathrm{score}_E^{\mathcal{R}^\infty}(c) \le \mathrm{score}_E^{\mathcal{R}}(c)$. This completes the proof. $\square$

In other words, any ℓ_∞-votewise rule is a $|V|$-approximation of a corresponding ℓ_1-votewise rule in the sense of Caragiannis et al. [CCF$^+$09, CKKP10]. It is easy to see that for the case of majority consensus we can slightly strengthen our result, using the fact that we only need the majority of the voters to rank a candidate first for him to be the $\mathcal{M}$-winner.

Corollary 5.2. *For any distance d over votes, let $\mathcal{R}$ be the rule that is $(\mathcal{M}, \widehat{d})$-rationalized and let $\mathcal{R}^\infty$ be the rule that is $(\mathcal{M}, \widehat{d^\infty})$-rationalized. Then for each election $E = (C, V)$ and each candidate $c \in C$ it holds that* $\mathrm{score}_E^{\mathcal{R}^\infty} \leq \mathrm{score}_E^{\mathcal{R}}(c) \leq \left(\left\lceil \frac{|V|}{2} + 1 \right\rceil\right) \mathrm{score}_E^{\mathcal{R}^\infty}(c)$.

Of course, these approximations are very weak as they depend linearly on the number of voters. The reason why we find them interesting is that they are obtained by a very general method and thus illustrate the power of the distance rationalizability framework. Further, for the Dodgson rule its ℓ_∞ variant is polynomial-time computable, and one may hope that this will be the case for other rules, too. However, so far, we have not been able to prove this for the genetal case. In particular, it would be interesting to resolve the following question.

Question 5.3. *Let R be $(\mathcal{S}, \widehat{d^\infty}_{\mathrm{swap}})$-rationalizable. Is the problem of deciding whether a given candidate is a winner of $\mathcal{R}$-elections in* P*?*

Note that the rule in the above question is an ℓ_∞ variant of Kemeny (we omit the formal definition here due to space constraints; effectively, Kemeny is the rule that is rationalized by $\mathcal{S}$ and $\widehat{d}_{\mathrm{swap}}$). Of course, there are much better approximation algorithms known for Kemeny [ACN08, CFR10, KMS07] and the value of resolving the above question lays in it enhancing our understanding of Kemeny and relations between ℓ_1- and ℓ_∞-votewise rules.

6 Monotonicity

There are two main reasons why one should study monotonicity in the context of distance-rationalizability. The first one is that monotonicity is believed to be a very desirable property of voting rules. Thus, it is important to know which (votewise) distance-rationalizable rules are monotone. The second reason regards the core nature of distance-rationalizability framework. The role of a preference order submitted by a voter is to indicate which candidates are more preferred by this voter (those ranked at higher positions) and which are less preferred (those ranked at lower positions). However, as illustrated by our discussion of $\mathcal{M}$-scoring rules, distance-rationalizable rules may ignore this intuition altogether (except for the requirements imposed by compatibility with a given consensus notion). In this section we attempt to formulate conditions on (votewise) distances that refine distance-rationalizability framework to respect the intuition. Corollary 6.12 is a sign of our success in this respect.

In this section we will not discuss the Condorcet consensus. The reason for it is as follows. It is well known that Dodgson rule is not monotone (see [Bra09]). Yet, it is $(\mathcal{C}, \widehat{d}_{\mathrm{swap}})$-rationalizable and $\widehat{d}_{\mathrm{swap}}$ is about the best-behaved distance one can think of. Thus, if even using $\widehat{d}_{\mathrm{swap}}$ does not ensure monotonicity for $\mathcal{C}$, then it appears that finding a reasonable condition on distances that ensures monotonicity for $\mathcal{C}$ may not be possible. The reason is that $\mathcal{C}$ is, in some sense, the least "local" of the consensus classes we consider. As a result, conditions regarding just the distance among votes may be very hard to translate to conditions regarding the whole profile (and monotonicity is a condition of that sort). For simplicity, we focus on ℓ_1-votewise rules and ℓ_∞-votewise rules.

Let C be a set of candidates and let d be a distance among votes. How can we specify a condition on d so that voting rules rationalized using this distance are monotone? Intuitively, the condition should ensure that if candidate c is a winner and some voter ranks him higher, then the distance to a consensus where c is the winner decreases more than the distance to a consensus with any other winner. The next definition tries to capture this intuition.

