Counting groups: gnus, moas and other exotica

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Abstract

The number of groups of a given order is a fascinating function. We report on its known values, discuss some of its properties, and study some related functions.

1 Introduction

How many distinct abstract groups have a given finite order n? We shall call this number the **g***roup* **nu***mber* of n, and denote it by $\text{gnu}(n)$. Given the long history of group constructions, a study initiated by Cayley [4] in 1854, it is perhaps surprising that only in this decade has a sizeable table of group numbers become available.

The table in the appendix, adapted and slightly extended from that which appeared in [2], tabulates $\text{gnu}(n)$ for $0 < n < 2048$. The next value, $\text{gnu}(2048)$, is still not precisely known, but it strictly exceeds 1774274116992170, which is the exact number of groups of order 2048 that have exponent-2 class 2, and can confidently be expected to agree with that number in its first 3 digits.

In this paper we study some properties of the gnu function. We also introduce and study a new and related function: $\text{moa}(n)$ is the smallest of the numbers m for which $\text{gnu}(m) = n$, provided any exist. (The name abbreviates **m***inimal* **o***rder* **a***ttaining* a given group number, and also honours the country in which this paper was written.)

We refer the reader to the recent survey of [2] for a detailed account of the history of the problem. Relying on this, we do not provide extensive references to the various contributions, usually citing only those that are immediately relevant or very recent.

After this paper was written, we learned that the recent book [3, Chapter 21] provides a more scholarly discussion of the group number function.

2 The gnu function and multiprimality

The first thing that influences $\text{gnu}(n)$ is the number of primes (counting repetitions) of which n is the product. This well-known function, $\Omega(n)$, which does not seem hitherto to have received a standard name, we call the *multiprimality* of n, and describe n as *prime, biprime, triprime,* etc., according as its multiprimality is $1, 2, 3$, etc. We let adjectives that usually apply to numbers also apply to groups; so for example a *square-free* group is one of square-free order.

We now display gnu(n) for n at most 100 according to the multiprimality m of n, together with what we call the *estimate*, which is the mth Bell number, defined as the number of equivalence relations on a set of m objects.

1. Primes (estimate 1): gnu $= 1$ for

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.$$

2. Biprimes (estimate 2): gnu $= 2$ for

$$4, 6, 9, 10, 14, 21, 22, 25, 26, 34, 38, 39, 46, 49, 55, 57, 58, 62, 74, 82, 86, 93, 94$$

   but gnu $= 1$ for

$$15, 33, 35, 51, 65, 69, 77, 85, 87, 91, 95.$$

3. Triprimes: (estimate 5):

8	12	18	20	27	28	30	42	44	45	50	52	63	66	68	70	75	76	78	92	98	99
5	5	5	5	5	4	4	6	4	2	5	5	4	4	5	4	3	4	6	4	5	2

4. Quadruprimes (estimate 15):

16	24	36	40	54	56	60	81	84	88	90	100
14	15	14	14	15	13	13	15	15	12	10	16

5. Quinqueprimes (estimate 52):

32	48	72	80
51	52	50	52

6. Sextiprimes (estimate 203):

64	96
267	231

These values show that when n and its multiprimality m are both small, gnu(n) does not differ much from the estimate. We think this remarkable approximation deserves an explanation, even though for larger numbers it ceases to hold.

Many other oddities will be noticed among the values in the table in the Appendix. For example, is it merely a coincidence that there are three numbers n with gnu(n) = 1387, and a fourth with gnu(n) = 1388?

3 Powerful gnus

If n is a power up to the fourth of some prime, then indeed $\mathrm{gnu}(n)$ equals its estimate, except that $\mathrm{gnu}(16) = 14$ rather than 15. But from the fifth power onwards the situation is very different. We summarise the known results. In addition to those cited in [2], the new sources are [14] and [15].

Theorem 3.1.

1. *There are*

$$51 \text{ groups of order } 2^5,$$

$$67 \text{ of order } 3^5$$

and

$$61 + 2p + 2\gcd(p-1,3) + \gcd(p-1,4)$$

of order p^5 *for prime* $p \geq 5$.

2. *There are*

$$267 \text{ groups of order } 2^6,$$

$$504 \text{ of order } 3^6$$

and

$$3p^2 + 39p + 344 + 24\gcd(p-1,3) + 11\gcd(p-1,4) + 2\gcd(p-1,5)$$

of order p^6 *for prime* $p \geq 5$.

3. *There are*

$$2328 \text{ groups of order } 2^7,$$

$$9310 \text{ groups of order } 3^7,$$

$$34297 \text{ groups of order } 5^7$$

and

$$\begin{aligned}&3p^5 + 12p^4 + 44p^3 + 170p^2 + 707p + 2455\\&+(4p^2 + 44p + 291)\gcd(p-1,3) + (p^2 + 19p + 135)\gcd(p-1,4)\\&+(3p + 31)\gcd(p-1,5) + 4\gcd(p-1,7) + 5\gcd(p-1,8) + \gcd(p-1,9)\end{aligned}$$

of order p^7 *for prime* $p \geq 7$.

