Cyclic m-cycle systems of near-complete graphs

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based on joint work with Heather Jordon

University of Lethbridge

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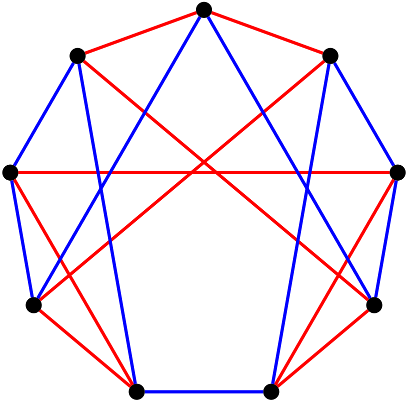

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Most work on this problem has been done on the case where the graph being decomposed is a complete graph, K_n, if n is odd, or $K_n - I$ if n is even, where I is any 1-factor (matching); the latter case is what is referred to in the title as a "near-complete graph."

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Theorem (Alspach, Gavlas; Šajna)

The "obvious" necessary conditions are also sufficient; that is, an m-cycle system of K_n or $K_n - I$ exists if and only if $n \geq m$, every vertex of K_n or $K_n - I$ has even degree, and m divides the number of edges in K_n or $K_n - I$, respectively.

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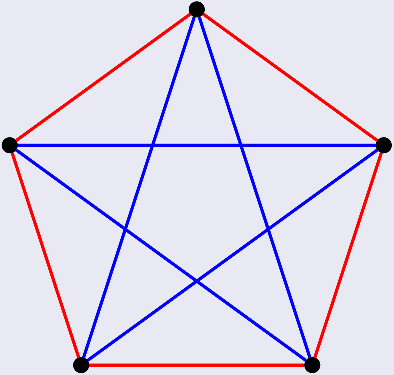
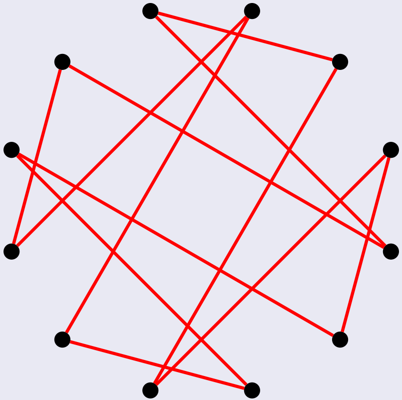
An m-cycle system $\mathcal{C}$ of a graph G with vertex set $\mathbb{Z}_n$ is *cyclic* if, for every cycle $C = (v_1, v_2, \ldots, v_m)$ in $\mathcal{C}$, the cycle $\rho(C) = (\rho(v_1), \rho(v_2), \ldots, \rho(v_m))$ is also in $\mathcal{C}$.

Example:



Fancier example: $K_{12} - I$



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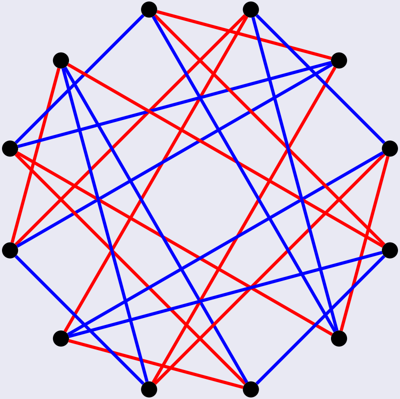

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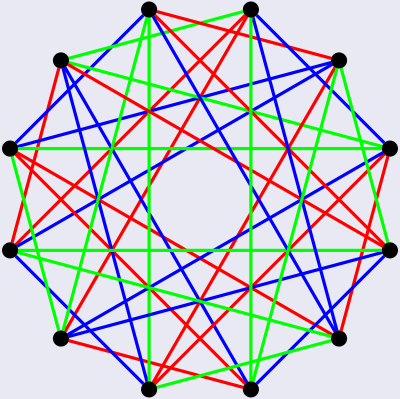

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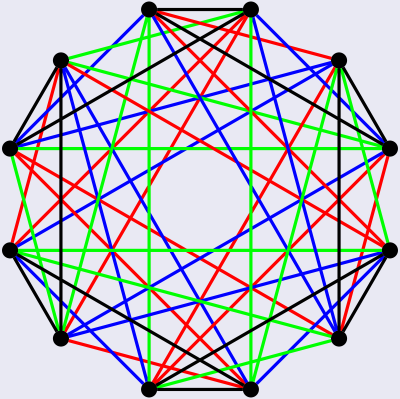

