SIMPLICIAL COMPLEXES OBTAINED FROM QUALITATIVE PROBABILITY ORDERS

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Abstract. Edelman & Fishburn (2006) initiated the study of abstract simplicial complexes which are initial segments of qualitative probability orders. By their nature they are combinatorial generalizations of threshold complexes, however they have not been studied before. In this paper we prove that the class of initial segments of qualitative probabilities is a new class, different from all the classic generalizations of threshold complexes. More precisely we construct a qualitative probability order on 26 atoms that has an initial segment which is not a threshold simplicial complex. Although 26 is probably not the minimal number for which such example exists we provide some evidence that it cannot be much smaller.

1. Introduction

The concept of qualitative (comparative) probability takes its origins in attempts of de Finetti (1931) to axiomatise probability theory. It also played an important role in the expected utility theory of Savage (1954, p.32). The essence of a qualitative probability is that it does not give us numerical probabilities but instead provides us with the information, for every pair of events, which one is more likely to happen. The class of qualitative probability orders is broader than the class of probability measures for any $n \geq 5$ (Kraft et al., 1959). Qualitative probability orders on finite sets are now recognised as an important combinatorial object (Kraft et al., 1959; Fishburn, 1996, 1997) that finds applications in areas as far apart from probability theory as the theory of Gröbner bases (e.g., Maclagan, 1999).

Another important combinatorial object, also defined on a finite set is an abstract simplicial complex. This is a set of subsets of a finite set, called faces, with the property that a subset of a face is also a face. This concept is dual to the concept of a simple game whose winning coalitions form a set of subsets of a finite set with the property that if a coalition is winning, then every superset of it is also a winning coalition. The most studied class of simplicial complexes is the class of threshold simplicial complexes. These arise when we assign weights to elements of a finite set, set a threshold and define faces as those subsets whose combined weight is not achieving the threshold. Given a qualitative probability order one may obtain a simplicial complex in another way. For this one has to choose a threshold—which now will be a subset of our finite set—and consider as faces all subsets that are earlier than the threshold in the given qualitative probability order. The new class of simplicial complexes contains threshold complexes and is contained in a well-studied class of shifted complexes (Klivans, 2005, 2007). A natural question is therefore to ask if this is indeed a new class and whether or not it is different from the class of threshold complexes.

In this paper we give answer to this question by presenting an initial segment of a qualitative probability order on 26 atoms that is not threshold. We also show that such example cannot be too small, in particular, it is unlikely that one can be found on less than 18 atoms.

The structure of this paper is as follows. In Section 2 we introduce the basics of qualitative probability orders. In Section 3 we give a construction that will further provide us with examples of qualitative probability orders that are not related to any probability measure. In Section 4 we consider abstract simplicial complexes and give necessary and sufficient conditions for them being threshold. Finally in Section 5 we present our main result which is an example of a qualitative probability order on 26 atoms that is not threshold. Section 6 concludes with some relevant results and a discussion.

2. Qualitative Probability Orders and Discrete Cones

In this paper all our objects are defined on the set $[n]=\{1,2,\ldots,n\}$. By $2^{[n]}$ we denote the set of all subsets of $[n]$. An order[1] $\preceq$ on $2^{[n]}$ is called a *qualitative probability order* on $[n]$, if it is not true that

$$A \preceq \emptyset \tag{1}$$

for every nonempty subset A of $[n]$, and $\preceq$ satisfies de Finetti's axiom, namely for all $A,B,C\in 2^{[n]}$

$$A\preceq B \iff A\cup C \preceq B\cup C \text{ whenever } (A\cup B)\cap C=\emptyset\,. \tag{2}$$

Note that if we have a probability measure $\mathbf{p}=(p_1,\ldots,p_n)$ on $[n]$, where p_i is the probability of i, then we know the probability $p(A)$ of every event A and $p(A)=\sum_{i\in A}p_i$. We may now define a relation $\preceq$ on $2^{[n]}$ by

$$A\preceq B \quad \text{if and only if} \quad p(A)\le p(B);$$

obviously $\preceq$ is a qualitative probability order on $[n]$, and any such order is called *representable* (e.g., Fishburn, 1996; Regoli, 2000). Those not obtainable in this way are called *non-representable.* The class of qualitative probability orders is broader than the class of probability measures for any $n\ge 5$ (Kraft et al., 1959). A non-representable qualitative probability order $\preceq$ on $[n]$ is said to *almost agree* with the measure $\mathbf{p}$ on $[n]$ if

$$A\preceq B \Longrightarrow p(A)\le p(B). \tag{3}$$

If such a measure $\mathbf{p}$ exists, then the order $\preceq$ is said to be *almost representable.* Since the arrow in (3) is only one-sided it is perfectly possible for an almost representable order to have $A\preceq B$ but not $B\preceq A$ while $p(A)=p(B)$.

We will start with some rather standard properties of qualitative probability orders which we will need further. Let $\preceq$ be a qualitative probability order on $2^{[n]}$. As usual the following two relations can be derived from it. We write $A\prec B$ if $A\preceq B$ but not $B\preceq A$ and $A\sim B$ if $A\preceq B$ and $B\preceq A$.

Lemma 1. *Suppose that $\preceq$ is a qualitative probability order on $2^{[n]}$, $A,B,C,D\in 2^{[n]}$, $A\preceq B$, $C\preceq D$ and $B\cap D=\emptyset$. Then $A\cup C\preceq B\cup D$. Moreover, if $A\prec B$ or $C\prec D$, then $A\cup C\prec B\cup D$.*

[1]An order in this paper is any reflexive, complete and transitive binary relation. If it is also anti-symmetric, it is called linear order.

Proof. Firstly, let us consider the case when $A \cap C = \emptyset$. Let $B' = B \setminus C$ and $C' = C \setminus B$ and $I = B \cap C$. Then by (2) we have

$$A \cup C' \preceq B \cup C' = B' \cup C \preceq B' \cup D$$

where we have $A \cup C' \prec B' \cup D$ if $A \prec B$ or $C \prec D$. Now we have

$$A \cup C' \preceq B' \cup D \Leftrightarrow A \cup C = (A \cup C') \cup I \preceq (B' \cup D) \cup I = B \cup D.$$

Now let us consider the case when $A \cap C \neq \emptyset$. Let $A' = A \setminus C$. By (1) and (2) we now have $A' \prec B$. Since now we have $A' \cap C = \emptyset$ so by the previous case

$$A \cup C = A' \cup C \prec B \cup C \preceq B \cup D.$$

In this case we always obtain a strict inequality. □

A weaker version of this lemma can be found in Maclagan (1999)[Lemma 2.2].

Definition 1. *A sequence of subsets* $(A_1, \dots, A_j; B_1, \dots, B_j)$ *of* $[n]$ *of even length* $2j$ *is said to be a* trading transform *of length* j *if for every* $i \in [n]$

$$|\{j \mid i \in A_j\}| = |\{j \mid i \in B_j\}|.$$

In other words, sets $A_1, \dots, A_j$ *can be converted into* $B_1, \dots, B_j$ *by rearranging their elements. We say that an order* $\preceq$ *on* $2^{[n]}$ *satisfies the* k*-th cancellation condition* CC_k *if there does not exist a trading transform* $(A_1, \dots, A_k; B_1, \dots, B_k)$ *such that* $A_i \preceq B_i$ *for all* $i \in [k]$ *and* $A_i \prec B_i$ *for at least one* $i \in [k]$.

The key result of Kraft et al. (1959) can now be reformulated as follows.

Theorem 1 (Kraft-Pratt-Seidenberg)**.** *A qualitative probability order* $\preceq$ *is representable if and only if it satisfies* CC_k *for all* $k = 1, 2, \dots$.

It was also shown in Fishburn (1996, Section 2) that CC_2 and CC_3 hold for linear qualitative probability orders. It follows from de Finetti's axiom and properties of linear orders. It can be shown that a qualitative probability order satisfies CC_2 and CC_3 as well. Hence CC_4 is the first nontrivial cancellation condition. As was noticed in Kraft et al. (1959), for $n < 5$ all qualitative probability orders are representable, but for $n = 5$ there are non-representable ones. For $n = 5$ all orders are still almost representable Fishburn (1996) which is no longer true for $n = 6$ Kraft et al. (1959).

To every such linear order $\preceq$, there corresponds a *discrete cone* $C(\preceq)$ in T^n, where $T = \{-1, 0, 1\}$, as defined in Fishburn (1996).

Definition 2. *A subset* $C \subseteq T^n$ *is said to be a discrete cone if the following properties hold:*

- D1. $\{\mathbf{e}_1, \dots, \mathbf{e}_n\} \subseteq C$ *and* $\{-\mathbf{e}_1, \dots, -\mathbf{e}_n\} \cap C = \emptyset$*, where* $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ *is the standard basis of* $\mathbb{R}^n$*,*
- D2. $\{-\mathbf{x}, \mathbf{x}\} \cap C \neq \emptyset$ *for every* $\mathbf{x} \in T^n$*,*
- D3. $\mathbf{x} + \mathbf{y} \in C$ *whenever* $\mathbf{x}, \mathbf{y} \in C$ *and* $\mathbf{x} + \mathbf{y} \in T^n$*.*

We note that in Fishburn (1996), Fishburn requires $\mathbf{0} \notin C$ because his orders are anti-reflexive. In our case, condition D2 implies $\mathbf{0} \in C$.

Given a qualitative probability order $\preceq$ on $2^{[n]}$, for every pair of subsets A, B satisfying $B \preceq A$ we construct a characteristic vector of this pair $\chi(A, B) = \chi_A - \chi_B \in T^n$. We define the set $C(\preceq)$ of all characteristic vectors $\chi(A, B)$, for $A, B \in$

$2^{[n]}$ such that $B \preceq A$. The two axioms of qualitative probability guarantee that $C(\preceq)$ is a discrete cone (see Fishburn, 1996, Lemma 2.1).

