Stability of closedness of convex cones under linear mappings

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Abstract. In this paper we reconsider the question of when the continuous linear image of a closed convex cone is closed in Euclidean space. In particular, we show that although it is not true that the closedness of the image is preserved under small perturbations of the linear mappings it is "almost" true that the closedness of the image is preserved under small perturbations, in the sense that, for "almost all" linear mappings from $\mathbb{R}^n$ into $\mathbb{R}^m$ if the image of the cone is closed then there is a small neighbourhood around it whose members also preserve the closedness of the cone.

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1 Introduction

We say that a nonempty subset K of a vector space V is a *cone* if for each $\lambda \in [0, \infty)$ and each $x \in K$, $\lambda x \in K$. If $\{a_1, a_2, \ldots, a_n\}$ is a finite subset of a vector space V then we shall denote by $\langle a_1, a_2, \ldots, a_n \rangle$ the *cone generated* by $\{a_1, a_2, \ldots, a_n\}$ i.e.,

$$\langle a_1, a_2, \ldots, a_n \rangle := \left\{ \sum_{k=1}^{n} \lambda_k a_k : 0 \le \lambda_k < \infty \text{ for all } 1 \le k \le n \right\}.$$

Further, we shall say that a convex cone K in a vector space V is *finitely generated* if there exists a finite set $\{a_1, a_2, \ldots, a_n\} \subseteq V$ such that $K = \langle a_1, a_2, \ldots, a_n \rangle$. By [2, page 25] we know that each finitely generated cone in a normed linear space X is closed. In fact, each finitely generated cone is a *polyhedral* set i.e., a finite intersection of closed half-spaces, [2, page 99].

If X and Y are finite dimensional normed linear spaces then we shall denote by, $L(X, Y)$ the set of all linear transformations from X into Y. Throughout this paper we shall assume that $L(X, Y)$ is endowed with a Hausdorff linear topology. Since all Hausdorff linear topologies on finite dimensional spaces are homeomorphic, [3, page 51] we shall, with out loss of generality, assume that the topology on $L(X, Y)$ is generated by the operator norm on $L(X, Y)$ and that the topologies on X and Y are generated by the Euclidean norms.

In this paper we will examine the question of whether the continuous linear image of a closed convex cone is closed. The motivation for this will be well known to many readers:

[1]This paper is dedicated to our friend and colleague Stephen Simons on the occasion of his 70th birthday.

the abstract version of the *Farkas lemma* [2, p. 24] or the *Krein-Rutman theorem* [2, Cor. 3.3.13] asserts that for a closed convex cone K in Y and A in $L(X, Y)$ one has

$$\left(A^{-1}K\right)^{+} = A^{*}(K^{+}) \tag{1}$$

if, and only, if $A^{*}(K^{+})$ is closed. Here $K^{+} := \{x \in Y\colon \langle x, y\rangle \geq 0, \text{ for all } y \in K\}$ is the positive *dual cone* and A^{*} denotes the transpose operator. Formula (1) in turn is the basis of strong duality in *abstract linear programming* [2, §5.3] and of the *Karush-Kuhn-Tucker theorem* [2, §6.1] and its generalizations.
A naive guess—based on two-dimensional reasoning—might be that the continuous linear image of a closed convex cone is always closed and in fact this is the case for finitely generated cones (or, equivalently, polyhedral cones). There are, however, simple examples (see for instance Example 1) that show that this naive guess is false. One might then speculate that the closedness of the image of a closed convex cone under a continuous linear mapping might, at least, be preserved under small perturbations of the linear map. This is made more plausible by the recent literature on the *distance to inconsistency* for abstract inequality systems [2, p.122] viewed as a generalization of the condition number. As pioneered by Renegar and others, there is a strictly positive distance to inconsistency for a system $\{x \in X\colon Ax - b \in K\}$ and one might hope this is true for closure.

2 Two Limiting Examples

Such speculation is also refuted by Example 1 which brings us to the other purpose of this paper. This is to (i) analyze when the continuous linear image of a closed convex cone is closed and to (ii) provide some sufficiency conditions under which the continuous linear image of a closed convex cone is closed. There is surprisingly little literature on the issue of when precisely a conical linear image is closed, see for example [4] and [1].