4. *For* $n \geq 8$,

$$\mathrm{gnu}(256) = 56092,\ \ \mathrm{gnu}(512) = 10494213,\ \ \mathrm{gnu}(1024) = 49487365422.$$

4 Great gnus

The greatest values of gnu are those just mentioned, namely its values at powers of 2, which dominate the others in a quite surprising way. For example, if a group is selected at random from all the groups of order < 2048, the odds are more than 100-to-1 that it will have order 1024. The 423171191 groups of all the other orders < 2048 are swamped by the 49487365422 of order 1024. The vast majority, 48803495722, of the latter have exponent-2 class 2, and it is the fact that the number of groups of this special type can be counted without explicit construction that has enabled gnu(1024) to be precisely calculated; an algorithm to perform this calculation is described in [7].

The asymptotic estimates of Higman [10] and Sims [17] show that the number of groups of order p^n is $p^{2n^3/27+O(n^{8/3})}$. M.F. Newman (private communication) and C. Seeley have shown that the exponent 8/3 can be reduced to 5/2. Pyber [16] has shown that

$$\mathrm{gnu}(n) \leq n^{(2/27+o(1))\mu(n)^2},$$

as $\mu(n)$, the largest exponent in the prime-power factorization of n, tends to infinity.

5 Powerless gnus

This title refers to the group numbers for square-free orders. The results for prime and biprime numbers are in every beginning course on group theory. There is a unique group of each prime order (agreeing with the estimate of 1 in that case); and either one or two groups of order pq, the number being two (which is the estimate) if and only if either $p = q$, or one of p and q is congruent to 1 modulo the other. The first group is cyclic, and the second, supposing that $q \equiv 1 \bmod p$, has presentation

$$\{A, B \mid 1 = A^p = B^q, A^{-1}BA = B^k\}$$

where k is some number having order p modulo q.

In such a case, we say that A *acts* on B, and call A the *actor*, and B the *reactor*. Since replacing A by A^j $(0 < j < p)$ replaces k by k^j, which is another number having order p (modulo q), all the possible choices for k yield the same group.

Hölder [11] generalized these results to all groups of square-free order. Namely every such group has a presentation with a generator A_p for each prime divisor p of n, while the generators for distinct primes p and q either commute or one (say A_p) acts on the other (A_q) by replacing it by its kth power for some k not congruent to 1 mod q. We shall say that "p (or A_p) acts as k on q (or A_q)."

There are some restrictions on these presentations. The same generator cannot be both an actor and a reactor. Moreover, p can only act on q if $q \equiv 1 \bmod p$, which condition we therefore call an *opportunity* (for action). Moreover, if p does act as k on q, then k must be a number that has order p (modulo q), and as before we can replace k by k^j $(0 < j < p)$ by replacing A_q by A_q^j. This entails that if A_p acts on several generators, then the value of k for one of them may be freely chosen, but then the rest are determined (by the given group).

The groups of square-free order $n = pqr\dots$ may therefore be specified by graphs having a node for each of the primes $p, q, r, \dots$, there being an arrow (directed edge) marked k from node p to node q just when A_p acts as k on A_q. If we replace A_p by its jth power, then all the marks k on the arrows from node p are replaced by their jth powers (modulo p), and so if there is only one arrow from node p, the mark on it is unimportant, and may be omitted.

The number of possibilities for such graphs, taking the above restrictions and equivalences into account, is therefore the number of groups of the given square-free order. It is completely determined by specifying the opportunities for action among the primes $p, q, r, \dots$, namely which of them are congruent to 1 modulo which others.

Hölder [11] summarized the results in an elegant (if somewhat opaque) formula.

Theorem 5.1.

$$\mathrm{gnu}(n) = \sum_{d|n} \prod_{p|d} \frac{(p^{\mathrm{opp}(p,e)} - 1)}{(p-1)}$$

where $de = n$ and $p|d$ and $\mathrm{opp}(p, e)$ is the number of opportunities for p to act on the primes dividing e.

Murty & Murty [12] generalized this to count all the groups of any prescribed order whose Sylow subgroups are all cyclic. Their result is the following.

Theorem 5.2. *The number of groups of order n, all of whose Sylow subgroups are cyclic, is:*

$$\sum_{d|n} \prod_{p^\alpha|d} \left(\sum_{j=1}^{\alpha} \frac{(p^{\mathrm{opp}(p^j,e)} - p^{\mathrm{opp}(p^{j-1},e)})}{p^{(j-1)}(p-1)} \right)$$

where α is the largest power of p dividing d, $de = n$, $\gcd(d, e) = 1$, and $p|d$, and now

$$p^{\mathrm{opp}(p^j,e)} = \prod_{q|m} \gcd(p^j, q-1)$$

where p and q denote primes and j is a positive integer.