Following Fishburn (1996), the cancellation conditions can be reformulated as follows:

Proposition 1. *A qualitative probability order $\preceq$ satisfies the k-th cancellation condition CC_k if and only if for no set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ of nonzero vectors in $C(\preceq)$ such that*

$$\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_m = \mathbf{0} \tag{4}$$

and $-\mathbf{x}_i \notin C(\preceq)$ for at least one i.

Geometrically, a qualitative probability order $\preceq$ is representable if and only if there exists a positive vector $\mathbf{u} \in \mathbb{R}^n$ such that

$$\mathbf{x} \in C(\preceq) \Longleftrightarrow (\mathbf{u}, \mathbf{x}) \geq 0 \quad \text{for all } \ \mathbf{x} \in T^n \setminus \{\mathbf{0}\},$$

where $(\cdot, \cdot)$ is the standard inner product; that is, $\preceq$ is representable if and only if every non-zero vector in the cone $C(\preceq)$ lies in the closed half-space $H_{\mathbf{u}}^{+} = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{u}, \mathbf{x}) \geq 0\}$ of the corresponding hyperplane $H_{\mathbf{u}} = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{u}, \mathbf{x}) = 0\}$.

Similarly, for a non-representable but *almost* representable qualitative probability order $\preceq$, there exists a vector $\mathbf{u} \in \mathbb{R}^n$ with non-negative entries such that

$$\mathbf{x} \in C(\preceq) \Longrightarrow (\mathbf{u}, \mathbf{x}) \geq 0 \quad \text{for all } \ \mathbf{x} \in T^n \setminus \{\mathbf{0}\}.$$

In the latter case we can have $\mathbf{x} \in C(\preceq)$ and $-\mathbf{x} \notin C(\preceq)$ despite $(\mathbf{u}, \mathbf{x}) = 0$.

In both cases, the normalised vector $\mathbf{u}$ gives us the probability measure, namely $\mathbf{p} = (u_1 + \ldots + u_n)^{-1}(u_1, \ldots, u_n)$, from which $\preceq$ arises or with which it almost agrees.

3. Constructing almost representable orders from nonlinear representable ones

Proposition 2. *Let $\preceq$ be a non-representable but almost representable qualitative probability order which almost agrees with a probability measure $\mathbf{p}$. Suppose that the mth cancellation condition CC_m is violated, and that for some non-zero vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\} \subseteq C(\preceq)$ the condition (4) holds. Then all of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ lie in the hyperplane $H_{\mathbf{p}}$.*

Proof. First note that for every $\mathbf{x} \in C(\preceq)$ which does not belong to $H_{\mathbf{p}}$, we have $(\mathbf{p}, \mathbf{x}) > 0$. Hence the condition (4) can hold only when all $\mathbf{x}_i \in H_{\mathbf{p}}$. □

We need to understand how we can construct new qualitative probability orders from old ones so we need the following investigation. Let $\preceq$ be a representable but not linear qualitative probability order which agrees with a probability measure $\mathbf{p}$.

Let $S(\preceq)$ be the set of all vectors of $C(\preceq)$ which lie in the corresponding hyperplane $H_{\mathbf{p}}$. Clearly, if $\mathbf{x} \in S(\preceq)$, then $-\mathbf{x}$ is a vector of $S(\preceq)$ as well. Since in the definition of discrete cone it is sufficient that only one of these vectors is in $C(\preceq)$ we may try to remove one of them in order to obtain a new qualitative probability order. The new order will almost agree with $\mathbf{p}$ and hence will be at least almost representable. The big question is: what are the conditions under which a set of vectors can be removed from $S(\preceq)$?

What can prevent us from removing a vector from $S(\preceq)$? Intuitively, we cannot remove a vector if the set comparison corresponding to it is a consequence of those remaining. We need to consider what a consequence means formally.

There are two ways in which one set comparison might imply another one. The first way is by means of the de Finetti condition. This however is already built in the definition of the discrete cone as $\chi(A,B) = \chi(A\cup C, B\cup C)$. Another way in which a comparison may be implied from two other is transitivity. This has a nice algebraic characterisation. Indeed, if $C \prec B \prec A$, then $\chi(A,C) = \chi(A,B) + \chi(B,C)$. This leads us to the following definition.

Following Christian et al. (2007) let us define a restricted sum for vectors in a discrete cone C. Let $\mathbf{u}, \mathbf{v} \in C$. Then

$$\mathbf{u} \oplus \mathbf{v} = \begin{cases} \mathbf{u}+\mathbf{v} & \text{if } \mathbf{u}+\mathbf{v} \in T^n, \\ \text{undefined} & \text{if } \mathbf{u}+\mathbf{v} \notin T^n. \end{cases}$$

It was shown in (Fishburn, 1996, Lemma 2.1) that the transitivity of a qualitative probability order is equivalent to closedness of its corresponding discrete cone with respect to the restricted addition (without formally defining the latter). The axiom D3 of the discrete cone can be rewritten as

D3. $\mathbf{x} \oplus \mathbf{y} \in C$ whenever $\mathbf{x}, \mathbf{y} \in C$ and $\mathbf{x} \oplus \mathbf{y}$ is defined.

Note that a restricted sum is not associative.

Theorem 2 (Construction method). *Let $\preceq$ be a representable non-linear qualitative probability order which agrees with the probability measure $\mathbf{p}$. Let $S(\preceq)$ be the set of all vectors of $C(\preceq)$ which lie in the hyperplane $H_{\mathbf{p}}$. Let X be a subset of $S(\preceq)$ such that*

- *$X \cap \{\mathbf{s}, -\mathbf{s}\} \neq \emptyset$ for every $\mathbf{s} \in S(\preceq)$.*
- *X is closed under the operation of restricted sum.*

Then $Y = S(\preceq) \setminus X$ may be dropped from $C(\preceq)$, that is $C_Y = C(\preceq) \setminus Y$ is a discrete cone.

Proof. We first note that if $\mathbf{x} \in C(\preceq) \setminus S(\preceq)$ and $\mathbf{y} \in C(\preceq)$, then $\mathbf{x} \oplus \mathbf{y}$, if defined, cannot be in $S(\preceq)$. So due to closedness of X under the restricted addition all axioms of a discrete cone are satisfied for C_Y. On the other hand, if for some two vectors $\mathbf{x}, \mathbf{y} \in X$ we have $\mathbf{x} \oplus \mathbf{y} \in Y$, then C_Y would not be a discrete cone and we would not be able to construct a qualitative probability order associated with this set. $\square$

Example 1 (Positive example). *The probability measure*

$$\mathbf{p} = \frac{1}{16}(6, 4, 3, 2, 1).$$

defines a qualitative probability order $\preceq$ on $[5]$ (which is better written from the other end):

$$\emptyset \prec 5 \prec 4 \prec 3 \prec 45 \prec 35 \sim 2 \prec 25 \sim 34 \prec 1 \prec 345 \sim 24 \prec 23 \sim 15 \prec 245 \prec 14 \sim 235 \ldots.$$

(Here only the first 17 terms are shown, since the remaining ones can be uniquely reconstructed. See (Kraft et al., 1959, Proposition 1) for details). There are only four equivalences here

$$35 \sim 2, \quad 25 \sim 34, \quad 23 \sim 15 \quad \textit{and} \quad 14 \sim 235,$$

and all other follow from them, that is:

$$35 \sim 2 \textit{ implies } 345 \sim 24,\ 135 \sim 12;$$
$$25 \sim 34 \textit{ implies } 125 \sim 134;$$
$$23 \sim 15 \textit{ implies } 234 \sim 145;$$
$$14 \sim 235 \textit{ has no consequences}$$

Let $\mathbf{u}_1 = \chi(2,35) = (0,1,-1,0,-1)$, $\mathbf{u}_2 = \chi(34,25) = (0,-1,1,1,-1)$, $\mathbf{u}_3 = \chi(15,23) = (1,-1,-1,0,1)$ *and* $\mathbf{u}_4 = \chi(235,14) = (-1,1,1,-1,1)$. *Then*

$$S(\preceq) = \{\pm\mathbf{u}_1, \pm\mathbf{u}_2, \pm\mathbf{u}_3, \pm\mathbf{u}_4\}$$

and $X = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ *is closed under the restricted addition as* $\mathbf{u}_i \oplus \mathbf{u}_j$ *is undefined for all* $i \neq j$. *Hence we can subtract from the cone* $C(\preceq)$ *any non-empty subset* Y *of* $-X = \{-\mathbf{u}_1, -\mathbf{u}_2, -\mathbf{u}_3, -\mathbf{u}_4\}$ *and still get a qualitative probability. Since*

$$\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}_4 = \mathbf{0}.$$

it will not be representable. The new order corresponding to the discrete cone C_{-X} *is linear.*

Example 2 (Negative example). *A certain qualitative probability order is associated with the Gabelman game of order 3. Nine players are involved each of whom we think as associated with a certain cell of a* 3×3 *square:*

1	*2*	*3*
4	*5*	*6*
7	*8*	*9*

The ith player is given a positive weight w_i, $i = 1, 2, \ldots, 9$, *such that in the qualitative probability order, associated with* $\mathbf{w} = (w_1, \ldots, w_9)$,

$$147 \sim 258 \sim 369 \sim 123 \sim 456 \sim 789$$

and all other equivalencies are consequences of these. Suppose that we want to construct a qualitative probability order $\preceq$ *for which*

$$147 \sim 258 \sim 369 \prec 123 \sim 456 \sim 789.$$

Then we would like to claim that it is not weighted since for the vectors

$$\mathbf{x}_1 = (0,1,1,-1,0,0,-1,0,0) = \chi(123,147),$$
$$\mathbf{x}_2 = (0,-1,0,1,0,1,0,-1,0) = \chi(456,258),$$
$$\mathbf{x}_3 = (0,0,-1,0,0,-1,1,1,0) = \chi(789,369)$$

we have $\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 = \mathbf{0}$. *Putting the sign* $\prec$ *instead of* $\sim$ *between* 369 *and* 123 *will also automatically imply* $147 \prec 123$, $258 \prec 456$ *and* $369 \prec 789$. *This means that we are dropping the set of vectors* $\{-\mathbf{x}_1, -\mathbf{x}_2, -\mathbf{x}_3\}$ *from the cone while leaving the set* $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ *there. This would not be possible since* $\mathbf{x}_1 \oplus \mathbf{x}_2 = -\mathbf{x}_3$. *So every* $X \supset \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ *with* $X \cap \{-\mathbf{x}_1, -\mathbf{x}_2, -\mathbf{x}_3\} = \emptyset$ *is not closed under* $\oplus$.