Example 1 Let

$$K := \{(w, x, y, z) \in \mathbb{R}^4 : 0 \leq w, 0 \leq x \text{ and } y^2 + (z - x)^2 \leq x^2\}.$$

Then K is an inverse linear image of the right-circular cone and so closed and convex. For each $\lambda \geq 0$ define the linear mapping $T_\lambda : \mathbb{R}^4 \to \mathbb{R}^3$ by, $T_\lambda(w, x, y, z) := (x - \lambda w, y, z)$, a rank-one linear perturbation of T_0. Note that if $\lambda = 0$ then $T_\lambda(K)$ is a closed cone in $\mathbb{R}^3$, but for every $\lambda > 0$ the image

$$T_\lambda(K) \equiv \{(x, y, z) : z > 0\} \cup \{(x, 0, 0) : x \in \mathbb{R}\}$$

which is convex but not closed. ☺

Example 1 shows that the closedness of the image of a closed convex cone under a linear mapping is not stable even under arbitrarily small rank-one perturbations. A more concrete, but closely related example, is given by the following abstract linear program.

Example 2 Let us consider the following closed convex cone in $\mathbb{R}^7$. We simplify things by letting $\overline{z} := (x_1, x_2, x_3, x_4, y_1, y_2, y_3)$ denote a point in $\mathbb{R}^7$ and let

$$K := \{\overline{z} \in \mathbb{R}^7 : 0 \le x_1, 0 \le x_2, x_3^2 + (x_4 - x_2)^2 \le x_2^2, 0 \le y_1, y_2^2 + (y_3 - y_1)^2 \le y_1^2\}$$

and let the linear mapping $z^* : \mathbb{R}^7 \to \mathbb{R}$ be defined by $z^*(\overline{z}) := x_4 + y_3$. For each $0 \le \lambda < \infty$ and $\mu \in \mathbb{R}$ let us define

$$A^\lambda := \begin{pmatrix} -\lambda & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } \beta^\mu := \begin{pmatrix} 1 \\ 1 \\ \mu \end{pmatrix}.$$

Then for each $0 \le \lambda < \infty$ and $\mu \in \mathbb{R}$ we can consider the optimization problem.

$$E(\lambda, \mu) := \inf\{z^*(x) : x \in K \text{ and } A^\lambda x = \beta^\mu\}.$$

It is easy to check that this infimum is obtained if, and only if, $\lambda = \mu = 0$. ☺

Example 2 shows that the existence of minima in abstract linear programming problems is not stable under arbitrarily small rank-one perturbations. This again highlights the difficulty of exactly characterizing closure of a conical linear image.

3 Preliminary Positive Results

We first collect and improve various known results. Versions of each are to be found in [2]. Our first result shows that the closedness of the image of a closed convex cone is related to the closedness of the sum of a closed convex cone with a finite dimensional subspace.

Proposition 1 *Let $T \in L(\mathbb{R}^n, \mathbb{R}^m)$ and let $\varnothing \ne K \subseteq \mathbb{R}^n$ be an arbitrary set. Then $T(K)$ is closed in $\mathbb{R}^m$ if, and only if, $K + \ker(T)$ is closed in $\mathbb{R}^n$.*

Proof: Suppose that $T(K)$ is closed. Then $T^{-1}(K) = K + \ker(T)$ is closed in $\mathbb{R}^n$, since T is continuous. Conversely, suppose that $K + \ker(T)$ is closed in $\mathbb{R}^n$. Then $C := [K + \ker(T)] \cap [\ker(T)]^\perp$ is also closed in $\mathbb{R}^n$, and moreover, $T(K) = T(C)$. Now, $T|_{\ker(T)^\perp}$ is a 1-to-1 linear mapping (and hence a homeomorphism) onto $T([\ker(T)]^\perp)$; which is a closed subspace of $\mathbb{R}^m$. Therefore, $T(K) = T(C)$ is closed in $T([\ker(T)]^\perp)$ and hence closed in $\mathbb{R}^m$. ☺

Proposition 2 *Suppose that K is a finitely generated convex cone in a vector space X. If $T : X \to Y$ is a linear mapping into a normed linear space Y then $T(K)$ is a closed convex cone in Y.*

Proof: Since K is finitely generated there exists a finite set $\{a_1, a_2, \ldots, a_n\}$ in X such that $K = \langle a_1, a_2, \ldots, a_n \rangle$. A simple calculation then reveals that $T(K) = \langle T(a_1), T(a_2), \ldots, T(a_n) \rangle$; which is finitely generated and hence closed, [2, page 25] ☺

Next we give some sufficiency conditions for the image of a closed convex cone to be closed.