6 All triprime groups

It is well-known that $\mathrm{gnu}(p^3) = 5$, the estimate. For square-free triprimes pqr, where we may assume $p < q < r$, the results are:

No opportunity	1 group
Just one opportunity	2 groups
Two consecutive opportunities	3 groups
Two inward opportunities	4 groups
Two outward opportunities	$p + 2$ groups
Three opportunities	$p + 4$ groups

To interpret and explain these results, it suffices to draw the possible *opportunity graphs*, having an arrow from p to q when p has the opportunity to act on q, and then to indicate the different ways for some of these opportunities to be seized. Such graphs also appear in [13]. The opportunity graphs for groups of order pqr appear in Figure 1.

For the remaining triprimes $n = pq^2$, Hölder found that $\mathrm{gnu}(n)$ is:

2	if we have none of $p \mid q-1, p \mid q+1, q \mid p-1$;
3	if both $p, q > 2$ and $p \mid q+1$;
4	if $p > 3$, $q \mid p-1$, but not $q^2 \mid p-1$;
5	if $p = 2 < q$ or $p = 3, q = 2$ or $q^2 \mid p-1$; and finally
$(p+9)/2$	if p is odd and divides $q-1$.

7 The baby gnus

It is not hard to prove the following theorems, which together find all the numbers n for which $\mathrm{gnu}(n) \leq 4$.

We list first the form of n where letters represent distinct primes and any repeated prime is explicitly shown. Thus "*form pqr* ..." means square-free. Recall that if primes p and q both divide n, we call the condition $p \mid (q-1)$ an *opportunity*, in which p is the *actor* and q the *reactor*; if p^2 also divides n, then $p^2 \mid (q-1)$ is a *double-opportunity* and $q \mid (p^2-1)$ a *half-opportunity.*

Theorem 7.1. $\mathrm{gnu}(n) = 1$ *if and only if* n *is a square-free number with no opportunity.*

This condition is equivalent to the coprimality of n with its Euler function $\phi(n)$.

Theorem 7.2. $\mathrm{gnu}(n) = 2$ *if and only if* n *has:*

- *form* $pqr\ldots$ *with just one opportunity;*
- *form* $p^2qr\ldots$ *with no opportunity or half-opportunity.*

Theorem 7.3. $\mathrm{gnu}(n) = 3$ *if and only if* n *has:*

- *form* $pqr\ldots$ *and just two consecutive opportunities;*
- *form* $p^2qr\ldots$ *with one half-opportunity but no opportunity.*

Theorem 7.4. $\mathrm{gnu}(n) = 4$ *if and only if* n *has:*

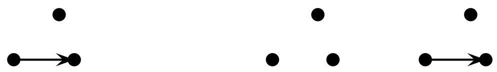
- *form* $pqr\ldots$ *with just two opportunities, that are either inward, disjoint, or both have* 2 *as the actor;*
- *form* $p^2qr\ldots$ *with no double or half-opportunity, and just one opportunity, whose reactor is distinct from* p*;*
- *form* $p^2q^2r\ldots$ *with no opportunity or half-opportunity.*

These results make it easy to compute the number of n below any reasonable bound for which $\mathrm{gnu}(n)$ is at most 4. The densities of $1, 2, 3, 4$ for $1 \leq n \leq 10^8$ are respectively $0.285, 0.132, 0.003, 0.093$. Table 1 records the number of occurrences of $1 \leq j \leq 10$ for $n \leq 2048$.

(i) No opportunities: 1 group

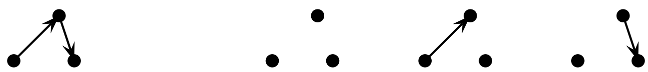
1

(ii) One opportunity: 2 groups



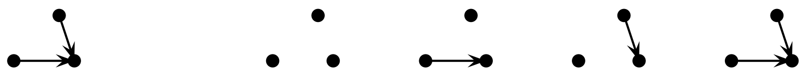
1 1

(iii) Two consecutive opportunities: 3 groups



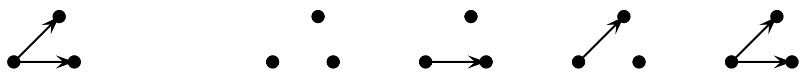
1 1 1

(iv) Two inward opportunities: 4 groups



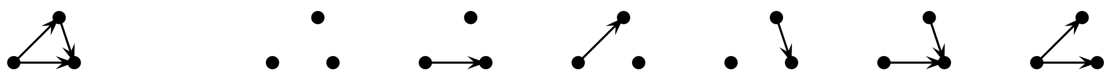
1 1 1 1

(v) Two outward opportunities: $p+2$ groups



1 1 1 $p-1$

(vi) Three opportunities: $p+4$ groups



1 1 1 1 1 $p-1$

Figure 1: Opportunity graphs for groups of order pqr

j	Number
1	656
2	393
3	11
4	232
5	102
6	62
7	4
8	14
9	12
10	39

Table 1: Occurrences of small values for $n \leq 2048$

8 Hunting gnus and moas

We now formulate the *gnu-hunting conjecture.*

Conjecture 8.1. *Every positive integer is a group number.*

For the history of this conjecture see [3, §21.6]. It seems very likely that every number is a value of gnu(n) for some square-free n, as has been conjectured by R. Keith Dennis [6]. He has established this for all numbers up to 10000000.