4. Simplicial complexes and their cancellation conditions

A subset $\Delta \subseteq 2^{[n]}$ is an *(abstract) simplicial complex* if it satisfies the condition:

$$\text{if } B \in \Delta \text{ and } A \subseteq B, \text{ then } A \in \Delta.$$

Subsets that are in Δ are called *faces*. Abstract simplicial complexes arose from geometric simplicial complexes in topology (e.g., Maunder, 1996). Indeed, for every geometric simplicial complex Δ the set of vertex sets of simplices in Δ is an abstract simplicial complex, also called the *vertex scheme* of Δ. In combinatorial optimization various abstract simplicial complexes associated with finite graphs (Jonsson (2005)) are studied, such as the independence complex, matching complex etc. Abstract simplicial complexes are also in one-to-one correspondence with *simple games* as defined by Neumann & Morgenstern (1944). A simple game is a pair $G = ([n], W)$, where W is a subset of the power set $2^{[n]}$ which satisfies the monotonicity condition:

$$\text{if } X \in W \text{ and } X \subseteq Y \subseteq [n], \text{ then } Y \in W.$$

The subsets from W are called *winning coalitions* and the subsets from $L = 2^{[n]} \setminus W$ are called *losing coalitions*. Obviously the set of losing coalitions L is a simplicial complex. The reverse is also true: if Δ is a simplicial complex, then the set $2^{[n]} \setminus \Delta$ is a set of winning coalitions of a certain simple game.

The most studied class of simplicial complexes is the *threshold* complexes (mostly as an equivalent concept to the concept of a weighted majority game but also as threshold hypergraphs (Reiterman et al., 1985)). A simplicial complex Δ is a threshold complex if there exist non-negative reals $w_1, \ldots, w_n$ and a positive constant q, such that

$$A \in \Delta \Longleftrightarrow w(A) = \sum_{i \in A} w_i < q.$$

The same parameters define a what is known as a *weighted majority game* by setting

$$A \in W \Longleftrightarrow w(A) = \sum_{i \in A} w_i \geq q.$$

This game has the standard notation $[q; w_1, \ldots, w_n]$.

A much larger but still well-understood class of simplicial complexes are *shifted* simplicial complexes (Klivans, 2005, 2007). A simplicial complex is shifted if there exists an order $\trianglelefteq$ on the set of vertices $[n]$ such that for any face F, replacing any of its vertices $x \in F$ with a vertex y such that $y \trianglelefteq x$ results in a subset $(F \setminus \{x\}) \cup \{y\}$ which is also a face. Shifted complexes correspond to complete[2] games (Freixas & Molinero, 2009). A complete game has an order $\trianglelefteq$ on players such that if a coalition W is winning, then replacing any player $x \in W$ with a player $x \trianglelefteq z$ results in a coalition $(W \setminus \{x\}) \cup \{z\}$ which is also winning.

A related concept is the so-called Isbel's desirability relation $\leq_I$ Taylor & Zwicker (1999). Given a game G the relation $\leq_I$ on $[n]$ is defined by setting $j \leq_I i$ if for every set $X \subseteq [n]$ not containing i and j

$$X \cup \{j\} \in W \Longrightarrow X \cup \{i\} \in W. \tag{5}$$

The idea is that if $j \leq_I i$, then i is more desirable as a coalition partner than j. The game is complete iff $\leq_I$ is an order on $[n]$.

Let $\preceq$ be a qualitative probability order on $[n]$ and $T \in 2^{[n]}$. We denote

$$\Delta(\preceq, T) = \{X \subseteq [n] \mid X \prec T\},$$

[2]sometimes also called linear

where $X \prec Y$ stands for $X \preceq Y$ but not $Y \preceq X$, and call it an *initial segment* of $\preceq$. Any initial segment of a qualitative probability order is a shifted simplicial complex. Similarly, the *terminal segment*

$$G(\preceq, T) = \{X \subseteq [n] \mid T \preceq X\}$$

of any qualitative probability order is a complete simple game.

Necessary and sufficient conditions for a simplicial complex to be a threshold complex arise in the similar manner as cancellation conditions for the qualitative probability orders.

Definition 3. *A simplicial complex Δ is said to satisfy CC_k^* if for no $k \geq 2$ there exists a trading transform $(A_1, \ldots, A_k; B_1, \ldots, B_k)$, such that $A_i \in \Delta$ and $B_i \notin \Delta$, for every $i \in [k]$.*

The Theorem 2.4.2 of the book Taylor & Zwicker (1999) can be reformulated to give necessary and sufficient conditions for the simplicial complex to be a threshold.

Theorem 3. *An abstract simplicial complex $\Delta \subseteq 2^{[n]}$ is a threshold complex if and only if the condition CC_k^* holds for all $k \geq 2$.*

Let us show the connection between CC_k and CC_k^*.

Theorem 4. *Suppose $\preceq$ is a qualitative probability order on $2^{[n]}$ and $\Delta(\preceq, T)$ is its initial segment. If $\preceq$ satisfies CC_k then $\Delta(\preceq, T)$ satisfies CC_k^*.*

This gives us some initial properties of initial segments. Since conditions CC_k, $k = 2, 3$, hold for all qualitative probability orders (Fishburn, 1996) we obtain

Theorem 5. *If an abstract simplicial complex $\Delta \subseteq 2^{[n]}$ is an initial segment of some qualitative probability order, then it satisfies CC_k^* for all $k \leq 3$.*

We will now show that for small values of n cancellation condition CC_4^* is satisfied for any initial segment. This will also give us invaluable information on how to construct a non-threshold initial segment later.

Definition 4. *Two pairs of subsets (A_1, B_1) and (A_2, B_2) are said to be compatible if the following two conditions hold:*

$$x \in A_1 \cap A_2 \Longrightarrow x \in B_1 \cup B_2, \text{ and}$$
$$x \in B_1 \cap B_2 \Longrightarrow x \in A_1 \cup A_2.$$

Lemma 2. *Let $\preceq$ be a qualitative probability order on $2^{[n]}$, $T \subseteq [n]$, and let $\Delta = \Delta_n(\preceq, T)$ be the respective initial segment. Suppose $(A_1, \ldots, A_s, B_1, \ldots, B_s)$ is a trading transform and $A_i \prec T \preceq B_j$ for all $i, j \in [s]$. If any two pairs (A_i, B_k) and (A_j, B_l) are compatible, then $\preceq$ fails to satisfy CC_{s-1}.*

Proof. Let us define

$$\bar{A}_i = A_i \setminus (A_i \cap B_k), \qquad \bar{B}_k = B_k \setminus (A_i \cap B_k), \tag{6}$$
$$\bar{A}_j = A_j \setminus (A_j \cap B_l), \qquad \bar{B}_l = B_l \setminus (A_j \cap B_l). \tag{7}$$

We note that

$$\bar{A}_i \cap \bar{A}_j = \bar{B}_k \cap \bar{B}_l = \emptyset. \tag{8}$$

Indeed, suppose, for example, $x \in \bar{A}_i \cap \bar{A}_j$, then also $x \in A_i \cap A_j$ and by the compatibility $x \in B_k$ or $x \in B_l$. In both cases it is impossible for x to be in $x \in \bar{A}_i \cap \bar{A}_j$. We note also that by Lemma 1 we have

$$\bar{A}_i \cup \bar{A}_j \prec \bar{B}_k \cup \bar{B}_l. \tag{9}$$

Now we observe that

$$(\bar{A}_i, \bar{A}_j, A_{m_1}, \ldots, A_{m_{s-2}}; \bar{B}_k, \bar{B}_l, B_{r_1}, \ldots, B_{r_{s-2}}).$$

is a trading transform. Hence, due to (8),

$$(\bar{A}_i \cup \bar{A}_j, A_{m_1}, \ldots, A_{m_{s-2}}; \bar{B}_k \cup \bar{B}_l, B_{r_1}, \ldots, B_{r_{s-2}})$$

is also a trading transform. This violates CC_{s-1} since (9) holds and $A_{m_t} \prec B_{r_t}$ for all $t = 1, \ldots, s-2$. □

By definition of a trading transform we are allowed to use repetitions of the same coalition in it. However we will show that to violate CC_4^* we need a trading transform $(A_1, \ldots, A_4; B_1, \ldots, B_4)$ where all A's and B's are different.