Proposition 3 *Let $T \in L(\mathbb{R}^n, \mathbb{R}^m)$ and let K be a closed cone (not necessarily convex) in $\mathbb{R}^n$. If*

$$K \cap \ker(T) = \{0\}$$

then there exists a neighbourhood $\mathcal{N}$ of T in $L(\mathbb{R}^n, \mathbb{R}^m)$ such that $S(K)$ is closed in $\mathbb{R}^m$ for each $S \in \mathcal{N}$.

Proof: Let $C := \{k \in K : \|k\| = 1\}$. Then both C and $T(C)$ are compact and $0 \notin T(C)$ therefore $\operatorname{dist}(0, T(C)) > 0$ and so there exists a neighbourhood $\mathcal{N}$ of T in $L(\mathbb{R}^n, \mathbb{R}^m)$ such that $\operatorname{dist}(0, S(C)) > 0$ for each $S \in \mathcal{N}$. Since

$$S(K) = \{\lambda c : c \in S(C) \text{ and } 0 \leq \lambda < \infty\}$$

and $S(C)$ is compact for each $S \in \mathcal{N}$, it follows that $S(K)$ is closed in $\mathbb{R}^m$ for each $S \in \mathcal{N}$, as claimed. ☺

Proposition 4 *Let $T \in L(\mathbb{R}^n, \mathbb{R}^m)$ and let K be a closed convex cone in $\mathbb{R}^n$. If $K \cap \ker(T)$ is a linear subspace then $T(K)$ is closed convex cone in $\mathbb{R}^m$.*

Proof: Let $M := K \cap \ker(T)$ and let $N := M^\perp \cap [K + M] = M^\perp \cap K$. Then $T(N) = T(K)$ and $N \cap \ker(T) = \{0\}$. Therefore, $T(K) = T(N)$ is a closed convex cone on application of Proposition 3. ☺

For a subset D of a vector space V, the *core* of D, denoted, $\operatorname{cor}(D)$, is the set of all points $d \in D$ where for each $x \in V \setminus \{d\}$ there exists an $0 < r < 1$ such that $\lambda x + (1 - \lambda)d \in D$ for all $0 \leq \lambda < r$. Clearly if the affine span $\operatorname{aff}(D) \neq V$ then $\operatorname{cor}(D) = \varnothing$. In this case the following concept is useful.

Given a subset C of a vector space V, the *intrinsic core* of C, denoted $\operatorname{icor}(A)$, is the set of all points $c \in C$ where for each $x \in \operatorname{aff}(C)$ there exists an $0 < r < 1$ such that $\lambda x + (1 - \lambda)c \in C$ for all $0 \leq \lambda < r$.

One of the most important properties of the intrinsic core is that if C is a convex subset of a finite dimensional vector space V then $\operatorname{icor}(C) \neq \varnothing$, [3, page 7]. In fact, if V is a finite dimensional topological vector space then $\operatorname{icor}(C)$ is dense in C for each convex subset C of the space V. Another important property of the core is that for a convex subset C of a finite dimensional topological vector space, $\operatorname{cor}(C) = \operatorname{int}(C)$, [2, Theorem 4.1.4].

The reason for our interest in the intrinsic core is based in the following result.

Proposition 5 *Let Y be a normed linear space, $T : \mathbb{R}^n \to Y$ be a linear transformation and let K be a closed cone in $\mathbb{R}^n$. If*

$$\ker(T) \cap \operatorname{icor}(K) \neq \varnothing$$

then $T(K)$ is a finite dimensional linear subspace of Y and hence a closed convex cone.

Proof: By [2, Problem 13 part (e)], $\{0\} = T(\ker(T) \cap \operatorname{icor}(K)) \subseteq T(\operatorname{icor}(K)) \subseteq \operatorname{icor}(T(K))$. Since $T(K)$ is a cone we see that $T(K) = \operatorname{aff}(T(K)) = \operatorname{span}(T(K))$. The result now follows, since every finite dimensional subspace of a normed linear space is closed. ☺

From Proposition 4 and Proposition 5 we see that:

Corollary 1 *The only way $T(K)$ can fail to be closed is if*

$$\ker(T) \cap K \subseteq K \setminus \operatorname{icor}(K)$$

and that at the same time $\ker(T) \cap K$ is not a linear subspace.