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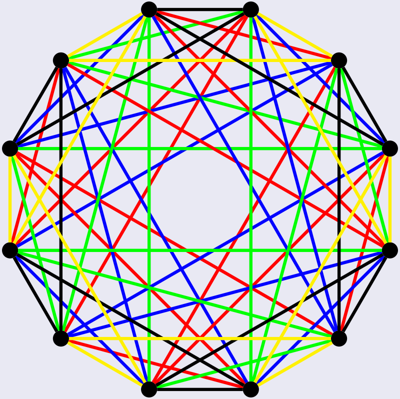

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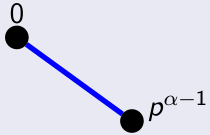
Suppose $n = p^\alpha$. Consider the edge from 0 to $p^{\alpha-1}$, and the cycle C containing this edge in K_n. Let k be the length of the orbit of C, and recall that we must have $k|n$, so k is a power of p.

Proof that K_{p^α} has no cyclic hamiltonian cycle system



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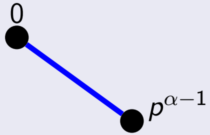

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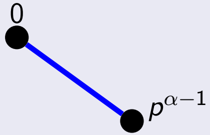

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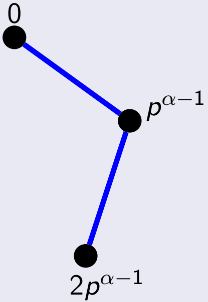

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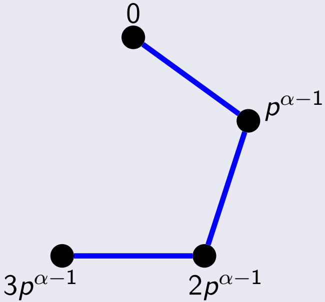

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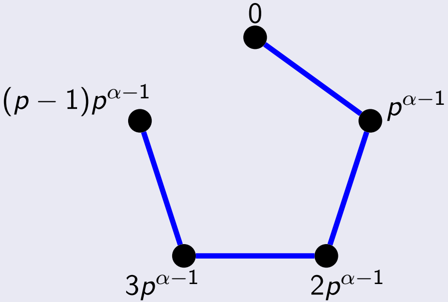

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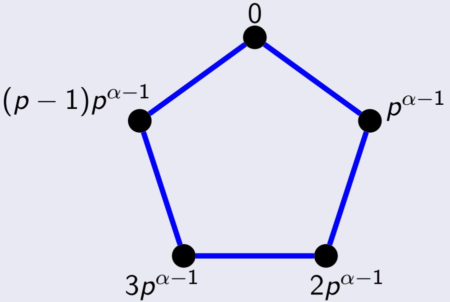

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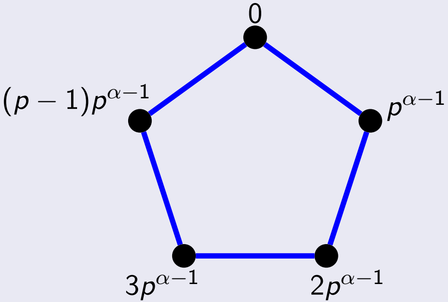

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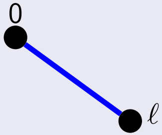

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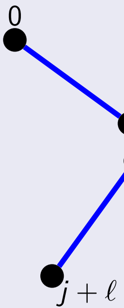

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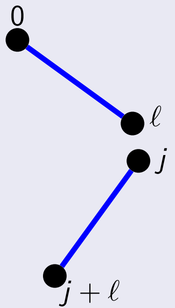

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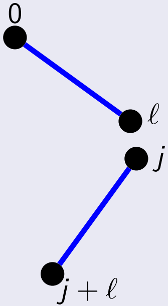

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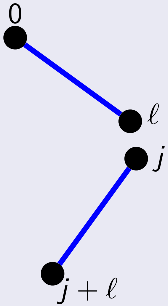

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There is a similar argument for $n = 15$.

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Theorem (Bryant, Gavlas, Ling)

There is a cyclic m-cycle system of $K_{2mk+2} - I$ if and only if $mk \equiv 0, 3 \pmod{4}$.

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Most of the cases where there is no system, are eliminated by parity conditions like those in the hamiltonian case.

Open questions

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Thank you!