Hemisystems and Relative Hemisystems of Generalised Quadrangles and their Generalisations

Joint work with John Bamberg and Michael Giudici

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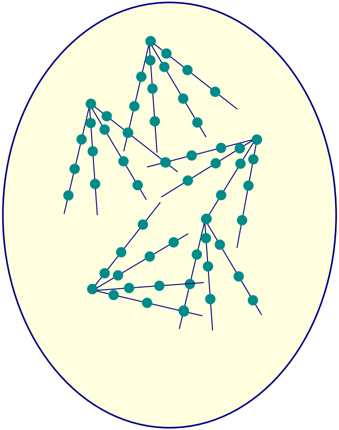
17th February 2016

A train analogy and *m*-covers

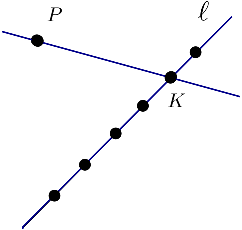
- Is it possible to choose a set of train lines such that every station is on exactly m of them?
- Such a set of lines is called an m-cover.
- When $m = 1$, it is called a spread.



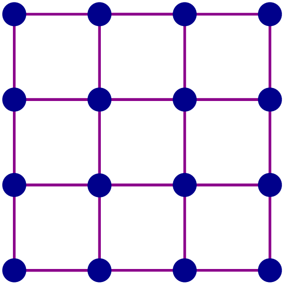
Generalised Quadrangles

A **generalised quadrangle** of order (s, t) is an incidence structure of points and lines such that:

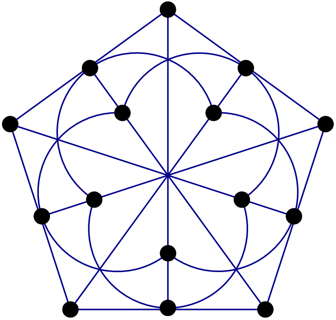
- Any two points are incident with at most one line.
- Every point is incident with $t + 1$ lines.
- Every line is incident with $s + 1$ points.
- For any point P and line ℓ that are not incident, there is a unique point K on ℓ that is collinear with P.



Examples of Generalised Quadrangles



$GQ(3,1)$



$GQ(2,2)$

Hermitian and Symplectic Spaces

- A **Hermitian space**, denoted $\mathrm{H}(3, q^2)$, is a generalised quadrangle of order (q^2, q).
- A **symplectic space**, denoted $\mathrm{W}(3, q)$, is a generalised quadrangle of order (q, q).

These are both examples of **polar spaces.**

Dualising

The **dual** of a generalised quadrangle of order (s, t) (achieved by swapping the points and lines) is a generalised quadrangle of order (t, s).

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- Any two points are incident with at most one line.
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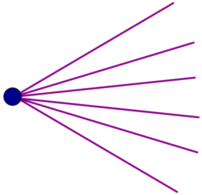
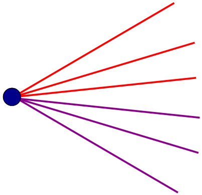
- Every **point** is incident with $s + 1$ **lines**.
- For any **point** P and **line** ℓ that are not incident, there is a unique **line** a on P that is collinear with ℓ.

Hemisystems

- In 1965, B. Segre proved that the only (non-trivial) m-covers on $\mathrm{H}(3, q^2)$ have $m = \frac{q+1}{2}$.
- Called these $\frac{q+1}{2}$-covers **hemisystems**.
- Gave an example of a hemisystem on $\mathrm{H}(3, 3^2)$.



$q + 1$ lines



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Structures arising from hemisystems

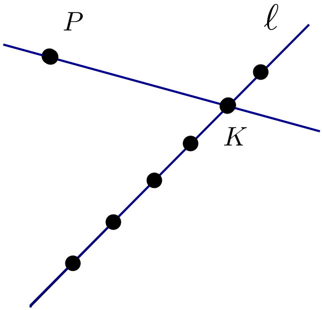
- Strongly regular graphs,
- Cometric Q-antipodal association schemes.
- Partial quadrangles with parameters $(\frac{q-1}{2}, q^2, \frac{(q-1)^2}{2})$,

What about Higher Dimensions?