What is the next term in the sequence $1, 4, 75, 28, 8, 42$? This is, of course, the moa sequence: there is 1 group of order 1, 2 of order 4, 3 of order 75, 4 of order 28, and so on. The required answer is therefore 375, since that is the smallest order for which there are exactly 7 groups.

It is distinctly harder to hunt moas than gnus, since to show that moa(n) $= m$ one must not only verify that there are exactly m groups of order n, but show that no order smaller than n has exactly this number of groups. Table 2 lists the moa values we know up to moa(100) $= 3822$, together with our guesses about the rest.

The known answers, say moa(g) $= m$, when not less than 2048, were found by computing, for each number $n < m$, either the exact value of gnu(n) or showing that it exceeds g. We used four methods to find (lower bounds for) gnu(n).

1. If gnu(d) is known for each divisor d of some number n, then, by the inclusion-exclusion method, the number of indecomposable groups for each such divisor can also be computed, and one lower bound for gnu(n) is obtained by multiplying such numbers over various factorizations of n. (Recall that a *decomposable* group is one that can be expressed as a non-trivial direct product.)

2. The formula of [12] for the number of groups of given order that have cyclic Sylow subgroups can be supplemented by estimates for those that don't; thence we obtain another useful bound.

	+0	+1	+2	+3	+4	+5	+6	+7	+8	+9
0		1	4	75	28	8	42	375	510	308
10	90	140	88	56	16	24	100	675	156	1029
20	820	1875	6321	294	546	2450	2550	1210	2156	1380
30	270	?11774	630	?163293	450	616	612	180	1372	264
40	280	420	176	112	392	108	252	120	2730	300
50	72	32	48	656	272	162	500	168	4650	6875
60	378	312	702	3630	1596	?59150	588	243	882	1215
70	4100	3660	1638	?9139263	?26010	2420	2964	1092	?51772	3612
80	6050	6820	?126945	?16807	2394	?35322	?18620	?34914	2028	4140
90	?13300	?12324	?24990	?23460	?28308	?85484	6930	6498	4950	1188
100	3822									

Table 2: Values and guesses for the moa function

3. If the number is cubefree, we use the group construction package CubeFree of Dietrich & Eick [5] to count explicitly the number of groups of this order.

4. Otherwise, the groups of order n can be explicitly enumerated by the GrpConst package of Besche & Eick [1] until all (or enough to establish a sufficient lower bound) are found.

The last two options can be expensive and are only practical for limited ranges: for those cubefree numbers $n \in \{2048, \ldots, 163293\}$ that gave lower bounds at most 30, we successfully calculated $\text{gnu}(n)$ using the CubeFree package in GAP [8].

The guesses in Table 2 are prefixed by "?". For 83, the guess is 7^5; for 73, it is the smallest square-free integer having this value of gnu; all our other guesses are the smallest cubefree possibilities.

9 Good gnus and bad gnus

Is there any hope of proving the gnu-hunting conjecture? We address Dennis' stronger form that every number arises as $\text{gnu}(n)$ for some square-free n.

Hölder's formula for the group number of a square-free number $pqr\ldots$ is a sum of products of powers of the primes $p, q, r, \ldots$, of which it is composed. For example, as we saw in Section 6, when $p < q < r$, the value of $\text{gnu}(pqr)$ is $1, 2, 3, 4, p+2$ or $p+4$ according to the number and nature of the opportunities among p, q, r.

A reason one can still hope to prove the conjecture is that as well as the bad forms (like $p+2$ and $p+4$ above) that involve unknown primes, there are good ones (like $1, 2, 3, 4$ above) that don't. It seems likely that the latter type already suffice to prove the conjecture.

Let $n = pqr\ldots$ be the prime factorization of a square-free number, for which we are given only the opportunities for action among the primes $p, q, r, \ldots$.

We say that gnu is *good* at $pqr\ldots$, or loosely that the expression $\text{gnu}(pqr\ldots)$ is a *good gnu*, if its value is a constant that does not depend on the particular primes involved. The bad gnus are those expressions gnu($pqr\ldots$) for which some prime, p say, has a number $k > 1$ of opportunities to act, since then the number of cases in which it seizes those opportunities involves the factor $(p-1)^{k-1}$.

The opportunity graph for a good gnu, having at most one arrow leading from any node, must therefore be a forest based on the primes involved – we call it *the primeval forest* – consisting of rooted trees (the roots corresponding to the primes that have no opportunities for action). We define the *uprooted forest* to be the smaller forest obtained from the primeval one by removing the roots of all these trees, along with the edges that led to them, and then also removing the arrowheads on any edges that remain.