4 The Main Results

We are now ready for our principle positive results.

Lemma 1 *Suppose that K is a closed convex cone in $\mathbb{R}^n$, $Y := K - K$, $T \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $T|_Y \in L(Y, \mathbb{R}^m)$ has rank m. If*

$$\ker(T) \cap \operatorname{icor}(K) \neq \varnothing$$

then there exists a neighbourhood $\mathcal{W}$ of T in $L(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\ker(S) \cap \operatorname{icor}(K) \neq \varnothing$$

for all $S \in \mathcal{W}$. In particular, $S(K)$ is a closed convex cone in $\mathbb{R}^m$ for each $S \in \mathcal{W}$.

Proof: Let $\mathcal{M} \subseteq L(Y, \mathbb{R}^m)$ be the family of all mappings with rank m. It is routine to show that $\mathcal{M}$ is a dense open subset of $L(Y, \mathbb{R}^m)$ since $T|_Y \in \mathcal{M}$ and so $m \leq \operatorname{Dim}(Y)$.
(a) We shall consider first the case when $Y = \mathbb{R}^n$. For each $S \in L(\mathbb{R}^n, \mathbb{R}^m)$, let A_S denote the matrix representation of S with respect to the standard bases on $\mathbb{R}^n$ and $\mathbb{R}^m$. Since $\operatorname{rank}(S) = m$ for all $S \in \mathcal{M}$, the rows of each A_S, $S \in \mathcal{M}$ are linearly independent. Therefore, for each $S \in \mathcal{M}$, the matrix $A_S^t(A_S A_S^t)^{-1} A_S$ is well-defined and represents the orthogonal projection of $\mathbb{R}^n$ onto the row space of A_S. Since the null space of A_S is perpendicular to the row space of A_S, $I_n - A_S^t(A_S A_S^t)^{-1} A_S$ is the matrix representation of the projection of $\mathbb{R}^n$ onto the null space of A_S. Here I_n denotes the $n \times n$ identity matrix.
Next, we shall consider $M_{(n,n)}$, the space of all $n \times n$ matrices, endowed with the linear topology of component-wise convergence. With respect to this topology the mapping $\varphi : \mathcal{M} \to M_{(n,n)}$ defined by, $\varphi(S) := I_n - A_S^t(A_S A_S^t)^{-1} A_S$ is continuous. Moreover, for any fixed $x \in \mathbb{R}^n$, the mapping $S \mapsto \varphi(S)(x)$ is continuous on $\mathcal{M}$ and $\varphi(S)(x) \in \ker(S)$ for all $S \in \mathcal{M}$. Therefore, if we choose $x \in \ker(T) \cap \operatorname{icor}(K) = \ker(T) \cap \operatorname{cor}(K) = \ker(T) \cap \operatorname{int}(K)$ then there exists a neighbourhood $\mathcal{W}$ of T in $L(\mathbb{R}^n, \mathbb{R}^m)$ such that $\varphi(S)(x) \in \ker(S) \cap \operatorname{int}(K) = \ker(S) \cap \operatorname{cor}(K) = \ker(S) \cap \operatorname{icor}(K)$ for all $S \in \mathcal{W}$. This completes the proof for the special case when $Y = \mathbb{R}^n$.
(b) In the general case, consider the mapping $R : L(\mathbb{R}^n, \mathbb{R}^m) \to L(Y, \mathbb{R}^m)$ defined by, $R(S)(x) := S(x)$ for all $x \in Y$. Then R is a continuous linear mapping from $L(\mathbb{R}^n, \mathbb{R}^m)$ into $L(Y, \mathbb{R}^m)$. We now apply the first part of the proof to $R(T) \in \mathcal{M}$ to obtain a neighbourhood $\mathcal{W}'$ of $R(T)$ in $L(Y, \mathbb{R}^m)$ such that $\ker(S) \cap \operatorname{icor}(K) \neq \varnothing$ for all $S \in \mathcal{W}'$. Therefore, if we let $\mathcal{W} := R^{-1}(\mathcal{W}')$ then $\mathcal{W}$ is an open neighbourhood of T in $L(\mathbb{R}^n, \mathbb{R}^m)$ and $\ker(S) \cap \operatorname{icor}(K) \supseteq \ker(R(S)) \cap \operatorname{icor}(K) \neq \varnothing$ for all $S \in \mathcal{W}$. ☺

