Equivelar toroids with few flag-orbits

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Symmetries and Covers of Discrete Objects
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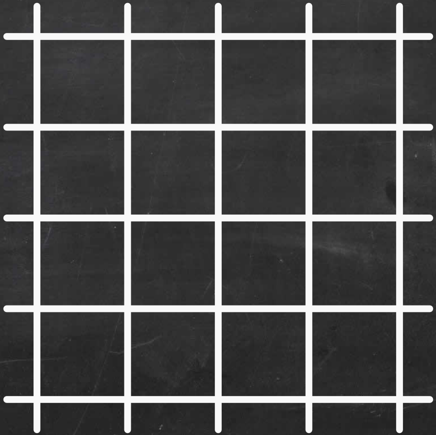
Tessellations

An Euclidean Tessellation $\mathcal{U}$ of $\mathbb{E}^n$ is a family of convex n-polytopes such that

* $\mathcal{U}$ is a cover of $\mathbb{E}^n$ and the cells tile $\mathbb{E}^n$ in a face-to-face manner.
* $\mathcal{U}$ is locally finite.

Tessellations



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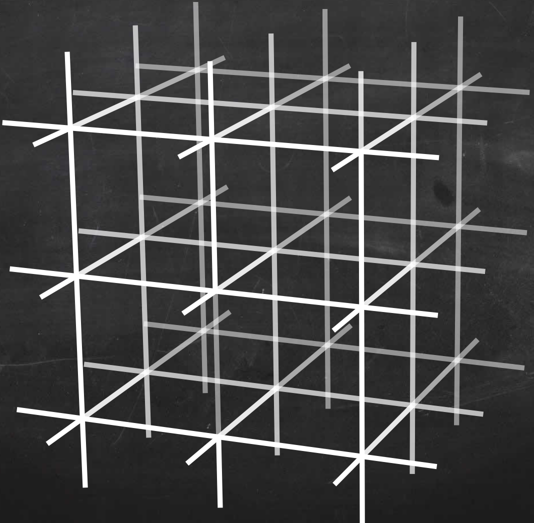

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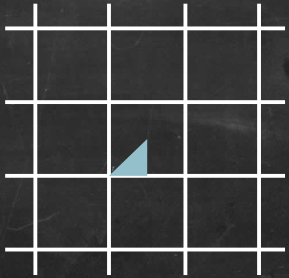

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* $G(\mathcal{U})$ acts on the set of flags of $\mathcal{U}$. We say that $\mathcal{U}$ is regular if this action is transitive.

Regular Tessellations

Regular tessellations are well-known:

* If $n = 2$:
  - Cubic tessellation $\{4, 4\}$.
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* If $n \in \{3, 5, 6 \ldots\}$:
  - Cubic tessellation $\{4, 3^{n-2}, 4\}$

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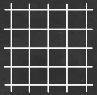

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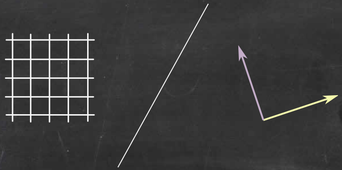

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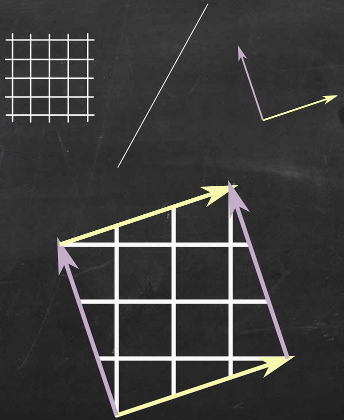

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* Provide examples of abstract polytopes.

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$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\gamma} & \mathcal{U} \\ \downarrow & & \downarrow \\ \mathcal{U}/\Lambda & \overset{\bar{\gamma}}{\dashrightarrow} & \mathcal{U}/\Lambda \end{array}$$

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* Define $Aut(\mathcal{U}/\Lambda) = Norm_{G(\mathcal{U})}(\Lambda)/\Lambda$.
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* $\mathcal{U}/\Lambda \cong \mathcal{U}/\Lambda'$ if and only if Λ and Λ' are conjugate.

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* A toroid $\mathcal{T}$ is equivelar if it is induced by a regular tessellation.

Toroids
What do we know?

* Regular toroids are classified:
  - If $n = 2$ there are two families. (Coxeter, 1948)
  - If $n \geqslant 3$ there are three families. (McMullen and Schulte, 1996)

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- Chiral toroids are classified, they only exist in dimension 2 (chiral maps). (Hartley, McMullen and Schulte, 1999)

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* Toroids of dimension three are classified (Hubard, Orbanić, Pellicer and Weiss, 2012)

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- Still useful...

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- Q: Can we classify (equivelar) 2-orbits $(n+1)$-toroids?
- Q: Do they even exist if $n > 3$?

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* If $n \geqslant 3$, all 2-orbits $(n+1)$-toroids are few-orbits toroids.

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  - If n is odd, there are no 2-orbit toroids.
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- If $n \geqslant 5$, there are no cubic toroids with k orbits if $2 < k < n$.

Few-orbits toroids

$(4+1)$-toroids

Cubic toroids:

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* 2-orbits toroids: one family in class $2_{\{1,2,3\}}$.

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- ∗ 3-orbits toroids: two families with different symmetry type.

Open problems/Future work

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* Study few-orbits structures in other Euclidean space forms.
* Achieve a complete classification of toroids.

Thank you!

And happy birthday conference to
Marston, Gareth and Steve.