Central automorphisms of finite Laguerre planes

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What is a Laguerre plane?

Definition

A finite Laguerre plane $\mathcal{L} = (P, \mathcal{C}, \mathcal{G})$ *of order* n *consists of a set* P *of* $n(n+1)$ *points, a set* $\mathcal{C}$ *of* n^3 *circles and a set* $\mathcal{G}$ *of* $n+1$ *generators (where circles and generators are both subsets of* P*) such that the following three axioms are satisfied:*

(G) $\mathcal{G}$ *partitions* P *and each generator contains* n *points.*

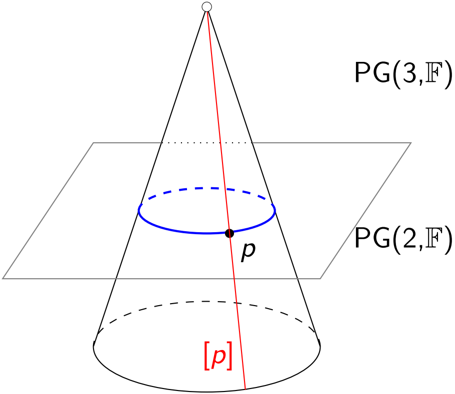
(C) *Each circle intersects each generator in precisely one point.*

(J) *Three points no two of which are on the same generator can be joined by a unique circle.*

A finite Laguerre plane of order n is a transversal design $TD_1(3, n+1, n)$, or equivalently, an orthogonal array of strength 3 on n symbols, $n+1$ constraints and index 1. In case n is odd the Laguerre plane corresponds to an antiregular generalized quadrangle of order (n, n).

Models of Laguerre planes

All known finite Laguerre planes are *ovoidal*, that is, they are obtained as the geometry of non-trivial plane sections of a cone, minus its vertex, over an oval in 3-dimensional projective space over a finite field $\mathbb{F}$. In case the oval is a conic one obtains the *miquelian Laguerre plane* over $\mathbb{F}$.



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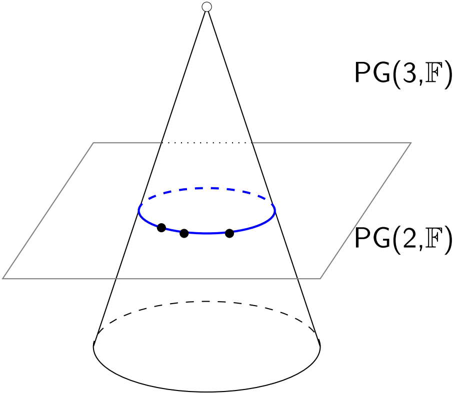

Derived incidence structures

The derived design at a point p of a finite Laguerre plane of order n is an affine plane of order n. Circles not passing through p induce ovals in the projective completion of the affine plane at p by adding the point ω at infinity of vertical lines that come from generators of the Laguerre plane.

A planar representation of an ovoidal Laguerre plane $\mathcal{L}(f)$ has point set $(\mathbb{F} \cup \{\infty\}) \times \mathbb{F}$ and circles are of the form

$$\{(x, af(x) + bx + c) \mid x \in \mathbb{F}\} \cup \{(\infty, a)\}$$

where $a, b, c \in \mathbb{F}$ and $f : \mathbb{F} \to \mathbb{F}$ is parabolic.

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Theorem

- *A finite Laguerre plane of odd order with a Desarguesian derivation is miquelian. (Chen, Kaerlein 1973, Payne, Thas 1976)*
- *A Laguerre plane of order at most ten is ovoidal and, in fact, miquelian except in case of order 8. (S. 1992, 2003)*

Laguerre homotheties

An *automorphism* of a Laguerre plane $\mathcal{L}$ is a permutation of the point set that takes generators to generators and circles to circles.
A *homothety* of $\mathcal{L}$ is an automorphism of $\mathcal{L}$ that is either the identity or fixes precisely two points on different generators and induces a homothety in the derived affine plane at each of these two fixed points. One speaks of a $\{p, q\}$-*homothety* if p, q are the two fixed points.

A group Γ of automorphisms of $\mathcal{L}$ is said to be $\{p, q\}$-transitive if Γ contains a subgroup of $\{p, q\}$-homotheties that acts transitively on each circle through p and q minus p and q.

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Ruth Kleinewillinghöfer investigated the possible configurations $\mathcal{H}$ of all unordered pairs of distinct points $\{p, q\}$ for which the automorphism group of $\mathcal{L}$ is $\{p, q\}$-transitive and found 13 feasible configurations.

One says that $\mathcal{L}$ is of type m if $\mathcal{H}$ is as in configuration m.

Kleinewillinghöfer types w.r.t. homotheties

1. $\mathcal{H} = \emptyset$.

⋮

5. There are a circle C and a fixed-point-free involution $\phi : C \to C$ such that $\mathcal{H} = \{\{p, \phi(p)\} \mid p \in C\}$.

⋮

8. There are two distinct generators F, G such that $\mathcal{H} = \{\{p, q\} \mid p \in F, q \in G\}$.
9. Each point of $\mathcal{L}$ is in exactly one pair in $\mathcal{H}$.

⋮

11. There is a point p such that $\mathcal{H} = \{\{p, q\} \mid q \in P \setminus [p]\}$.
12. There is a generator G such that $\mathcal{H} = \{\{p, q\} \mid p \in G, q \in P \setminus G\}$.
13. $\mathcal{H}$ consists of all unordered pairs of points on different generators.

Examples

A finite ovoidal Laguerre plane has Kleinewillinghöfer type 1, 8, 12 or 13.

The respective types are obtained as $\mathcal{L}(f)$ over $\mathrm{GF}(2^h)$ when

$$f(x) = \begin{cases} x^{1/6} + x^{3/6} + x^{5/6} & \text{where } h \geq 5 \text{ is odd;} \\ x^6 & \text{where } h \geq 5 \text{ is odd;} \\ x^{2^i} & \text{where } \gcd(i, h) = 1; \\ x^2 & \text{any } h. \end{cases}$$

Characterisations and exclusions

Theorem

- *A Laguerre plane is of Kleinewillinghöfer type 13 if and only if it is miquelian. (Hartmann, 1982)*
- *A finite Laguerre plane has Kleinewillinghöfer type 12 if and only if it has even order and is ovoidal over a proper translation oval (not a conic). (Hartmann, 1982, S. 2015)*
  ($\mathcal{H} = \{\{p, q\} \mid p \in G, q \in P \setminus G\}$)
- *A finite Laguerre plane of Kleinewillinghöfer type 5 or 9 has odd order. (Kleinewillinghöfer, 1979)*
  (type 5: $\mathcal{H} = \{\{p, \phi(p)\} \mid p \in C\}$, ϕ fixed-point-free involution on C,
  type 9: each point is in exactly one pair in $\mathcal{H}$)

Characterisations and exclusions, cont.

Theorem

- *A finite Laguerre plane that contains a group of automorphisms of Kleinewillinghöfer type 11 is miquelian or ovoidal over a translation oval; the plane then is of type 13 or 12.*
  ($\mathcal{H} = \{\{p, q\} \mid q \in P \setminus [p]\}$)

- *A finite Laguerre plane of type 8 is an elation Laguerre plane, that is, the plane admits a group of automorphisms that acts trivially on the set of generators and regularly on the set of circles.*
  ($\mathcal{H} = \{\{p, q\} \mid p \in F, q \in G\}$)

- *A finite non-ovoidal elation Laguerre plane has type 1 or 8.*