Lemma 3. *Let $\preceq$ be a qualitative probability order on $2^{[n]}$, $T \subseteq [n]$, and let $\Delta = \Delta_n(\preceq, T)$ be the respective initial segment. Suppose $(A_1, \ldots, A_4, B_1, \ldots, B_4)$ is a trading transform and $A_i \prec T \preceq B_j$ for all $i, j \in [4]$. Then*

$$|\{A_1, \ldots, A_4\}| = |\{B_1, \ldots, B_4\}| = 4.$$

Proof. Note that every pair $(A_i, B_j), (A_l, B_k)$ is not compatible. Otherwise by Lemma 2 the order $\preceq$ fails CC_3, which contradicts to the fact that every qualitative probability satisfies CC_3. Assume, to the contrary, that we have at least two identical coalitions among $A_1, \ldots, A_4$ or $B_1, \ldots, B_4$. Without loss of generality we can assume $A_1 = A_2$. Clearly all A's or all B's cannot coincide and there are at least two different A's and two different B's. Suppose $A_1 \neq A_3$ and $B_1 \neq B_2$. The pair $(A_1, B_1), (A_3, B_2)$ is not compatible. It means one of the following two statements is true: either there is $x \in A_1 \cap A_3$ such that $x \notin B_1 \cup B_2$ or there is $y \in B_1 \cap B_2$ such that $y \notin A_1 \cup A_3$. Consider the first case the other one is similar. We know that $x \in A_1 \cap A_3$ and we have at least three copies of x among $A_1, \ldots, A_4$. At the same time $x \notin B_1 \cup B_2$ and there could be at most two copies of x among $B_1, \ldots, B_4$. This is a contradiction. □

Theorem 6. *CC_4^* holds for $\Delta = \Delta_n(\preceq, T)$ for all $n \leq 17$.*

Proof. Let us consider the set of column vectors

$$U = \{\mathbf{x} \in \mathbb{R}^8 \mid x_i \in \{0,1\} \text{ and } x_1 + x_2 + x_3 + x_4 = x_5 + x_6 + x_7 + x_8 = 2\}. \tag{10}$$

This set has an involution $\mathbf{x} \mapsto \bar{\mathbf{x}}$, where $\bar{x}_i = 1 - x_i$. Say, if $\mathbf{x} = (1,1,0,0,0,0,1,1)^T$, then $\bar{\mathbf{x}} = (0,0,1,1,1,1,0,0)^T$. There are 36 vectors from U which are split into 18 pairs $\{\mathbf{x}, \bar{\mathbf{x}}\}$.

Suppose now $\mathcal{T} = (A_1, A_2, A_3, A_4; B_1, B_2, B_3, B_4)$ is a trading transform, $A_i \prec T \preceq B_j$ and no two coalitions in the trading transform coincide. Let us write the characteristic vectors of A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4 as rows of $8 \times n$ matrix M, respectively. Since $\preceq$ satisfies CC_3, by Lemma 2 we know that no two pairs (A_i, B_a) and (A_j, B_b) are compatible. The same can be said about the complementary pair of pairs (A_k, B_c) and (A_l, B_d), where $\{a, b, c, d\} = \{i, j, h, l\} = [4]$. We have

$$A_i \prec B_a,\ A_j \prec B_b,\ A_h \prec B_c,\ A_l \prec B_d,$$

Since (A_i, B_a) and (A_j, B_b) are not compatible one of the following two statements is true: either there exists $x \in A_i \cap A_j$ such that $x \notin B_a \cup B_b$ or there exists $y \in B_a \cap B_b$ such that $x \notin A_i \cup A_j$. As $\mathcal{T}$ is the trading transform in the first case we will also have $x \in B_c \cap B_d$ such that $x \notin A_h \cup A_l$; in the second $y \in A_h \cap A_l$ such that $y \notin B_c \cup B_d$.

Let us consider two columns M_x and M_y of M that corresponds to elements $x, y \in [n]$. The above considerations show that both belong to U and $M_x = \bar{M}_y$.

In particular, if $(i,j,k,l) = (a,b,c,d) = (1,2,3,4)$, then the columns M_x and M_y will be as in the following picture

$$\begin{array}{cccc} & & & \begin{array}{cc} x & y \end{array} \\ M = & \begin{bmatrix} \chi(A_1) \\ \chi(A_2) \\ \chi(A_3) \\ \chi(A_4) \\ \chi(B_1) \\ \chi(B_2) \\ \chi(B_3) \\ \chi(B_4) \end{bmatrix} & = & \left[\begin{array}{cc} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ \hline 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{array}\right] \end{array}$$

(we emphasize however that we have only one such column in the matrix, not both). We saw that one pairing of indices $(i,a), (j,b), (k,c), (k,d)$ gives us a column from one of the 18 pairs of U. It is easy to see that a vector from every pair of U can be obtained by the appropriate choice of the pairing of indices. This means that the matrix contains at least 18 columns. That is $n \geq 18$. $\square$

5. An example of a nonthreshold initial segment of a linear qualitative probability order

In this section we shall construct an almost representable linear qualitative probability order $\sqsubseteq$ on $2^{[26]}$ and a subset $T \subseteq [26]$, such that the initial segment $\Delta(\sqsubseteq, T)$ of $\sqsubseteq$ is not a threshold complex as it fails to satisfy the condition CC_4^*.

The idea of the example is as follows. We will start with a representable linear qualitative probability order $\preceq$ on $[18]$ defined by weights $w_1, \ldots, w_{18}$ and extend it to a representable but nonlinear qualitative probability order $\preceq'$ on $[26]$ with weights $w_1, \ldots, w_{26}$. A distinctive feature of $\preceq'$ will be the existence of eight sets $A_1', \ldots, A_4'$, $B_1', \ldots, B_4'$ in $[26]$ such that:

(1) The sequence $(A_1', \ldots, A_4'; B_1', \ldots, B_4')$ is a trading transform.
(2) The sets $A_1', \ldots, A_4'$, $B_1', \ldots, B_4'$ are tied in $\preceq'$, that is,

$$A_1' \sim' \ldots A_4' \sim' B_1' \sim' \ldots \sim' B_4'.$$

(3) If any two distinct sets $X, Y \subseteq [26]$ are tied in $\preceq'$, then $\chi(X,Y) = \chi(S,T)$, where $S, T \in \{A_1', \ldots, A_4', B_1', \ldots, B_4'\}$. In other words all equivalences in $\preceq'$ are consequences of $A_i' \sim' A_j'$, $A_i' \sim' B_j'$, $B_i' \sim' B_j'$, where $i, j \in [4]$.

Then we will use Theorem 2 to untie the eight sets and to construct a comparative probability order $\sqsubseteq$ for which

$$A_1' \sqsubset A_2' \sqsubset A_3' \sqsubset A_4' \sqsubset B_1' \sqsubset B_2' \sqsubset B_3' \sqsubset B_4',$$

where $X \sqsubset Y$ means that $X \sqsubseteq Y$ is true but not $Y \sqsubseteq X$.

This will give us an initial segment $\Delta(\sqsubseteq, A_4)$ of the linear qualitative probability order $\sqsubseteq$, which is not threshold since CC_4^* fails to hold.

Let $\preceq$ be a representable linear qualitative probability order on $2^{[18]}$ with weights $w_1, \ldots, w_{18}$ that are linearly independent (over $\mathbb{Z}$) real numbers in the interval $[0, 1]$. Due to the choice of weights, no two distinct subsets $X, Y \subseteq [18]$ have equal weights relative to this system of weights, i.e.,

$$X \neq Y \Longrightarrow w(X) = \sum_{i \in X} w_i \neq w(Y) = \sum_{i \in Y} w_i.$$

Let us consider again the set U defined in (10). Let M be a subset of U with the following properties: $|M| = 18$ and $\mathbf{x} \in M$ if and only if $\bar{\mathbf{x}} \notin M$. In other words M contains exactly one vector from every pair into which U is split. By M we will also denote an 8×18 matrix whose columns are all the vectors from M taken in arbitrary order. By $A_1, \ldots, A_4, B_1, \ldots, B_4$ we denote the sets with characteristic vectors equal to the rows $M_1, \ldots, M_8$ of M, respectively. The way M was constructed secures that the following lemma is true.

Lemma 4. *The subsets $A_1, \ldots, A_4, B_1, \ldots, B_4$ s of $[18]$ satisfy:*

(1) *$(A_1, \ldots, A_4; B_1, \ldots, B_4)$ is a trading transform;*
(2) *for any choice of $i, k, j, m \in [4]$ with $i \neq k$ and $j \neq m$ the pair $(A_i, B_j), (A_k, B_m)$ is not compatible.*

We shall now embed $A_1, \ldots, A_4, B_1, \ldots, B_4$ into $[26]$ and add new elements to them forming $A_1', \ldots, A_4'$, $B_1', \ldots, B_4'$ in such a way that the characteristic vectors $\chi(A_1'), \ldots, \chi(A_4'), \chi(B_1'), \ldots, \chi(A_1')$ are the rows $M_1', \ldots, M_8'$ of the following matrix

$$M' = \begin{array}{c} \begin{array}{c|c|c} 1\ldots18 & 19\;\;20\;\;21\;\;22 & 23\;\;24\;\;25\;\;26 \end{array} \\ \left[\begin{array}{c|c|c} \chi(A_1) & & \\ \chi(A_2) & I & I \\ \chi(A_3) & & \\ \chi(A_4) & & \\ \hline \chi(B_1) & & \\ \chi(B_2) & J & I \\ \chi(B_3) & & \\ \chi(B_4) & & \end{array}\right] \end{array}, \tag{11}$$

respectively. Here I is the 4×4 identity matrix and

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Note that if X belongs to $[18]$, it also belongs to $[26]$, so the notation $\chi(X)$ is ambiguous as it may be a vector from $\mathbb{Z}^{18}$ or from $\mathbb{Z}^{26}$, depending on the circumstances. However the reference set will be always clear from the context and the use of this notation will create no confusion.

One can see that $(A_1', \ldots, A_4'; B_1', \ldots, B_4')$ is again a trading transform and there are no compatible pairs $(A_i', B_j'), (A_k', B_m')$, where $i, k, j, m \in [4]$ and $i \neq k$ or $j \neq m$. We shall now choose weights $w_{19}, \ldots, w_{26}$ of new elements $19, \ldots, 26$ in such a way that the sets $A_1', A_2', A_3', A_4', B_1', B_2', B_3', B_4'$ all have the same weight N, which is a sufficiently large number. It will be clear from the proof how large it should be.