The next result shows—as promised—that although it is not true that, if $T(K)$ is closed for some closed convex cone K then $S(K)$ is closed for all S in some neighbourhood of T, it is "almost" true, in the sense that for "almost all" $T \in L(\mathbb{R}^n, \mathbb{R}^m)$ if $T(K)$ is closed then there exists a neighbourhood $\mathcal{W}$ of T such that $S(K)$ is closed for all $S \in \mathcal{W}$.

Theorem 1 *Suppose that K is a closed convex cone in $\mathbb{R}^n$ then*

$$\mathrm{int}\{T \in L(\mathbb{R}^n, \mathbb{R}^m) : T(K) \,\text{is closed}\}$$

is a dense open subset of $L(\mathbb{R}^n, \mathbb{R}^m)$.

Proof: Let $Y := K - K$, and let $\mathcal{M} \subseteq L(\mathbb{R}^n, \mathbb{R}^m)$ be the family of all linear mappings T such that $T|_Y$ has maximal rank (i.e., $\mathrm{rank}(T|_Y) = \min\{m, \mathrm{Dim}(Y)\}$). It is standard that $\mathcal{M}$ is a dense open subset of $L(\mathbb{R}^n, \mathbb{R}^m)$. Hence it will be sufficient to show that

$$\mathrm{int}\{T \in L(\mathbb{R}^n, \mathbb{R}^m) : T(K) \text{ is closed}\}$$

is dense in $\mathcal{M}$. If $\mathrm{Dim}(Y) \leq m$ then $T|_Y$ is one-to-one for each member of $\mathcal{M}$ and so $\ker(T) \cap K = \{0\}$ for each $T \in \mathcal{M}$ and thus we are done by Proposition 3. Hence we shall assume that $m < \mathrm{Dim}(Y)$. Let T be any element of $\mathcal{M}$ and let $\mathcal{N}$ be any neighbourhood of T in $\mathcal{M}$. If $\ker(T) \cap K = \{0\}$ then we are again done by Proposition 3. So let us suppose that $\{0\} \neq \ker(T) \cap K$. If $\ker(T) \cap \mathrm{icor}(K) \neq \varnothing$ then by Lemma 1 there exists a neighbourhood $\mathcal{N}'$ of T in $\mathcal{N}$ such that $S(K)$ is closed for each $S \in \mathcal{N}'$. Thus, we will suppose that

$$\{0\} \neq \ker(T) \cap K \subseteq K \setminus \mathrm{icor}(K).$$

Choose $k_0 \in [\ker(T) \cap K] \setminus [\{0\} \cup \mathrm{icor}(K)]$. Then since $\inf\{\|T(k)\| : k \in \mathrm{icor}(K) \cap \mathcal{V}\} = 0$ for each neighbourhood $\mathcal{V}$ of k_0 there exists a $k' \in \mathrm{icor}(K)$ and $S \in \mathcal{N}$ such that $S(k') = 0$. We now re-apply Lemma 1 to obtain a neighbourhood $\mathcal{U}$ of S in $\mathcal{N}$ such that $\ker(S') \cap \mathrm{icor}(K) \neq \varnothing$ for all $S' \in \mathcal{U}$ and so $S'(K)$ is a closed subspace for each $S' \in \mathcal{U}$. ☺

This result should be compared to a corresponding result on negligibility in Hausdorff measure in [5]. These results are largely motivated by semi-definite programming, [6].

Corollary 2 *For any given closed convex cone K, the abstract Farkas lemma of equation (1) holds for a dense open set of operators.*

Proof: The adjoint mapping between $L(\mathbb{R}^n, \mathbb{R}^m)$ and $L(\mathbb{R}^m, \mathbb{R}^n)$ preserves both density and openness. Indeed, for a dense open set of operators both $A(K)$ and $A^*(K^+)$ are simultaneously closed. ☺

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