Definition 6.1. *Let C be a set of candidates and let d be a distance between votes over C. We say that d is relatively monotone if for every candidate $c \in C$, each two preference orders y and y' such that y' is identical to y except that y' ranks c higher than y, and every two preference orders x and z such that x ranks c first and z does not, it holds that*

$$d(x, y) - d(x, y') \geq d(z, y) - d(z, y').$$

As a quick sanity check, we note that the swap distance, d_{swap}, satisfies the relative monotonicity condition.

Proposition 6.2. *Distance d_{swap} is relatively monotone.*

Proof. Let $d = d_{\mathrm{swap}}$ and let C be a set of candidates, c be a candidate in C, and let y, y', x, and z be as in the definition of relative monotonicity. In addition, let k be a positive integer such that y' is identical to y except in y' candidate c is ranked k positions higher. We need k swaps to get y' from y so $d(y, y') = k$. We first note that $d(x, y) - d(x, y') = k$. This is so because the swap distance measures the number of inverses between two preference orders. As x ranks c on top and y' ranks it k positions higher than y does (without any other changes), the number of inverses between x and y' is the same as that between x and y less k. By the triangle inequality. We have $d(z, y) \leq d(z, y') + d(y', y) = d(z, y') + k$, hence $d(z, y) - d(z, y') \leq k$ and this completes the proof. □

Relative monotonicity of an ℓ_1-votewise distance (or, strictly speaking, of the distance among votes that underlies this ℓ_1-votewise distance) naturally translates to the monotonicity of a resulting voting rule, provided we use of either $\mathcal{S}$ or $\mathcal{U}$ as a consensus.

Theorem 6.3. *Let $\mathcal{R}$ be a voting rule rationalized by $(\mathcal{K}, \widehat{d})$, where $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}\}$ and d is a relatively monotone distance on votes. Then $\mathcal{R}$ is monotone.*

Proof. Let $E = (C, V)$ be an election, where $V = (v_1, \ldots, v_n)$, and $c \in C$ be a candidate such that $c \in \mathcal{R}(E)$. Let $E' = (C, V')$, where $V' = (v'_1, \ldots, v'_n)$, be an arbitrary election that is identical to E except one voter, say v'_1, ranks c higher ceteris paribus. It suffices to show that $c \in \mathcal{R}(E')$.

To show this, we give a proof by contradiction. Let $(C, U) \in \mathcal{K}$, where $U = (u_1, \ldots, u_n)$, be a consensus witnessing that $c \in \mathcal{R}(E)$, and let (C, W), where $W = (w_1, \ldots, w_n)$, be any consensus in $\mathcal{K}$ such that c is not a consensus winner of (C, W). For the sake of contradiction, let us assume that $\widehat{d}(U, V') > \widehat{d}(W, V')$. If $\mathcal{K}$ is either $\mathcal{U}$ or $\mathcal{S}$, then we know that u_1 ranks c first and that w_1 does not rank c first. By relative monotonicity, this means that

$$d(u_1, v_1) - d(u_1, v_1') \geq d(w_1, v_1) - d(w_1, v_1'). \tag{3}$$

However, since $\widehat{d}(U, V') > \widehat{d}(W, V')$ and V differs from V' only on the first voter, it holds that

$$d(u_1, v_1') + \sum_{i=2}^{n} d(u_i, v_i) > d(w_1, v_1') + \sum_{i=2}^{n} d(w_i, v_i). \tag{4}$$

If we add inequality (3) sideways to inequality (4), we obtain

$$d(u_1, v_1) + \sum_{i=2}^{n} d(u_i, v_i) > d(w_1, v_1) + \sum_{i=2}^{n} d(w_i, v_i).$$

That is, $\widehat{d}(U, V) > \widehat{d}(W, V)$, which is a contradiction by our choice of U. □

Relative monotonicity is a remarkably strong condition, not satisfied even by some very natural distances that, intuitively, should be monotone.