To find weights $w_{19}, \ldots, w_{26}$ that satisfy this condition we need to solve the following system of linear equations

$$\begin{pmatrix} I & I \\ J & I \end{pmatrix} \begin{pmatrix} w_{19} \\ \vdots \\ w_{26} \end{pmatrix} = N\mathbf{1} - M \cdot \mathbf{w}, \tag{12}$$

where $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^8$ and $\mathbf{w} = (w_1, \ldots, w_{18})^T \in \mathbb{R}^{18}$.

The matrix from (12) has rank 7, and the augmented matrix of the system has the same rank. Therefore, the solution set is not empty, moreover, there is one free variable (and any one can be chosen for this role). Let this free variable be w_{26} and let us give it value K, such that K is large but much smaller than N. In particular, $126 < K < N$. Now we can express all other weights $w_{19}, ..., w_{25}$ in terms of $w_{26} = K$ as follows:

$$\begin{aligned}
w_{19} &= N{-}K - (\chi(A_4) - \chi(B_1) + \chi(A_1)) \cdot \mathbf{w} \\
w_{20} &= N{-}K - (\chi(A_4) - \chi(B_1) + \chi(A_1) - \chi(B_2) + \chi(A_2)) \cdot \mathbf{w} \\
w_{21} &= N{-}K - (\chi(A_4) - \chi(B_1) + \chi(A_1) - \chi(B_2) + \chi(A_2) - \\
&\qquad \chi(B_3) + \chi(A_3)) \cdot \mathbf{w} \\
w_{22} &= N{-}K - \chi(A_4) \cdot \mathbf{w} \\
w_{23} &= K{-}(-\chi(A_4) + \chi(B_1)) \cdot \mathbf{w} \\
w_{24} &= K{-}(-\chi(A_4) + \chi(B_1) - \chi(A_1) + \chi(B_2)) \cdot \mathbf{w} \\
w_{25} &= K{-}(-\chi(A_4) + \chi(B_1) - \chi(A_1) + \chi(B_2) - \chi(A_2) + \chi(B_3)) \cdot \mathbf{w}.
\end{aligned} \tag{13}$$

By choice of N and K weights $w_{19}, ..., w_{25}$ are positive. Indeed, all "small" terms in the right-hand-side of (13) are strictly less then $7 \cdot 18 = 126 < \min\{K, N-K\}$.

Let $\preceq'$ be the representable qualitative probability order on $[26]$ defined by the weight vector $\mathbf{w}' = (w_1, \ldots, w_{26})$. Using $\preceq'$ we would like to construct a linear qualitative probability order $\sqsubseteq$ on $2^{[26]}$ that ranks the subsets A'_i and B'_j in the sequence

$$A'_1 \sqsubset A'_2 \sqsubset A'_3 \sqsubset A'_4 \sqsubset B'_1 \sqsubset B'_2 \sqsubset B'_3 \sqsubset B'_4. \tag{14}$$

We will make use of Theorem 2 now. Let $H_{\mathbf{w}'} = \{x \in \mathbb{R}^n | (\mathbf{w}', x) = 0\}$ be the hyperplane with the normal vector $\mathbf{w}'$ and $S(\preceq')$ be the set of all vectors of the respective discrete cone $C(\preceq')$ that lie in $H_{\mathbf{w}'}$. Suppose

$$X' = \{\chi(C, D) \mid C, D \in \{A'_1, \ldots, A'_4, B'_1, \ldots, B'_4\} \text{ and } D \text{ earlier than } C \text{ in (14)}\}.$$

This is a subset of T^{26}, where $T = \{-1, 0, 1\}$. Let also $Y' = S(\preceq') \setminus X'$. To use Theorem 2 with the goal to achieve (14) we need to show, that

- $S(\preceq') = X' \cup -X'$ and
- X' is closed under the operation of restricted sum.

If we could prove this, then $C(\sqsubseteq) = C(\preceq') \setminus Y'$ is a discrete cone of a linear qualitative probability order $\sqsubseteq$ on $[26]$ satisfying (14). Then the initial segment $\Delta(\sqsubseteq, B'_1)$ will not be a threshold complex, because the condition CC_4^* will fail for it.

Let Y be one of the sets $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4$. By $\breve{Y}$ we will denote the corresponding superset of Y from the set $\{A'_1, A'_2, A'_3, A'_4, B'_1, B'_2, B'_3, B'_4\}$.

Proposition 3. *The subset*

$$X = \{\chi(C, D) \mid C, D \in \{A_1, \dots, A_4, B_1, \dots, B_4\} \text{ with } \breve{D} \text{ earlier than } \breve{C} \text{ in (14)}\}.$$

of T^{18} *is closed under the operation of restricted sum.*

Proof. Let $\mathbf{u}$ and $\mathbf{v}$ be any two vectors in X. As we will see the restricted sum $\mathbf{u} \oplus \mathbf{v}$ is almost always undefined. Without loss of generality we can consider only five cases.

Case 1. $\mathbf{u} = \chi(B_i, A_j)$ and $\mathbf{v} = \chi(B_k, A_m)$, where $i \neq k$ and $j \neq m$. In this case by Lemma 4 the pairs (B_i, A_j) and (B_k, A_m) are not compatible. It means that there exists $p \in [18]$ such that either $p \in B_i \cap B_k$ and $p \notin A_j \cup A_m$ or $p \in A_j \cap A_m$ and $p \notin B_i \cup B_k$. The vector $\mathbf{u} + \mathbf{v}$ has 2 or -2 at pth position and $\mathbf{u} \oplus \mathbf{v}$ is undefined. This is illustrated in the table below:

	$\chi(B_i)$	$\chi(B_k)$	$\chi(A_j)$	$\chi(A_m)$	$\chi(B_i, A_j)$	$\chi(B_k, A_m)$	$\mathbf{u}+\mathbf{v}$
pth	1	1	0	0	1	1	2
coordinate	0	0	1	1	-1	-1	-2

Case 2. $\mathbf{u} = \chi(B_i, A_j)$, $\mathbf{v} = \chi(B_i, A_m)$ or $\mathbf{u} = \chi(B_j, A_i)$, $\mathbf{v} = \chi(B_m, A_i)$, where $j \neq m$. In this case choose $k \in [4] \setminus \{i\}$. Then the pairs (B_i, A_j) and (B_k, A_m) are not compatible. As above, the vector $\chi(B_i, A_j) + \chi(B_k, A_m)$ has 2 or -2 at some position p. Suppose $p \in B_i \cap B_k$ and $p \notin A_j \cup A_m$. Then B_i has a 1 in pth position and each of the vectors $\chi(B_i, A_j)$ and $\chi(B_i, A_m)$ has a 1 in pth position as well. Therefore, $\mathbf{u} \oplus \mathbf{v}$ is undefined because $\mathbf{u} + \mathbf{v}$ has 2 in pth position. Similarly, in the case when $p \in A_j \cap A_m$ and $p \notin B_i \cup B_k$ the pth coordinate of $\mathbf{u} + \mathbf{v}$ is -2. The case when $\mathbf{u} = \chi(B_j, A_i)$ and $\mathbf{v} = \chi(B_m, A_i)$ is similar.

Case 3. $\mathbf{u} = \chi(B_i, B_j)$, $\mathbf{v} = \chi(B_k, B_m)$ or $\mathbf{u} = \chi(A_i, A_j)$, $\mathbf{v} = \chi(A_k, A_m)$, where $\{i, j, k, m\} = [4]$. By construction of M there exists $p \in [18]$ such that $p \in B_i \cap B_k$ and $p \notin B_j \cup B_m$ or $p \notin B_i \cup B_k$ and $p \in B_j \cap B_m$. So there is $p \in [18]$, such that $\mathbf{u} + \mathbf{v}$ has 2 or -2 in pth position. Thus $\mathbf{u} \oplus \mathbf{v}$ is undefined.

Case 4. $\mathbf{u} = \chi(B_i, B_j)$, $\mathbf{v} = \chi(B_k, B_m)$ or $\mathbf{u} = \chi(A_i, A_j)$, $\mathbf{v} = \chi(A_k, A_m)$, where $i = k$ or $j = m$. If $i = k$ and $j = m$, then $\mathbf{u} \oplus \mathbf{v}$ is undefined. Consider the case $i = k$, $j \neq m$ and $\mathbf{u} = \chi(B_i, B_j)$, $\mathbf{v} = \chi(B_i, B_m)$. Let $s = [4] \setminus \{i, j, m\}$. By construction of M either we have $p \in [18]$ such that $p \in B_i \cap B_s$ and $p \notin B_j \cup B_m$ or $p \notin B_i \cup B_s$ and $p \in B_j \cap B_m$. In both cases $\mathbf{u} + \mathbf{v}$ has 2 or -2 in position p.