The following surprising result is remarkably easy to prove.

Theorem 9.1. *The value of a good* gnu *expression* gnu($pqr\ldots$) *depends only on the shape of the uprooted forest. In particular, it is independent of the number and arrangement of the roots, and of the directions of the arrows on any of the remaining edges.*

Proof. Since each prime has at most one opportunity for action, the groups correspond in a one-to-one manner to the possible sets of acting primes, and these sets are just the independent subsets of the uprooted forest, whose definition doesn't need the directions of its edges. (An independent subset of the vertices of a graph is a set that does not contain both endpoints of any edge.) □

For gnu-hunters, this theorem is an unfortunate one, since it forces many different numbers to yield just one value of gnu. It implies, for instance, that the four primeval trees of Figure 2 all give the same gnu value of 14.

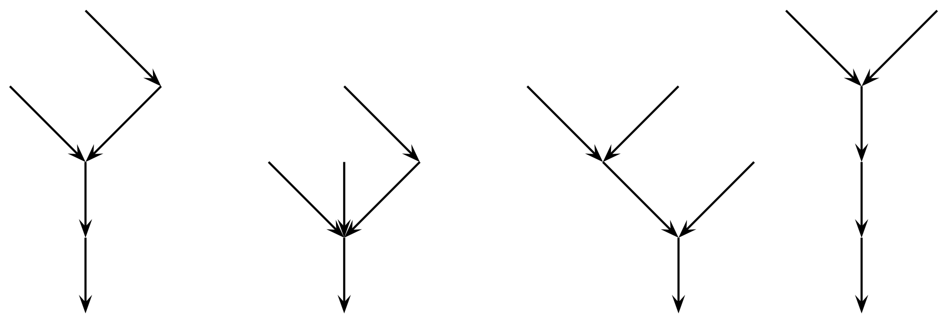


Figure 2: Primeval forests

The theorem has an interesting corollary.

Corollary 9.2. *The set of values of good* gnu*s (of square-free numbers) is closed under multiplication.*

Proof. The number of independent sets in a forest is the product of those numbers for its component trees. Any forest can be realised since Dirichlet's Theorem [9, p. 13] implies that any shape of tree can be realised by a square-free number with arbitrarily large prime divisors. □

Now the number of independent sets for a tree with e edges varies in the range from f_{e+3} (the $(e+3)$-rd Fibonacci number) to 2^e+1. The numbers in the first few cases are the following.

0 edges:	2
1 edge:	3
2 edges:	5
3 edges:	$8, 9$
4 edges:	$13, 14, 17$
5 edges:	$21, 22, 23, 24, 26, 33$
6 edges:	$34, 35, 36, 37, 38, 40, 41, 43, 44, 50, 65$
7 edges:	$55, 57-62, 64-66, 68-70, 76, 77, 80, 83, 84, 98, 129$
8 edges:	$89, 92-102, 104-110, 112-114, 116, 118, 120-122,$ $124, 126, 128, 133, 134, 145, 148, 149, 152, 163, 164, 194, 257$

These results imply that any product of the displayed numbers is the value of $\text{gnu}(n)$ for infinitely many square-free numbers n. However, the observant reader will notice that the numbers $7, 11, 19, 29, 31, \ldots$ are missing. These do not arise as "good gnus" of any square-free numbers.

This might not matter. Dennis (see [3, §21.6]) lists 508 numbers that he conjectures are the only ones missing from the continued form of the above table, and he has verified that each of these is the (bad) gnu of some square-free number. All that remains is to prove that this list is complete!

We have already mentioned that the first 10 million integers are values of gnu; it is not hard to prove that every number whose prime factors are all smaller than 140 is of the form $\text{gnu}(n)$ for infinitely many numbers n.

10 Galloping gnus

Recall that a number n is traditionally called *perfect, abundant*, or *deficient* according as the sum of its proper divisors equals, exceeds, or falls short of n. We mirror this by calling a number n *group-perfect, group-abundant*, or *group-deficient* according as $\text{gnu}(n) = n$, $\text{gnu}(n) > n$, or $\text{gnu}(n) < n$.

Murty & Murty [12] prove that $\text{gnu}(n) \leq \phi(n)$ for square-free n, and so all square-free numbers greater than 1 are group-deficient. We do not know if there is any group-perfect number other than 1, but there are plenty of group-abundant ones, for instance all numbers of the form $1024n$ below 49487365422. However, it seems that the proportion of group-abundant numbers gradually falls to zero.

Again, we do not know if there is any *group-amicable* pair of numbers (that is, $m > n$ with $\text{gnu}(m) = n$ and $\text{gnu}(n) = m$). A negative answer would follow from the *galloping gnus conjecture* which we now formulate:

Conjecture 10.1. *For every positive integer n, the sequence*

$$n \longmapsto \text{gnu}(n) \longmapsto \text{gnu}^2(n) = \text{gnu}(\text{gnu}(n)) \longmapsto \text{gnu}^3(n) \longmapsto \ldots$$

consists ultimately of 1*s*.