Example 6.4. Consider a scoring vector $\alpha = (0, 1, 2, 3, 4, 5)$, i.e., the 6-candidate scoring vector for Borda rule and a candidate set $C = \{c, d, x_1, x_2, x_3, x_4\}$. Distance d_α does not satisfy the relative monotonicity condition. Indeed, let us consider the following four votes:

$$\begin{aligned}
x &: c > d > x_1 > x_2 > x_3 > x_4, \\
z &: x_1 > c > x_2 > x_3 > x_4 > d, \\
y &: x_1 > x_2 > d > c > x_3 > x_4, \\
y' &: x_1 > x_2 > c > d > x_3 > x_4.
\end{aligned}$$

Note that y and y' are identical except that in y' candidate c is ranked one position higher, and that c is ranked on top x and is not ranked on top z. We can easily verify that $d(x, y) - d(x, y') = 0$ but $d(z, y) - d(z, y') = 2$. Thus, d_α is not relatively monotone.

For the case of $\mathcal{U}$ we can weaken the assumptions of Theorem 6.3 to relative min-monotonicity from the definition below. It is easy to modify the proof of Theorem 6.3 to work for $\mathcal{U}$ and relative min-monotone distances.

Definition 6.5. *Let C be a set of candidates and let d be a distance between votes over C. We say that d is* relatively min-monotone *if for each candidate $c \in C$ and each two preference orders y and y' such that y' is identical to y except that y' ranks c higher than y, it holds that for each candidate $e \in C \setminus \{c\}$:*

$$\min_{x \in \mathcal{P}(C,c)} d(x, y) - \min_{x' \in \mathcal{P}(C,c)} d(x', y') \geq \min_{z \in \mathcal{P}(C,e)} d(z, y) - \min_{z' \in \mathcal{P}(C,e)} d(z', y').$$

Corollary 6.6. *Let $\mathcal{R}$ be a voting rule rationalized by $(\mathcal{U}, \widehat{d})$, where d is relatively min-monotone distance on votes. Then $\mathcal{R}$ is monotone.*

On the other hand, if we want to obtain monotonicity for voting rules based on votewise distances and the majority consensus, it seems that we have to give up the "relative" part of the definition of relative min-monotone distances. The next definition explains this formally.

Definition 6.7. *Let C be a set of candidates and let d be a distance between votes over C. We say that d is* min-monotone *if for every candidate $c \in C$ and every two preference orders y and y' such that y' is the same as y except that it ranks c higher, for each $e \in C \setminus \{c\}$ it holds that:*

$$\begin{aligned} \min_{x \in \mathcal{P}(C,c)} d(x,y) &\geq \min_{x' \in \mathcal{P}(C,c)} d(x',y'), \\ \min_{z \in \mathcal{P}(C,e)} d(z,y) &\leq \min_{z' \in \mathcal{P}(C,e)} d(z',y'). \end{aligned}$$

In other words, a distance is min-monotone if ranking a certain candidate c higher never increases his distance from the closest vote where he is ranked first, and, for every candidate e other than c, never decreases his distance from a vote where e is ranked first. It is easy to see that min-monotonicity is a relaxation of relative min-monotonicity and so Corollary 6.6 applies to min-monotone distances as well.

Proposition 6.8. *Each min-monotone distance d over votes is relatively min-monotone.*

Proof. Immediate from the definition. □

Using min-monotone distances, we can now show an analog of Theorem 6.3 for the case of the majority consensus.

Theorem 6.9. *Let $\mathcal{R}$ be a voting rule rationalized by $(\mathcal{M}, \widehat{d})$ such that d is a min-monotone distance on votes. Then $\mathcal{R}$ is monotone.*

Proof. Let $\mathcal{R}$ and $\widehat{d}$ be as in the statement of the theorem. Let $E = (C, V)$ be an election where $V = (v_1, \dots, v_n)$ and let $c \in \mathcal{R}(E)$ be one of the winners of E. Let (C, U) be a majority consensus witnessing that c is an $\mathcal{R}$ winner of E. Let $E' = (C, V')$, where $V' = (v'_1, v'_2, \dots, v'_n)$, be an election where v'_1 is identical to v_1 except that it ranks c higher and for each i, $2 \leq i \leq n$, $v'_i = v_i$.

For the sake of contradiction, we assume that c is not an $\mathcal{R}$ winner of E', but that some candidate $e \in C \setminus \{c\}$ is. Let (C, W'), where $W' = (w'_1, \dots, w'_n)$ be a majority consensus witnessing that e is a winner of E'. Let us form two new $\mathcal{M}$ consensuses, U' and W.