Case 5. $\mathbf{u} = \chi(B_i, B_j)$, $\mathbf{v} = \chi(B_k, B_m)$ or $\mathbf{u} = \chi(A_i, A_j)$, $\mathbf{v} = \chi(A_k, A_m)$, where $j = k$ or $i = m$. Suppose $j = k$. Since $i > j$ and $j > m$ we have $i > m$. This implies that $\chi(B_i, B_m)$ belongs to X. On the other hand $\mathbf{u}+\mathbf{v} = \chi(B_i) - \chi(B_m) = \chi(B_i, B_m)$. Therefore $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v} \in X$. □

Corollary 1. *X' is closed under restricted sum.*

Proof. We will have to consider the same five cases as in the Proposition 3. As above in the first four cases the restricted sum of vectors will be undefined. In the fifth case, when $\mathbf{u} = \chi(B'_i, B'_j)$, $\mathbf{v} = \chi(B'_k, B'_m)$ or $\mathbf{u} = \chi(A'_i, A'_j)$, $\mathbf{v} = \chi(A'_k, A'_m)$, where $j = k$ or $i = m$, we will have $\mathbf{u} + \mathbf{v} = \chi(B'_i) - \chi(B'_m) = \chi(B'_i, B'_m) \in X'$ or $\mathbf{u} + \mathbf{v} = \chi(A'_i) - \chi(A'_m) = \chi(A'_i, A'_m) \in X'$. □

To satisfy conditions of Theorem 2 we need also to show that the intersection of the discrete cone $C(\preceq')$ and the hyperplane $H_{\mathbf{w}'}$ equals to $X' \cup -X'$. More explicitly we need to prove the following:

Proposition 4. *Suppose $C, D \subseteq [26]$ are tied in $\preceq'$, that is $C \preceq' D$ and $D \preceq' C$. Then $\chi(C, D) \in X' \cup -X'$.*

Proof. Assume to the contrary that there are two sets $C, D \in 2^{[26]}$ that have equal weights with respect to the corresponding system of weights defining $\preceq'$ but $\chi(C, D) \notin X' \cup -X'$. The sets C and D have to contain some of the elements from $[26] \setminus [18]$ since $w_1, \ldots, w_{18}$ are linearly independent. Thus $C = C_1 \cup C_2$ and $D = D_1 \cup D_2$, where $C_1, D_1 \subseteq [18]$ and $C_2, D_2 \subseteq [26] \setminus [18]$ with C_2 and D_2 being nonempty. We have

$$0 = \chi(C, D) \cdot \mathbf{w}' = \chi(C_1, D_1) \cdot \mathbf{w} + \chi(C_2, D_2) \cdot \mathbf{w}^+,$$

where $\mathbf{w}^+ = (w_{19}, \ldots, w_{26})^T$. By (13), we can express weights $w_{19}, \ldots, w_{26}$ as linear combinations with integer coefficients of N, K and $w_1, \ldots, w_{18}$ obtaining

$$\chi(C_2, D_2) \cdot \mathbf{w}^+ = \left(\sum_{i=1}^{4} \gamma_i \chi(A_i) + \sum_{i=1}^{4} \gamma_{4+i} \chi(B_i) \right) \cdot \mathbf{w} + \beta_1 N + \beta_2 K,$$

where $\gamma_i, \beta_j \in \mathbb{Z}$.

Clearly the expression in the bracket on the right-hand-side is just a vector with integer entries. Let us denote it α. Then

$$\chi(C_2, D_2) \cdot \mathbf{w}^+ = \alpha \cdot \mathbf{w} + \beta_1 N + \beta_2 K, \tag{15}$$

where $\alpha \in \mathbb{Z}^{18}$. We can now write $\chi(C, D) \cdot \mathbf{w}'$ in terms of $\mathbf{w}, K$ and N:

$$0 = \chi(C, D) \cdot \mathbf{w}' = (\chi(C_1, D_1) + \alpha) \cdot \mathbf{w} + \beta_1 N + \beta_2 K.$$

We recap that K was chosen to be much greater then $\sum_{i \in [18]} w_i$ and N is much greater then K. So if β_1, β_2 are different from zero then $|\beta_1 N + \beta_2 K|$ is a very big number, which cannot be canceled out by $(\chi(C_1, D_1) + \alpha) \cdot \mathbf{w}$. Weights $w_1, \ldots, w_{18}$ are linearly independent, so for arbitrary $\mathbf{b} \in Z^{18}$ the dot product $\mathbf{b} \cdot \mathbf{w}$ can be zero if and only if $\mathbf{b} = \mathbf{0}$. Hence

$$w(C) = w(D) \text{ iff } \chi(C_1, D_1) = -\alpha \text{ and } \beta_1 = 0, \beta_2 = 0.$$

Taking into account that $\chi(C_1, D_1)$ is a vector from T^{18}, we get

$$\alpha \notin T^{18} \Longrightarrow w(C) \neq w(D). \tag{16}$$

We need the following two claims to finish the proof, their proofs are delegated to the next section.

Claim 1. *Suppose $\chi(C_1, D_1)$ belongs to $X \cup -X$. Then $\chi(C, D)$ belongs to $X' \cup -X'$.*

Claim 2. *If $\alpha \in T^{18}$, then α belongs to $X \cup -X$.*

Now let us show how with the help of these two claims the proof of Proposition 4 can be completed. The sets C and D have the same weight and this can happen only if α is a vector in T^{18}. By Claim 2 $\alpha \in X \cup -X$. The characteristic vector $\chi(C_1, D_1)$ is equal to $-\alpha$, hence $\chi(C_1, D_1) \in X \cup -X$. By Claim 1 we get $\chi(C, D) \in X' \cup -X'$, a contradiction. $\square$

Theorem 7. *There exists a linear qualitative probability order $\sqsubseteq$ on $[26]$ and $T \subset [26]$ such that the initial segment $\Delta(\sqsubseteq, T)$ is not a threshold complex.*

Proof. By Corollary 1 and Proposition 4 all conditions of Theorem 2 are satisfied. Therefore $C(\preceq')\setminus(-X')$ is a discrete cone $C(\sqsubseteq)$, where $\sqsubseteq$ is a almost representable linear qualitative probability order. By construction $A_1' \sqsubset A_2' \sqsubset A_3' \sqsubset A_4' \sqsubset B_1' \sqsubset B_2' \sqsubset B_3' \sqsubset B_4'$ and thus $\Delta(\sqsubseteq, B_1')$ is an initial segment, which is not a threshold complex. □

Note that we have a significant degree of freedom in constructing such an example. The matrix M can be chosen in 2^{18} possible ways and we have not specified the linear qualitative probability order $\preceq$.

6. Proofs of Claim 1 and Claim 2

Lets fix some notation first. Suppose $\mathbf{b} \in \mathbb{Z}^k$ and $\mathbf{x}_i \in \mathbb{Z}^n$ for $i \in [k]$. Then we define the product

$$\mathbf{b}\cdot(\mathbf{x}_1,\ldots,\mathbf{x}_k) = \sum_{i\in[k]} b_i\mathbf{x}_i.$$

It resembles the dot product (the difference is that the second argument is a sequence of vectors) and is denoted in the same way. For a sequence of vectors $(\mathbf{x}_1,\ldots,\mathbf{x}_k)$ we also define $(\mathbf{x}_1,\ldots,\mathbf{x}_k)_p = (\mathbf{x}_1^{(p)},\ldots,\mathbf{x}_k^{(p)})$, where $\mathbf{x}_i^{(j)}$ is the jth coordinate of vector $\mathbf{x}_i$.

We start with the following lemma.

Lemma 5. *Let* $\mathbf{b} \in \mathbb{Z}^6$. *Then*

$$\mathbf{b}\cdot(\chi(B_1,A_4),\chi(B_2,A_1),\chi(B_3,A_2),\chi(A_2,A_1),\chi(A_3,A_1),\chi(A_4,A_1)) = \mathbf{0}$$

if and only if $\mathbf{b} = \mathbf{0}$.

Proof. We know that the pairs (B_1, A_4) and (B_2, A_1) are not compatible. So there exists an element p that lies in the intersection $B_1\cap B_2$ (or $A_1\cap A_4$), but $p \notin A_4\cup A_1$ ($p \notin B_1 \cup B_2$, respectively). We have exactly two copies of every element among $A_1,\ldots,A_4$ and $B_1,\ldots,B_4$. Thus, the element p belongs to $A_2\cap A_3$ ($B_3\cap B_4$) and doesn't belong to $B_3 \cup B_4$ ($A_2 \cup A_3$). The following table illustrates this:

	$\chi(A_1)$	$\chi(A_2)$	$\chi(A_3)$	$\chi(A_4)$	$\chi(B_1)$	$\chi(B_2)$	$\chi(B_3)$	$\chi(B_4)$
pth	0	1	1	0	1	1	0	0
coordinate	1	0	0	1	0	0	1	1

Then at pth position we have

$$(\chi(B_1,A_4),\chi(B_2,A_1),\chi(B_3,A_2),\chi(A_2,A_1),\chi(A_3,A_1),\chi(A_4,A_1))_p = \pm(1,1,-1,1,1,0)$$

and hence

$$b_1 + b_2 - b_3 + b_4 + b_5 = 0.$$

From the fact that other pairs are not compatible we can get more equations relating $b_1, \ldots, b_6$:

$$\begin{array}{cll}
b_1 - b_2 + b_3 - b_4 - b_6 = 0 & \text{from} & (B_1,A_4),(B_3,A_2);\\
-b_1 + b_2 + b_3 + b_5 + b_6 = 0 & \text{from} & (B_1,A_4),(B_4,A_3);\\
b_2 + b_5 + b_6 = 0 & \text{from} & (B_1,A_1),(B_2,A_2);\\
b_4 + b_6 = 0 & \text{from} & (B_1,A_1),(B_3,A_3);\\
b_3 + b_5 + b_6 = 0 & \text{from} & (B_1,A_1),(B_3,A_2).
\end{array}$$

The obtained system of linear equations has only the zero solution. □

Lemma 6. *Let* $\mathbf{a} = (a_1, \ldots, a_8)$ *be a vector in* $\mathbb{Z}^8$ *whose every coordinate* a_i *has absolute value which is at most* 100. *Then* $\mathbf{a} \cdot \mathbf{w}^+ = 0$ *if and only if* $\mathbf{a} = \mathbf{0}$.