To check this for all starting numbers $n < 2048$ it suffices to follow the 47 group-abundant numbers among them. We do this in Table 3 in descending order of gnu^2 except that we have omitted two more numbers $48, 448$ with $\text{gnu}^2 = 5$; also $160, 432, 832, 1408, 1458, 1920, 2016$ with $\text{gnu}^2 = 4$; also $96, 288, 1088, 1296$ with $\text{gnu}^2 = 2$; and finally 17 further numbers with $\text{gnu}^2 = 1$. In fact every number less than 2048 reaches 1 after at most 5 steps.

672	$\longmapsto$	1280	$\longmapsto$	1116461	$\longmapsto$	1				
1024	$\longmapsto$	49487365422	$\longmapsto$	240	$\longmapsto$	208	$\longmapsto$	51	$\longmapsto$	1
720	$\longmapsto$	840	$\longmapsto$	186	$\longmapsto$	6	$\longmapsto$	2	$\longmapsto$	1
320	$\longmapsto$	1640	$\longmapsto$	68	$\longmapsto$	5	$\longmapsto$	1		
384	$\longmapsto$	20169	$\longmapsto$	67	$\longmapsto$	1				
128	$\longmapsto$	2328	$\longmapsto$	64	$\longmapsto$	267	$\longmapsto$	1		
960	$\longmapsto$	11394	$\longmapsto$	60	$\longmapsto$	13	$\longmapsto$	1		
864	$\longmapsto$	4725	$\longmapsto$	51	$\longmapsto$	1				
1344	$\longmapsto$	11720	$\longmapsto$	49	$\longmapsto$	2	$\longmapsto$	1		
1440	$\longmapsto$	5958	$\longmapsto$	16	$\longmapsto$	14	$\longmapsto$	2	$\longmapsto$	1
1248	$\longmapsto$	1460	$\longmapsto$	15	$\longmapsto$	1				
256	$\longmapsto$	56092	$\longmapsto$	11	$\longmapsto$	1				
1728	$\longmapsto$	47937	$\longmapsto$	6	$\longmapsto$	2	$\longmapsto$	1		
512	$\longmapsto$	10494213	$\longmapsto$	5	$\longmapsto$	1				
1536	$\longmapsto$	408641062	$\longmapsto$	4	$\longmapsto$	2	$\longmapsto$	1		
1664	$\longmapsto$	21507	$\longmapsto$	2	$\longmapsto$	1				
1280	$\longmapsto$	1116461	$\longmapsto$	1						

Table 3: Verifying the galloping gnu conjecture

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A The number of groups for each order < 2048

	+0	+1	+2	+3	+4	+5	+6	+7	+8	+ 9
0		1	1	1	2	1	2	1	5	2
10	2	1	5	1	2	1	14	1	5	1
20	5	2	2	1	15	2	2	5	4	1
30	4	1	51	1	2	1	14	1	2	2
40	14	1	6	1	4	2	2	1	52	2
50	5	1	5	1	15	2	13	2	2	1
60	13	1	2	4	267	1	4	1	5	1
70	4	1	50	1	2	3	4	1	6	1
80	52	15	2	1	15	1	2	1	12	1
90	10	1	4	2	2	1	231	1	5	2
100	16	1	4	1	14	2	2	1	45	1
110	6	2	43	1	6	1	5	4	2	1
120	47	2	2	1	4	5	16	1	2328	2
130	4	1	10	1	2	5	15	1	4	1
140	11	1	2	1	197	1	2	6	5	1
150	13	1	12	2	4	2	18	1	2	1
160	238	1	55	1	5	2	2	1	57	2
170	4	5	4	1	4	2	42	1	2	1
180	37	1	4	2	12	1	6	1	4	13
190	4	1	1543	1	2	2	12	1	10	1
200	52	2	2	2	12	2	2	2	51	1
210	12	1	5	1	2	1	177	1	2	2
220	15	1	6	1	197	6	2	1	15	1
230	4	2	14	1	16	1	4	2	4	1
240	208	1	5	67	5	2	4	1	12	1
250	15	1	46	2	2	1	56092	1	6	1
260	15	2	2	1	39	1	4	1	4	1
270	30	1	54	5	2	4	10	1	2	4
280	40	1	4	1	4	2	4	1	1045	2
290	4	2	5	1	23	1	14	5	2	1
300	49	2	2	1	42	2	10	1	9	2
310	6	1	61	1	2	4	4	1	4	1
320	1640	1	4	1	176	2	2	2	15	1
330	12	1	4	5	2	1	228	1	5	1
340	15	1	18	5	12	1	2	1	12	1
350	10	14	195	1	4	2	5	2	2	1
360	162	2	2	3	11	1	6	1	42	2
370	4	1	15	1	4	7	12	1	60	1
380	11	2	2	1	20169	2	2	4	5	1
390	12	1	44	1	2	1	30	1	2	5
400	221	1	6	1	5	16	6	1	46	1
410	6	1	4	1	10	1	235	2	4	1
420	41	1	2	2	14	2	4	1	4	2
430	4	1	775	1	4	1	5	1	6	1
440	51	13	4	1	18	1	2	1	1396	1
450	34	1	5	2	2	1	54	1	2	5
460	11	1	12	1	51	4	2	1	55	1
470	4	2	12	1	6	2	11	2	2	1
480	1213	1	2	2	12	1	261	1	14	2
490	10	1	12	1	4	4	42	2	4	1

