Generalised quadrangles with primitive automorphism groups

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- A generalised quadrangle (GQ) of order (s, t) is a point–line incidence geometry $\mathcal{Q} = (\mathcal{P}, \mathcal{L})$ such that
  - (i) every point (line) is incident with $t + 1$ lines ($s + 1$ points);
  - (i) every two points are incident with at most one common line;
  - (ii) for every non-incident point–line pair (P, ℓ), there is a unique line concurrent with ℓ and incident with P.
- Assume $\mathcal{Q}$ is thick, i.e. $s \geqslant 2$ and $t \geqslant 2$.
- Introduced by Tits (1959) in an attempt to find geometric models for simple groups of Lie type.
- Classical examples: low-rank polar spaces, admitting PSp$(4, q)$, PSU$(4, q)$, PSU$(5, q)$.
- Examples constructed from “hyperovals” in PG$(2, 2^f)$.
- Other ‘synthetic’ constructions.

- The classical GQs have automorphism groups acting primitively on both $\mathcal{P}$ and $\mathcal{L}$, and transitively on flags.
- Only two non-classical flag-transitive GQs are known: both from hyperovals, both point-primitive but line-imprimitive.
- It is conjectured that there are no other flag-transitive GQs (e.g. Kantor 1991; possibly earlier).
- Classification is a hard problem, but it also makes sense to ask about primitivity (where we have O'Nan–Scott, CFSG).
- Bamberg–Giudici–Morris–Royle–Spiga, 2012:
  (i) if $G \leqslant \mathrm{Aut}(\mathcal{Q})$ is point- and line-primitive, then G is almost simple;
  (ii) if G is point-primitive, flag-transitive and almost simple, then $\mathrm{soc}(G)$ is not alternating or sporadic.

- Here $\mathcal{P}$ is identified with a vector space $N = \mathbb{F}_p^d$, and $G \leqslant N \rtimes G_0 \leqslant \mathrm{AGL}(d,p)$ with $G_0 \leqslant \mathrm{GL}(d,p)$ irreducible.
- BGPP, 2014: if G is point-primitive and line-transitive, then $\mathcal{Q}$ is one of the two flag-transitive 'hyperoval' examples.
- Idea of proof:
  - (i) Show that $d = 3n$ and $p = 2$.
  - (ii) Lines incident with $0 \in \mathcal{P}$ comprise a 'pseudo-hyperoval' in $\mathrm{PG}(3n-1, 2^f)$ (corresponds to a hyperoval in $\mathrm{PG}(2, 2^{nf})$).
  - (iii) Classify pseudo-hyperovals with irred. transitive stabiliser.
- Problem seems too hard without line-transitivity, because every hyperoval yields a GQ.

- G has two point-regular normal subgroups isomorphic to $N \cong T^k$, with T a non-abelian finite simple group, $k \geqslant 1$.
- $N \rtimes \mathrm{Inn}(N) \leqslant G \leqslant N \rtimes \mathrm{Aut}(N)$.
- BPP, 2015: if $G \leqslant \mathrm{Aut}(\mathcal{Q})$ is point-primitive and line-transitive, then G cannot have holomorph type.
- Idea of proof:
  - (i) Again show that lines incident with $1 \in N = \mathcal{P}$ are subgroups (use $G_1 \geqslant \mathrm{Inn}(N)$ instead of N abelian).
  - (ii) After some arguments, this forces $k \leqslant 2$.
  - (iii) $k \leqslant 2$ handled using CFSG: (i) implies inequalities of the form $|T| \leqslant c|\mathrm{Out}(T)|^4$, which usually fail.

- Simple diagonal: $\mathrm{soc}(G) = T^k$, $N = T^k/\mathrm{Diag}(T^k) \cong T^{k-1}$ point-regular, $\mathrm{Inn}(T) \leqslant G_1 \leqslant \mathrm{Aut}(T) \times \mathrm{Sym}(k)$.
- Compound diagonal: $G \leqslant H \text{ wr } \mathrm{Sym}(r)$ for some SD-type primitive group H and some $r \geqslant 2$.
- Results so far (BPP, 2016):
  - If a CD-type example exists, then $r = 2$ or 3.
  - Moreover, every conjugacy class of T must have size at least $|T|^{3/5}$, which rules out arbitrarily large Lie rank.
  - e.g. if $T \cong \mathrm{PSL}(n, q)$ or $\mathrm{PSU}(n, q)$ then $n \leqslant 5$ (roughly).
  - SD is harder, but adding flag-transitivity implies $k \leqslant 6$.

(i) Lemma: let $(\mathcal{P}', \mathcal{L}')$ be the substructure of $\mathcal{Q}$ fixed by some $\theta \in \mathrm{Aut}(\mathcal{Q})$. Then (with some assumptions) $|\mathcal{P}'| \leqslant |\mathcal{P}|^{4/5}$.

(ii) For CD-type with $r \geqslant 4$, we can always find some θ that fixes enough points of $\mathcal{Q}$ to contradict the lemma.

(iii) For $r \leqslant 3$ (and also for SD case), use the lemma to show that all conjugacy classes of T must be 'large'.

(iv) The hardest case is SD:
- If the primitive group P permuting the simple direct factors of T^k contains $\mathrm{Alt}(k)$, the lemma implies $k \leqslant 6$.
- Else P is small (e.g. Maróti, 2002), flag-transitivity implies $|T| \leqslant$ polynomial in $|\mathrm{Out}(T)|$ when $k \geqslant 7$, use CFSG.

To do (remaining O'Nan–Scott types)

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