1. $U' = (u'_1, \dots, u'_n)$. For each i, $2 \leq i \leq n$, $u'_i = u_i$. If u_1 ranks c first then $u'_1 \in \arg\min_{x' \in \mathcal{P}(C,c)} d(x', v'_1)$, and otherwise $u'_1 = v'_1$.
2. $W = (w_1, \dots, w_n)$. For each i, $2 \leq i \leq n$, $w_i = w'_i$. If w'_1 ranks e first then $w_1 \in \arg\min_{z \in \mathcal{P}(C,e)} d(z, v_1)$, and otherwise $w_1 = v_1$.

Thus, by Lemma 3.5 and min-monotonicity of d it is easy to see that:

$$d(u'_1, v'_1) \leq d(u_1, v_1), \tag{5}$$
$$d(w'_1, v'_1) \geq d(w_1, v_1). \tag{6}$$

Now, using the fact that V and V' agree on all voters but the first one, our choice of W, and the two above inequalities, we can see that the following inequality holds:

$$\begin{aligned}\widehat{d}(U,V) &= d(u_1,v_1) + \sum_{i=2}^{n} d(u_i,v_i) \geq d(u'_1,v'_1) + \sum_{i=2}^{n} d(u_i,v_i) \\ &> d(w'_1,v'_1) + \sum_{i=2}^{n} d(u_i,v_i) \geq d(w_1,v_1) + \sum_{i=2}^{n} d(u_i,v_i) = \widehat{d}(W,V).\end{aligned}$$

However, this is a contradiction because by our choice of U, $\widehat{d}(U,V)$ is a minimal distance between V and any majority consensus with n voters. □

We can use essentially the same proof for the case of ℓ_∞-votewise distances.

Corollary 6.10. *Let $\mathcal{R}$ be a voting rule rationalized by $(\mathcal{M}, \widehat{d^\infty})$, where d is a min-monotone distance on votes. Then $\mathcal{R}$ is monotone.*

Note that it is hard to apply the notion of min-monotone distances to prove monotonicity of voting rules that are distance rationalized via strong unanimity consensus. The reason is that given a profile V of voters over some candidate set C, finding a $\mathcal{S}$ consensus closest to V requires finding a single preference order u that minimizes the aggregated distance from V to this order. However, it may be the case that for neither of the votes in V, u is a preference order that minimizes the distance from this vote to a preference order that ranks $\mathrm{top}(u)$ first.

The next proposition, together with Example 6.4, shows that indeed min-monotonicity is a considerably weaker condition than relatvie monotonicity.

Proposition 6.11. *Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a normalized scoring vector. (Pseudo)distance d_α is min-monotone if and only if α is nondecreasing.*

Proof. Let us fix some two distinct candidates $c = c_i$ and $e = c_j$ in C. Let y and y' be two votes that are identical, except that c is ranked on some position k in y and in y' candidate c is shifted to position k', where $k' < k$. By Lemma 2.2, it holds that $\min_{x \in \mathcal{P}(C,c_i)} d(x,y) = 2|\alpha_k - \alpha_1|$ and $\min_{x' \in \mathcal{P}(C,c_i)} d(x',y') = 2|\alpha_{k'} - \alpha_1|$. If α is nondecreasing then $2|\alpha_k - \alpha_1| = 2\alpha_k \geq 2\alpha_{k'} = 2|\alpha_{k'} - \alpha_1|$. On the other hand, if α is not nondecreasing, then it is possible to choose k and k', where $k' < k$, such that $2|\alpha_k - \alpha_1| < 2|\alpha_{k'} - \alpha_1|$. Thus, the first inequality from the definition of min-monotonicity is satisfied if and only if α is nondecreasing. One can analogously show that the same holds for the second inequality (in essence, the proof works by arguing that either $\mathrm{rank}(y,e) = \mathrm{rank}(y',e)$ or $\mathrm{rank}(y,e) = \mathrm{rank}(y',e) - 1$, and then showing that pushing a candidate back does not decrease his distance from being ranked first if and only if α is nondecreasing). □

The above proposition, combined with the proof of Theorem 4.9 of [EFS10b] gives the next corollary.