Proof. We first rewrite (13) in more convenient form:

$$
\begin{aligned}
w_{19} &= N - K - (-\chi(B_1, A_4) + \chi(A_1)) \cdot \mathbf{w} \\
w_{20} &= N - K - (-\chi(B_1, A_4) - \chi(B_2, A_1) + \chi(A_2)) \cdot \mathbf{w} \\
w_{21} &= N - K - (-\chi(B_1, A_4) - \chi(B_2, A_1) - \chi(B_3, A_2) + \chi(A_3)) \cdot \mathbf{w} \\
w_{22} &= N - K - \chi(A_4) \cdot \mathbf{w} \\
w_{23} &= K - \chi(B_1, A_4) \cdot \mathbf{w} \\
w_{24} &= K - (\chi(B_1, A_4) + \chi(B_2, A_1)) \cdot \mathbf{w} \\
w_{25} &= K - (\chi(B_1, A_4) + \chi(B_2, A_1) + \chi(B_3, A_2)) \cdot \mathbf{w} \\
w_{26} &= K
\end{aligned}
\tag{17}
$$

We calculate the dot product $\mathbf{a} \cdot \mathbf{w}^+$ substituting the values of $w_{19}, \ldots, w_{26}$ from (17):

$$
\begin{aligned}
0 = \mathbf{a} \cdot \mathbf{w}^+ = N \sum_{i \in [4]} a_i - K &\left(\sum_{i \in [4]} a_i - \sum_{i \in [4]} a_{4+i} \right) \\
&- \Bigg[\chi(B_1, A_4) \left(\sum_{i=5}^{7} a_i - \sum_{i=1}^{3} a_i \right) + \chi(B_2, A_1) \left(\sum_{i=6}^{7} a_i - \sum_{i=2}^{3} a_i \right) \\
&+ \chi(B_3, A_2)(-a_3 + a_7) + \sum_{i \in [4]} a_i \chi(A_i) \Bigg] \cdot \mathbf{w}.
\end{aligned}
\tag{18}
$$

The numbers N and K are very big and $\sum_{i \in [18]} w_i$ is small. Also $|a_i| \le 100$. Hence the three summands cannot cancel each other. Therefore $\sum_{i \in [4]} a_i = 0$ and $\sum_{i \in [4]} a_{4+i} = 0$. The expression in the square brackets should be zero because the coordinates of $\mathbf{w}$ are linearly independent.

We know that $a_1 = -a_2 - a_3 - a_4$, so the expression in the square brackets in (18) can be rewritten in the following form:

$$
\begin{aligned}
&b_1 \chi(B_1, A_4) + b_2 \chi(B_2, A_1) + b_3 \chi(B_3, A_2) + \\
&a_2 \chi(A_2, A_1) + a_3 \chi(A_3, A_1) + a_4 \chi(A_4, A_1),
\end{aligned}
\tag{19}
$$

where $b_1 = \sum_{i=5}^{7} a_i - \sum_{i=1}^{3} a_i$, $b_2 = \sum_{i=6}^{7} a_i - \sum_{i=2}^{3} a_i$ and $b_3 = a_7 - a_3$.

By Lemma 5 we can see that expression (19) is zero iff $b_1 = 0$, $b_2 = 0, b_3 = 0$ and $a_2 = 0, a_3 = 0, a_4 = 0$ and this happens iff $\mathbf{a} = \mathbf{0}$. □

Proof of Claim 1. Assume to the contrary that $\chi(C_1, D_1) \in X \cup -X$ and $\chi(C, D)$ does not belong to $X' \cup -X'$. Consider $\chi(\breve{C}_1, \breve{D}_1) \in X' \cup -X'$. We know that the weight of C is the same as the weight of D, and also that the weight of $\breve{C}_1$ is the same as the weight of $\breve{D}_1$. This can be written as

$$
\begin{aligned}
&\chi(C_1, D_1) \cdot \mathbf{w} + \chi(C_2, D_2) \cdot \mathbf{w}^+ = 0, \\
&\chi(C_1, D_1) \cdot \mathbf{w} + \chi(\breve{C}_1 \setminus C_1, \breve{D}_1 \setminus D_1) \cdot \mathbf{w}^+ = 0.
\end{aligned}
$$

We can now see that

$$(\chi(\breve{C}_1 \setminus C_1, \breve{D}_1 \setminus D_1) - \chi(C_2, D_2)) \cdot \mathbf{w}^+ = 0.$$

The left-hand-side of the last equation is a linear combination of weights $w_{19}, \dots, w_{26}$. Due to Lemma 6 we conclude from here that

$$\chi(\breve{C}_1 \setminus C_1, \breve{D}_1 \setminus D_1) - \chi(C_2, D_2) = \mathbf{0}.$$

But this is equivalent to $\chi(C, D) = \chi(\breve{C}_1, \breve{D}_1) \in X$, which is a contradiction. □

Proof of Claim 2. We remind the reader that α was defined in (15). Sets C and D has the same weight and we established that $\beta_1 = \beta_2 = 0$. So

$$\chi(C_2, D_2) \cdot \mathbf{w}^+ = \alpha \cdot \mathbf{w}.$$

If we look at the representation of the last eight weights in (17), we note that the weights w_{19}, w_{20}, w_{21}, w_{22} are much heavier than the weights w_{23}, w_{24}, w_{25}, w_{26}. Hence $w(C) = w(D)$ implies

$$\begin{aligned} |C_2 \cap \{19, 20, 21, 22\}| &= |D_2 \cap \{19, 20, 21, 22\}| \text{ and} \\ |C_2 \cap \{23, 24, 25, 26\}| &= |D_2 \cap \{23, 24, 25, 26\}|. \end{aligned} \tag{20}$$

That is C and D have equal number of super-heavy weights and equal number of heavy ones.

Without loss of generality we can assume that $C_2 \cap D_2$ is empty. Similar to derivation in the proof of Lemma 6, the vector α can be expressed as

$$\alpha = a_1\chi(B_1, A_4) + a_2\chi(B_2, A_1) + a_3\chi(B_3, A_2) + \sum_{i \in [4]} b_i\chi(A_i) \tag{21}$$

for some $a_i, b_j \in \mathbb{Z}$. The characteristic vectors $\chi(A_1), \dots, \chi(A_4)$ participate in the representations of super-heavy elements $w_{19}, \dots, w_{22}$ only. Hence $b_i = 1$ iff element $18 + i \in C_2$ and $b_i = -1$ iff element $18 + i \in D_2$. Without loss of generality we can assume that $C_2 \cap D_2 = \emptyset$. By (20) we can see that if C_2 contains some super-heavy element $p \in \{19, \dots, 22\}$ with $\chi(A_k)$, $k \in [4]$, in the representation of w_p, then D_2 has a super-heavy $q \in \{19, \dots, 22\}$, $q \neq p$ with $\chi(A_t), t \in [4] \setminus \{k\}$ in representation of w_q. In such case $b_k = -b_t = 1$ and

$$b_k\chi(A_k) + b_t\chi(A_t) = \chi(A_k, A_t).$$

By (20) the number of super-heavy element in C_2 is the same as the number of super-heavy elements in D_2. Therefore (21) can be rewritten in the following way:

$$\alpha = a_1\chi(B_1, A_4) + a_2\chi(B_2, A_1) + a_3\chi(B_3, A_2) + a_4\chi(A_i, A_p) + a_5\chi(A_k, A_t), \tag{22}$$

where $a_1, a_2, a_3 \in \mathbb{Z}$; $a_4, a_5 \in \{0, 1\}$ and $\{i, k, t, p\} = [4]$.

Now the series of technical facts will finish the proof.

Fact 1. *Suppose* $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{Z}^3$ *and* $|\{i, k, t\}| = |\{j, m, s\}| = 3$. *Then*

$$a_1\chi(B_j, A_i) + a_2\chi(B_m, A_k) + a_3\chi(B_s, A_t) \in T^{18}$$

if and only if

$$\mathbf{a} \in \{(0, 0, 0),\ (\pm 1, 0, 0),\ (0, \pm 1, 0),\ (0, 0, \pm 1),\ (1, 1, 1),\ (-1, -1, -1)\}. \tag{23}$$

Proof. The pairs $((B_j, A_i), (B_m, A_k))$, $((B_j, A_i), (B_s, A_t))$ and $((B_m, A_k), (B_s, A_t))$ are not compatible. Using the same technique as in the proofs of Proposition 3 and Lemma 5 and watching a particular coordinate we get

$$(a_1 + a_2 - a_3),\ (a_1 - a_2 + a_3),\ (-a_1 + a_2 + a_3) \in T,$$

respectively. The absolute value of the sum of every two of these terms is at most two. Add the first term to the third. Then $|2a_2| \le 2$ or, equivalently, $|a_2| \le 1$. In a similar way we can show that $|a_3| \le 1$ and $|a_1| \le 1$. The only vectors that satisfy all the conditions above are those listed in (23). □

Fact 2. *Suppose* $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{Z}^3$ *and* $|\{i,k,t\}| = |\{j,m,s\}| = 3$. *Then*

$$a_1\chi(B_j, A_i) + a_2\chi(B_m, A_k) + a_3\chi(B_s, A_t) + \chi(A_k, A_t) \in T^{18}$$

if and only if

$$\mathbf{a} \in \{(0,0,0),\ (0,1,0),\ (0,0,-1),\ (0,1,-1)\}. \tag{24}$$

Proof. Considering non-compatible pairs $((B_m, A_k), (B_s, A_t))$, $((B_j, A_i), (B_m, A_k))$, $((B_j, A_i), (B_s, A_t))$, $((B_j, A_k), (B_s, A_i))$, $((B_j, A_t), (B_m, A_i))$, we get the inclusions

$$(-a_1 + a_2 + a_3),\ (a_1 + a_2 - a_3 - 1),\ (a_1 - a_2 + a_3 + 1),\ (a_1 - 1),\ (a_1 + 1) \in T,$$

respectively. We can see that $|2a_2 - 1| \le 2$ and $|2a_3 + 1| \le 2$ and $a_1 = 0$. So a_2 can be only 0 or 1 and a_3 can have values -1 or 0. □

Fact 3. *Suppose* $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{Z}^3$ *and* $\{i,k,t,p\} = [4]$ *and* $|\{j,m,s\}| = 3$. *Then*

$$a_1\chi(B_j, A_i) + a_2\chi(B_m, A_k) + a_3\chi(B_s, A_t) + \chi(A_i, A_p) \in T^{18}$$

if and only if

$$a \in \{(0,0,0),\ (1,0,0),\ (1,1,1),\ (2,1,1)\}.$$

Proof. Let $\ell \in [4] \setminus \{j,m,s\}$. From consideration of the following non-compatible pairs

$((B_j, A_i), (B_m, A_k))$, $((B_j, A_i), (B_s, A_t))$, $((B_m, A_k), (B_s, A_t))$, $((B_j, A_i), (B_m, A_t))$, $((B_j, A_i), (B_m, A_p))$, $((B_j, A_i), (B_s, A_p))$, $((B_s, A_t), (B_\ell, A_i))$

we get the following inclusions

$$\begin{gathered}(a_1 + a_2 - a_3 - 1),\ (a_1 - a_2 + a_3 - 1),\ (-a_1 + a_2 + a_3),\\ (a_1 - 1),\ (a_1 - a_3),\ (a_1 - a_2),\ (a_2 - a_3 + 1) \in T,\end{gathered}$$

respectively. So we have $|2a_3 - 1| \le 2$ (from the second and the third inclusions) and $|2a_2 - 1| \le 2$ (from the first and the third inclusions) from which we immediately get $a_2, a_3 \in \{1, 0\}$. We also get $a_1 \in \{2,1,0\}$ (by the forth inclusion).