	+0	+1	+2	+3	+4	+5	+6	+7	+8	+ 9
500	56	1	2	1	202	2	6	6	4	1
510	8	1	10494213	15	2	1	15	1	4	1
520	49	1	10	1	4	6	2	1	170	2
530	4	2	9	1	4	1	12	1	2	2
540	119	1	2	2	246	1	24	1	5	4
550	16	1	39	1	2	2	4	1	16	1
560	180	1	2	1	10	1	2	49	12	1
570	12	1	11	1	4	2	8681	1	5	2
580	15	1	6	1	15	4	2	1	66	1
590	4	1	51	1	30	1	5	2	4	1
600	205	1	6	4	4	7	4	1	195	3
610	6	1	36	1	2	2	35	1	6	1
620	15	5	2	1	260	15	2	2	5	1
630	32	1	12	2	2	1	12	2	4	2
640	21541	1	4	1	9	2	4	1	757	1
650	10	5	4	1	6	2	53	5	4	1
660	40	1	2	2	12	1	18	1	4	2
670	4	1	1280	1	2	17	16	1	4	1
680	53	1	4	1	51	1	15	2	42	2
690	8	1	5	4	2	1	44	1	2	1
700	36	1	62	1	1387	1	2	1	10	1
710	6	4	15	1	12	2	4	1	2	1
720	840	1	5	2	5	2	13	1	40	504
730	4	1	18	1	2	6	195	2	10	1
740	15	5	4	1	54	1	2	2	11	1
750	39	1	42	1	4	2	189	1	2	2
760	39	1	6	1	4	2	2	1	1090235	1
770	12	1	5	1	16	4	15	5	2	1
780	53	1	4	5	172	1	4	1	5	1
790	4	2	137	1	2	1	4	1	24	1
800	1211	2	2	1	15	1	4	1	14	1
810	113	1	16	2	4	1	205	1	2	11
820	20	1	4	1	12	5	4	1	30	1
830	4	2	1630	2	6	1	9	13	2	1
840	186	2	2	1	4	2	10	2	51	2
850	10	1	10	1	4	5	12	1	12	1
860	11	2	2	1	4725	1	2	3	9	1
870	8	1	14	4	4	5	18	1	2	1
880	221	1	68	1	15	1	2	1	61	2
890	4	15	4	1	4	1	19349	2	2	1
900	150	1	4	7	15	2	6	1	4	2
910	8	1	222	1	2	4	5	1	30	1
920	39	2	2	1	34	2	2	4	235	1
930	18	2	5	1	2	2	222	1	4	2
940	11	1	6	1	42	13	4	1	15	1
950	10	1	42	1	10	2	4	1	2	1
960	11394	2	4	2	5	1	12	1	42	2
970	4	1	900	1	2	6	51	1	6	2
980	34	5	2	1	46	1	4	2	11	1
990	30	1	196	2	6	1	10	1	2	15

