Skew-morphisms of Groups and Regular Cayley maps

Jun-Yang Zhang

School of mathematics and statistics, Minnan Normal University

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Outline

Skew-morphism

Skew-morphism: A *skew-morphism* φ of a finite group G is a permutation on G such that $\varphi(1) = 1$ and $\varphi(gh) = \varphi(g)\varphi^{\pi(g)}(h)$ for all $g, h \in G$, where π is a function from G to the cyclic group $\mathbf{Z}_{|\varphi|}$, called the *power function* of φ. (R. Jajcay and J. Širáň, 2002)

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Core of a skew-morphism: The set

$$\mathrm{Core}\,\varphi := \{x \in G \mid \pi\big(\varphi^i(x)\big) = 1, i = 0, 1, 2, \ldots\}$$

is a normal subgroup of G, called the *core* of φ. (J. Y. Zhang, 2015)

Important formulas

$\mathrm{Aut}(G) \subseteq \mathrm{Skew}(G) \subseteq \langle \mathrm{Skew}(G) \rangle \subseteq \mathrm{Sym}(G)$

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For a φ-orbit X satisfying $X = X^{-1}$, let $\chi(x)$ be the smallest nonnegative integer such that $\varphi^{\chi(x)}(x) = x^{-1}$. Then

$$\pi(x) \equiv \chi\left(\varphi(x)\right) - \chi(x) + 1 \pmod{|X|} \text{ for all } x \in X.$$

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Skew-product

Skew-product: Let $L_G := \{L_g \mid g \in G\}$ be the left regular representation of a finite group G and let φ be a permutation on G. Then φ is a skew-morphism of G if and only if $L_G\langle\varphi\rangle$ is a subgroup of $\mathrm{Sym}(G)$. For a skew-morphism φ of G with power function π, we have

$$\varphi L_g = L_{\varphi(g)}\varphi^{\pi(g)} \text{ for any } g \in G.$$

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Proposition (J.Y Zhang & S. F. Du, 2015).

Set $T = L_G\langle\varphi\rangle$ *and write* $L_{\mathrm{Core}\,\varphi} := \{L_x \mid x \in \mathrm{Core}\,\varphi\}$. *Then*

$$\mathrm{Core}_T(L_G) = L_{\mathrm{Core}\,\varphi}.$$

Some general results

Theorem A (J.Y Zhang & S. F. Du, 2015).

Suppose that $G = \langle x_i \mid 1 \leq i \leq t \rangle$, φ *is a skew-morphism of* G *with the power* π. *Then: (i)* $|\varphi| = \mathrm{lcm}\{|\mathcal{O}_{x_1}|, |\mathcal{O}_{x_2}|, \ldots, |\mathcal{O}_{x_t}|\}$*; (ii) for any* $c \in G$, $c \in \mathrm{Ker}\,\varphi$ *if and only if* $\pi(c) \equiv 1 \pmod{|\mathcal{O}_{x_i}|}$ *for all* $i = 1, 2, \ldots, t$.

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Corollary (J.Y Zhang & S. F. Du, 2015).

Suppose that A *is a cyclic group with two subgroups* K *and* M *such that* $A = \langle K, M \rangle$. *Let* φ *be a skew-morphism of* A *preserving both* K *and* M. *If* $\varphi|_K \in \mathrm{Aut}(K)$ *and* $\varphi|_M \in \mathrm{Aut}(M)$, *then* $\varphi \in \mathrm{Aut}(A)$.

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Theorem B (J.Y Zhang & S. F. Du, 2015).

Let H *be a normal subgroup of* G *and write* $\overline{G} = G/H$. *Let* φ *be a skew-morphism of* G. *If* φ *preserves* H, *then it induces a permutation* $\overline{\varphi} : \overline{G} \to \overline{G}$, $\overline{g} \mapsto \overline{\varphi(g)}$, *which defines a skew-morphism of* $\overline{G}$.

Regular Cayely maps

Map: A *map* is a 2-cell embedding of a connected graph into a closed surface. An *automorphism* of a map is an automorphism of the underlying graph which can be extended to a self-homeomorphism of the supporting surface. For a map on an orientable surface, the group of all its orientation-preserving automorphisms acts always semi-regularly on the set of its arcs. If it acts regularly, then the map is called *orientably-regular* (or regular for simplicity).

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Cayely map: A *Cayley map* $\mathrm{CM}(G, X, \sigma)$ is a 2-cell embedding of the Cayley graph $\mathrm{C}(G, X)$ into an orientable surface with the same local rotation induced by the permutation σ at every vertex, where σ is a cyclic permutation on X.

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Proposition (R. Jajcay & J. Širáň, 2002)

A Cayley map $\mathrm{CM}(G, X, \sigma)$ *is regular if and only if there exists a skew-morphism* φ *of* G *such that* $\varphi|_X = \sigma$.

Skew-morphisms of dihedral groups I

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Corollary 2 (J.Y Zhang & S. F. Du, 2015).

If n is an odd number not divisible by 3, then φ must be an automorphism.

Skew-morphisms of dihedral groups II

Let φ be a skew-morphism of the group D_{2n} not preserving $\langle a\rangle$ and let X be the orbit of a under φ. Then $X\cap\langle a\rangle b\neq\emptyset$, $X^{-1}=X$, $D_{2n}=\langle X\rangle$ and $\mathrm{CM}(D_{2n},X,\varphi|_X)$ is a regular Cayley map.

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Problem

Classify all skew-morphisms of dihedral groups.

Thank you very much !