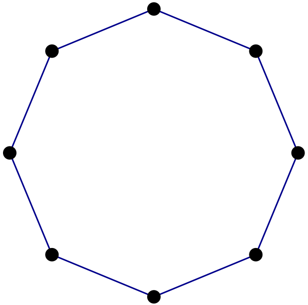
A **regular near** $2d$-**gon** of order (s, t) is an incidence structure of points and lines such that

- Any two points are incident with at most one line.
- The point graph of the structure is connected with diameter $d \geq 1$.
- For each point P and line ℓ, there is a unique point Q on ℓ that is "closest" to P.

Examples of closeness



GQ axiom



Regular n-gon

Dual polar spaces

We can construct a **dual polar space** from a polar space like $\mathrm{H}(2d-1, q^2)$ in a similar way to dualising generalised quadrangles.

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points	maximals
lines	next to maximals
k-dimensional subspaces	$(d-k)$-dimensional subspaces

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- A dual polar space is a regular near polygon.
- $\mathrm{DH}(2d-1, q^2)$ is a regular near $2d$-gon

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Spreads to ovoids

- An m-**ovoid** S is a set of points such that every line meets S in m points.
- An m-cover of $\mathrm{H}(3, q^2)$ is an m-ovoid of $\mathrm{DH}(3, q^2)$.
- **Not true** in higher dimensions because the lines of $\mathrm{H}(2d-1, q^2)$ are no longer the points of $\mathrm{DH}(2d-1, q^2)$

What about $(q+1)/2$-ovoids in higher dimensions?

Theorem (Vanhove 2011)

Suppose S is a $\frac{q+1}{2}$-ovoid of $\mathrm{DH}(2d-1, q^2)$, q odd. Then the subgraph induced by S on the point graph of $\mathrm{DH}(2d-1, q^2)$ is **distance regular**, with classical parameters

$$(d, b, \alpha, \beta) = \left(d, -q, -\frac{q+1}{2}, -\left(\frac{(-q)^d+1}{2}\right)\right)$$

An open question

Vanhove (2011)

Does there exist any $\frac{q+1}{2}$-ovoids of the dual polar space $\mathrm{DH}(2d-1, q^2)$ when $d \geq 3$?

- This is a hard problem!
- An example would give us a new distance regular graph with classical parameters $\left(d, -q, -\frac{q+1}{2}, -\left(\frac{(-q)^d+1}{2}\right)\right)$.

The search for $\frac{q+1}{2}$-ovoids

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There are **no** $\frac{q+1}{2}$-ovoids of $\mathrm{DH}(5, 3^2)$.

Simplifyng the problem

Lemma (L. 2015)

A $\frac{q+1}{2}$-ovoid of $\mathrm{DH}(5, q^2)$ induces a $\frac{q+1}{2}$-ovoid of an embedded $\mathrm{DW}(5, q)$.

- So no $\frac{q+1}{2}$-ovoid of an embedded $\mathrm{DW}(5, q) \implies$ no $\frac{q+1}{2}$-ovoid of $\mathrm{DH}(5, q^2)$.
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- So no $\frac{q+1}{2}$-ovoid of an embedded $\mathrm{DW}(5, q) \Longrightarrow$ no $\frac{q+1}{2}$-ovoid of $\mathrm{DH}(5, q^2)$.
- Use the same technique as before in GAP and Gurobi.
- $\mathrm{DW}(5, 3)$ and $\mathrm{DW}(5, 5)$ have no $\frac{q+1}{2}$-ovoids.

Open problems

Conjecture

There are no $\frac{q+1}{2}$-ovoids of $\mathrm{DH}(5, q^2)$ for any q odd.

Other questions

- Are there $\frac{q+1}{2}$-ovoids of $\mathrm{DH}(2d-1, q^2)$ for $d > 3$?
- Are there $\frac{q+1}{2}$-ovoids of $\mathrm{DW}(5, q)$?
- Are there $\frac{q+1}{2}$-ovoids of regular near $2d$-gons that are not dual polar spaces?