Corollary 6.12. *A voting rule $\mathcal{R}$ is $(U, \widehat{d})$-rationalizable for some min-monotone neutral pseudodistance d on votes if and only if $\mathcal{R}$ can be defined via a family of nondecreasing scoring vectors (one for each number of candidates).*

In essence, Proposition 6.11 ensures that for every nondecreasing scoring vector α, $\mathcal{R}_\alpha$ is ℓ_1-votewise with respect to $\mathcal{U}$ via a min-monotone distance over votes, and the definition of min-monotonicity ensures that the scoring vector derived in the proof of Theorem 4.9 of [EFS10b] is nondecreasing.

7 Conclusions

We discussed homogeneity and monotonicity of voting rules that are distance-rationalizable via votewise distances, focusing on ℓ_p-votewise rules, $p \geq 1$, and ℓ_∞-votewise rules. Monotonicity and homogeneity are among the most essential properties of voting rules. Monotonicity guarantees that a voting rule is—at least to some degree—responsive to voter preferences, and homogeneity ensures that the winners of an election depend on the proportions of each possible vote cast and not on the specific counts for each possible vote.

Regarding homogeneity, we have shown that with respect to strong unanimity consensus and weak unanimity consensus all ℓ_p-votewise rules, $p \geq 1$, and all ℓ_∞-votewise rules are homogeneous. For majority consensus we have provided a complete dichotomy theorem for homogeneity of neutral ℓ_1-votewise rules and have shown that all ℓ_∞-votewise rules are homogeneous. We have also given a simple criterion for consensus classes, split-homogeneity, such that if a given consensus class is split-homogeneous then with respect to that class any ℓ_∞-votewise rule is homogeneous. Finally, we have shown that Condorcet consensus is not split-homogeneous and we have shown several examples of interesting behaviors of votewise rules with respect to the Condorcet consensus.

Studying monotonicity under the distance-rationalizability framework is much more difficult than studying homogeneity. The reason is that homogeneity is, to a large extent, a property of the underlying norm and the consensus notion, whereas monotonicity is mostly a property of the underlying distance among votes (though, of course, the particular norm and consensus notion also play a role). Thus, for the case of monotonicity we have shown several conditions on distances that, when used with a matching consensus notion, ensure monotonicity, and—to show that our notions are not vacuous—have shown that practically useful distances satisfy our definitions.

In addition to studying monotonicity as a voting-rule property, our study is deeply motivated by the role of monotonicity in making distance-rationalizability a framework fitting one's intuition more closely. In particular, we have used our study of monotonicity to refine a result of Elkind, Faliszewski, and Slinko [EFS10b] characterizing the class of scoring rules in terms of distance-rationalizability (our Corollary 6.12).

As a "side effect" of our study of monotonicity and homogeneity, we also discovered several other results and notions. In particular, we have identified the family of $\mathcal{M}$-scoring rules, which constitute a (provably distinct) variant of scoring rules that, when counting points for a given candidate, ignore the less favorable half of the votes. Similar mechanisms are often used in practice (e.g., some grading methods can be interpreted in a similar manner). During our study of $\mathcal{M}$-voting rules we also closed an open issue regarding distance-rationalizability of scoring rules asked in [EFS09] (namely, we have shown how to convert the pseudodistances of [EFS09] into distances).

We have also explored relations between votewise rules rationalized using the same underlying distance over votes, but a different norm. For example, we have shown that ℓ_∞-votewise rules form weak approximations of corresponding ℓ_1-votewise rules, and we have shown that every voting rule that is N-votewise with respect to $\mathcal{M}$, where N is a symmetric norm that is monotone in the positive orthant, is also M-votewise with respect to $\mathcal{U}$, where M is also a symmetric norm monotone in the positive orthant. Using this observation we answered, in the negative, an open question regarding votewise distance-rationalizability of STV with respect to $\mathcal{M}$ asked by Elkind, Faliszewski, and Slinko [EFS10a].

Our work leads to several open problems. First, we are very much interested in characterizing for which votewise distances are rules rationalized with respect to the Condorcet consensus homogeneous. Are there any such ℓ_1-votewise distances? Which ℓ_∞-votewise lead to homogeneous rules? This question might be difficult to answer because, as opposed to the case of the weak unanimity consensus and the majority consensus, we do not have good characterizations of rules votewise with respect to $\mathcal{C}$, and the nonlocal nature of Condorcet consensus makes it hard to analyze in terms of homogeneity (as opposed to strong unanimity consensus which also is nonlocal, but much easier to work with).

We are also very much interested in finding less-demanding, yet practically useful, conditions on distances that lead to monotone rules.

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