	+0	+1	+2	+3	+4	+5	+6	+7	+8	+ 9
1000	199	1	4	1	4	2	2	1	954	1
1010	6	2	13	1	23	2	12	2	2	1
1020	37	1	4	2	49487365422	4	66	2	5	19
1030	4	1	54	1	4	2	11	1	4	1
1040	231	1	2	1	36	2	2	2	12	1
1050	40	1	4	51	4	2	1028	1	5	1
1060	15	1	10	1	35	2	4	1	12	1
1070	4	4	42	1	4	2	5	1	10	1
1080	583	2	2	6	4	2	6	1	1681	6
1090	4	1	77	1	2	2	15	1	16	1
1100	51	2	4	1	170	1	4	5	5	1
1110	12	1	12	2	2	1	46	1	4	2
1120	1092	1	8	1	5	14	2	2	39	1
1130	4	2	4	1	254	1	42	2	2	1
1140	41	1	2	5	39	1	4	1	11	1
1150	10	1	157877	1	2	4	16	1	6	1
1160	49	13	4	1	18	1	4	1	53	1
1170	32	1	5	1	2	2	279	1	4	2
1180	11	1	4	3	235	2	2	1	99	1
1190	8	2	14	1	6	1	11	14	2	1
1200	1040	1	2	1	13	2	16	1	12	5
1210	27	1	12	1	2	69	1387	1	16	1
1220	20	2	4	1	164	4	2	2	4	1
1230	12	1	153	2	2	1	15	1	2	2
1240	51	1	30	1	4	1	4	1	1460	1
1250	55	4	5	1	12	2	14	1	4	1
1260	131	1	2	2	42	3	6	1	5	5
1270	4	1	44	1	10	3	11	1	10	1
1280	1116461	5	2	1	10	1	2	4	35	1
1290	12	1	11	1	2	1	3609	1	4	2
1300	50	1	24	1	12	2	2	1	18	1
1310	6	2	244	1	18	1	9	2	2	1
1320	181	1	2	51	4	2	12	1	42	1
1330	8	5	61	1	4	1	12	1	6	1
1340	11	2	4	1	11720	1	2	1	5	1
1350	112	1	52	1	2	2	12	1	4	4
1360	245	1	4	1	9	5	2	1	211	2
1370	4	2	38	1	6	15	195	15	6	2
1380	29	1	2	1	14	1	32	1	4	2
1390	4	1	198	1	4	8	5	1	4	1
1400	153	1	2	1	227	2	4	5	19324	1
1410	8	1	5	4	4	1	39	1	2	2
1420	15	4	16	1	53	6	4	1	40	1
1430	12	5	12	1	4	2	4	1	2	1
1440	5958	1	4	5	12	2	6	1	14	4
1450	10	1	40	1	2	2	179	1	1798	1
1460	15	2	4	1	61	1	2	5	4	1
1470	46	1	1387	1	6	2	36	2	2	1
1480	49	1	24	1	11	10	2	1	222	1
1490	4	3	5	1	10	1	41	2	4	1

	+0	+1	+2	+3	+4	+5	+6	+7	+8	+ 9
1500	174	1	2	2	195	2	4	1	15	1
1510	6	1	889	1	2	2	4	1	12	2
1520	178	13	2	1	15	4	4	1	12	1
1530	20	1	4	5	4	1	408641062	1	2	60
1540	36	1	4	1	15	2	2	1	46	1
1550	16	1	54	1	24	2	5	2	4	1
1560	221	1	4	1	11	1	30	1	928	2
1570	4	1	10	2	2	13	14	1	4	1
1580	11	2	6	1	697	1	4	3	5	1
1590	8	1	12	5	2	2	64	1	4	2
1600	10281	1	10	1	5	1	4	1	54	1
1610	8	2	11	1	4	1	51	6	2	1
1620	477	1	2	2	56	5	6	1	11	5
1630	4	1	1213	1	4	2	5	1	72	1
1640	68	2	2	1	12	1	2	13	42	1
1650	38	1	9	2	2	2	137	1	2	5
1660	11	1	6	1	21507	5	10	1	15	1
1670	4	1	34	2	60	2	4	5	2	1
1680	1005	2	5	2	5	1	4	1	12	1
1690	10	1	30	1	10	1	235	1	6	1
1700	50	309	4	2	39	7	2	1	11	1
1710	36	2	42	2	2	5	40	1	2	2
1720	39	1	12	1	4	3	2	1	47937	1
1730	4	2	5	1	13	1	35	4	4	1
1740	37	1	4	2	51	1	16	1	9	1
1750	30	2	64	1	2	14	4	1	4	1
1760	1285	1	2	1	228	1	2	5	53	1
1770	8	2	4	2	2	4	260	1	6	1
1780	15	1	110	1	12	2	4	1	12	1
1790	4	5	1083553	1	12	1	5	1	4	1
1800	749	1	4	2	11	3	30	1	54	13
1810	6	1	15	2	2	9	12	1	10	1
1820	35	2	2	1	1264	2	4	6	5	1
1830	18	1	14	2	4	1	117	1	2	2
1840	178	1	6	1	5	4	4	1	162	2
1850	10	1	4	1	16	1	1630	2	2	2
1860	56	1	10	15	15	1	4	1	4	2
1870	12	1	1096	1	2	21	9	1	6	1
1880	39	5	2	1	18	1	4	2	195	1
1890	120	1	9	2	2	1	54	1	4	4
1900	36	1	4	1	186	2	2	1	36	1
1910	6	15	12	1	8	1	4	5	4	1
1920	241004	1	5	1	15	4	10	1	15	2
1930	4	1	34	1	2	4	167	1	12	1
1940	15	1	2	1	3973	1	4	1	4	1
1950	40	1	235	11	2	1	15	1	6	1
1960	144	1	18	1	4	2	2	2	203	1
1970	4	15	15	1	12	2	39	1	4	1
1980	120	1	2	2	1388	1	6	1	13	4
1990	4	1	39	1	2	5	4	1	66	1

	+0	+1	+2	+3	+4	+5	+6	+7	+8	+ 9
2000	963	1	8	1	10	2	4	4	12	2
2010	12	1	4	2	4	2	6538	1	2	2
2020	20	1	6	2	46	63	2	1	88	1
2030	12	1	42	1	10	2	5	5	2	1
2040	175	2	2	2	11	1	12	1		