- If $a_1 = 2$, then by the fifth and sixth inclusions $a_3 = 1$ and $a_2 = 1$.
- If $a_1 = 1$, then a_2 can be either zero or one. If $a_2 = 0$ then we have $\chi(B_j, A_i) + a_3\chi(B_s, A_t) + \chi(A_i, A_p) = \chi(B_j, A_p) + a_3\chi(B_s, A_t)$. By Fact 1, a_3 can be zero only. On the other hand, if $a_2 = 1$, then $a_3 = 1$ by the seventh inclusion.
- If $a_1 = 0$ then a_2 can be a 0 or a 1. Suppose $a_2 = 0$. Then $a_3 = 0$ by the first two inclusions. Assume $a_2 = 1$. Then $a_3 = 0$ by the third inclusion and on the other hand $a_3 = 1$ by the second inclusion, a contradiction.

This proves the statement. □

Fact 4. *Suppose* $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{Z}^3$ *and* $\{i,k,t,p\} = [4]$ *and* $|\{j,m,s\}| = 3$. *Then*

$$a_1\chi(B_j, A_i) + a_2\chi(B_m, A_k) + a_3\chi(B_s, A_t) + \chi(A_i, A_p) + \chi(A_k, A_t) \notin T^{18}.$$

Proof. Let $\ell \in [4] \setminus \{j,m,s\}$. Using the same technique as above from consideration of non-compatible pairs

$$((B_j,A_i),(B_m,A_t)),\ ((B_s,A_t),(B_j,A_k)),\ ((B_j,A_i),(B_s,A_t)),$$
$$((B_m,A_k),(B_s,A_t)),\ ((B_j,A_i),(B_m,A_p)),\ ((B_j,A_i),(B_\ell,A_k))$$

we obtain inclusions:

$$a_1,\ a_3,\ (a_1-a_2+a_3),\ (-a_1+a_2+a_3),\ (a_1-a_3),\ (a_1-a_3-2) \in T,$$

respectively.

From the last two inclusions we can see that $a_1-a_3=1$. This, together with the first and the second inclusions, imply $(a_1,a_3) \in \{(1,0),(0,-1)\}$. Suppose $(a_1,a_3) = (1,0)$. Then

$$\chi(B_j,A_i)+a_2\chi(B_m,A_k)+\chi(A_i,A_p)+\chi(A_k,A_t) = \chi(B_j,A_p)+a_2\chi(B_m,A_k)+\chi(A_k,A_t).$$

By Fact 3, it doesn't belong to T^{18} for any value of a_2.

Suppose now that $(a_1,a_3)=(0,-1)$. Then by the third and the forth inclusions a_2 can be only zero. Then $\mathbf{a}=(0,0,-1)$ and

$$-\chi(B_s,A_t)+\chi(A_i,A_p)+\chi(A_k,A_t) = -\chi(B_s,A_k)+\chi(A_i,A_p).$$

However, by Fact 3 the right-hand-side of this equation is not a vector of T^{18}. □

Fact 5. *Suppose* $\mathbf{a} \in \mathbb{Z}^5$ *and*

$$\mathbf{v} = a_1\chi(B_j,A_i)+a_2\chi(B_m,A_k)+a_3\chi(B_s,A_t)+a_4\chi(A_i,A_p)+a_5\chi(A_k,A_t).$$

If $a_4,a_5 \in \{0,1,-1\}$ *and* $v \in T^{18}$, *then* v *belongs to* X *or* $-X$.

Proof. First of all, we will find the possible values of $\mathbf{a}$ in case $\mathbf{v} \in T^{18}$. By Facts 1 –4 one can see that $\mathbf{v} \in T^{18}$ iff $\mathbf{a}$ belongs to the set

$$\begin{aligned} Q = \{&(0,0,0,0,0),\ (\pm1,0,0,0,0),\ (0,\pm1,0,0,0),\ (0,0,\pm1,0,0),\ (1,1,1,0,0),\\ &(0,0,0,\pm1,0),\ (\pm1,0,0,\pm1,0),\ (\pm1,\pm1,\pm1,\pm1,0),\ (\pm2,\pm1,\pm1,\pm1,0),\\ &(0,0,0,0,\pm1),\ (0,\pm1,0,0,\pm1),\ (0,0,\mp1,0,\pm1),\ (0,\pm1,\mp1,0,\pm1)\}.\end{aligned}$$

By the construction of $\preceq$ the sequence $(A_1,\ldots,A_4;B_1,\ldots,B_4)$ is a trading transform. So for every $\{i_1,\ldots,i_4\}=\{j_1,\ldots,j_4\}=[4]$ the equation

$$\chi(B_{i_1},A_{j_1})+\chi(B_{i_2},A_{j_2})+\chi(B_{i_3},A_{j_3})+\chi(B_{i_4},A_{j_4})=0. \tag{25}$$

holds. Taking (25) into account one can show, that for every $\mathbf{a} \in Q$, vector $\mathbf{v}$ belongs to X or $-X$. For example, if $\mathbf{a}=(2,1,1,1,0)$ then

$$\begin{aligned} &2\chi(B_j,A_i)+\chi(B_m,A_k)+\chi(B_s,A_t)+\chi(A_i,A_p) = \\ &\qquad\qquad \chi(B_j,A_i)-\chi(B_\ell,A_p)+\chi(A_i,A_p) = \chi(B_j,B_\ell),\end{aligned}$$

where $\ell \in [4] \setminus \{j,m,s\}$. □

One can see that $\mathbf{v}$ from the Fact 5 is the general form of α. Hence $\alpha \in T^{18}$ if and only if $\alpha \in X \cup -X$ which is Claim 2. □

7. Discussion and further research

We note that for any qualitative probability order $\preceq$, both $\Delta(\preceq, T)$ and $G(\preceq, T)$ satisfy a much stronger condition than completeness. To formulate it for a game $G = ([n], W)$ we define the *Winder desirability relation* $\preceq_W$ on coalitions. We set $A \preceq_W B$ if and only if

$$(A \setminus B) \cup Y \in W \Longrightarrow (B \setminus A) \cup Y \in W.$$

for every $Y \subseteq [n] \setminus ((A \setminus B) \cup (B \setminus A))$.

Furthermore, the *Winder existential ordering*, $\prec_W$, on $2^{[n]}$ is defined as follows. We set $A \prec_W B$ iff it is not the case that $B \preceq_W A$, which means that there exists $Z \subseteq [n] \setminus ((A \setminus B) \cup (B \setminus A))$ such that

$$(A \setminus B) \cup Z \in L \quad \text{and} \quad (B \setminus A) \cup Z \in W.$$

A simple game G is *strongly acyclic* if there are no k-cycles in the Winder existential ordering, that is, the situation

$$A_1 \prec_W A_2 \prec_W \ldots \prec_W A_k \prec_W A_1$$

is not possible for any $k \geq 2$. Note that the absence of 2-cycles in $\prec_W$ implies completeness of $\preceq_W$. Winder (1962)[3] and Taylor & Zwicker (1999) constructed examples of strongly acyclic non-representable simple games with the second one being even a constant-sum game.

The concept of strongly acyclic simplicial complex Δ can be defined straightforwardly. For this, the relation $\prec_W$ have to be defined as follows. Given a simplicial complex Δ we set $A \prec_W B$ for two subsets $A, B \subseteq [n]$ if there exists a subset $Z \subseteq [n] \setminus ((A \setminus B) \cup (B \setminus A))$ such that

$$(A \setminus B) \cup Z \in \Delta \quad \text{and} \quad (B \setminus A) \cup Z \notin \Delta.$$

It is also easy to show that any initial segment $\Delta(\preceq, T)$ of a qualitative probability order $\preceq$ is strongly acyclic so this construction seems like a natural way of obtaining strongly acyclic simplicial complexes which are not threshold. However it appeared to be not so easy: even if a qualitative probability order $\preceq$ is non-weighted this does not automatically imply that $\Delta(\preceq, T)$ is not threshold; Christian et al. (2006) computationally checked that all initial segments of qualitative probability orders for $n \leq 6$ atoms are threshold simplicial complexes while numerous non-representable qualitative probability orders for $n \geq 5$ exist. Up to date there were no known examples of non-threshold initial segments. It is unclear whether or not the simplicial complex corresponding to the strongly acyclic non-weighted game from Taylor & Zwicker (1999) is an initial segment of a comparative probability but, even if it was, it is very large.

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[3]This publication is not accessible to us.

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