Classification of Regular balanced Cayley maps of minimal non-abelian metacyclic groups

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Cayley graph

- A *Cayley graph* $\Gamma = \mathrm{Cay}(G, X)$ will be a graph based on a group G and a finite set $X \subseteq G$, say $X = \{x_1, x_2, \ldots, x_k\}$, such that:
  (1) $1_G \notin X$;
  (2) $X = X^{-1}$;
  (3) $G = \langle X \rangle$.
  $V(\Gamma) = G$, $E(\Gamma) = \{\{u, ux\} \mid u \in G, x \in X\}$.
- Let ρ be any cyclic permutation of the elements of X of order k. (or an ordering of the elements in X)

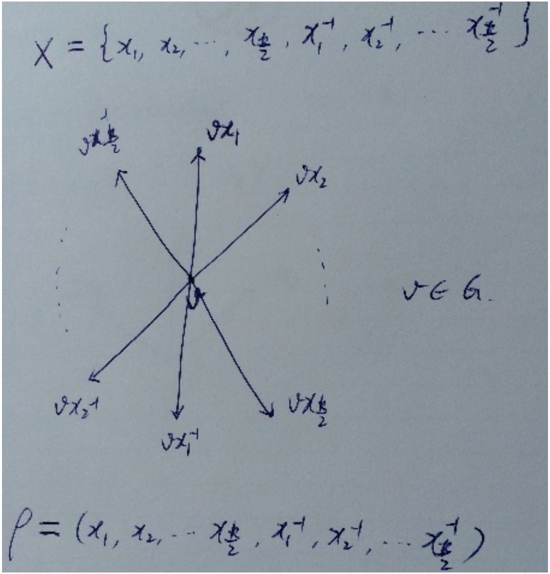
Cayley map

- The Cayley map $\mathcal{M} = \mathrm{CM}(G, X, \rho)$ is the 2-cell embedding of the Cayley graph $\mathrm{Cay}(G, X)$ in an orientable surface for which the orientation-induced local ordering of the darts emanating from any vertex $g \in G$ is always the same as the ordering of generators in X induced by ρ. That is, the neighbors of any vertex g are always spread counterclockwise around g in the order $(gx, g\rho(x), g\rho^2(x), \ldots, g\rho^{k-1}(x))$.

Regular balanced Cayley map

- A Cayley map $\mathrm{CM}(G, X, \rho)$ is called balanced if $\rho(x^{-1}) = \rho(x)^{-1}$ for every $x \in X$.

Local Image of Balanced Map

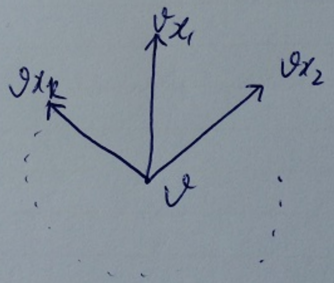


Local Image of Balanced Map

$X = \{x_1, \ldots x_k\}$,

x_i – involutions, $1 \le i \le k$.

$\rho = (x_1, x_2, \ldots, x_k)$



Some Known Results of Regular Balanced Cayley Maps

- A map $\mathcal{M}$ is (orientably) regular if $\mathrm{Aut}\,(\mathcal{M})$ acts regularly on the dart set.
- M. *Škoviera*, J. *Širáň*, Regular Maps from Cayley Graphs, Part 1: Balanced Cayley maps, Discrete Math., 109 (1992), pp. 265-276.
- A Cayley map $\mathrm{CM}(G, X, \rho)$ is regular and balanced iff there exists a group automorphism σ such that $\sigma|_X = \rho$.

Some Known Results of Regular Balanced Cayley Maps

- Each odd order abelian group possesses at least one regular balanced Cayley map

  "M. Conder, R. Jajcay, T.W. Tucker, Regular Cayley maps for finite abelian groups, Journal of Algebraic Combinatorics, Vol. 25 No. 3 (2007), pp. 259-283."

Continued

- Non-existence of regular balanced Cayley maps with semi-dihedral groups.

  "J.M. Oh, Regular t-balanced Cayley maps on semi-dihedral groups, Journal of Combinatorial Theory, Series B, Vol. 99, (2009), pp. 480-493."
- Y. Wang, R.Q. Feng, Regular balanced Cayley maps for cyclic, dihedral and generalized quternion groups, Acta Math. Sinica Vol. 21 No. 4 (2005), pp. 773-778.

Minimal non-abelian metacyclic groups

There are three classes of minimal non-abelian metacyclic groups:

(1) the quaternion group Q_8;

(2) $M_{p,q}(m,r) = \langle a, b \mid a^p = 1, b^{q^m} = 1, b^{-1}ab = a^r \rangle$, where p and q are distinct prime numbers, m is a positive integer and $r \not\equiv 1 \pmod p$ but $r^q \equiv 1 \pmod p$;

(3) $M_p(n,m) = \langle a, b \mid a^{p^n} = b^{p^m} = 1, b^{-1}ab = a^{1+p^{n-1}}, n \geq 2, m \geq 1 \rangle$.

Some References on these groups

(1) G.A. Miller, H.C. Moreno, Non-Abelian Groups in Which Every Subgroup is Abelian, Tran. of the American Math. Soc., Vol. 4, No. 4 (1903), pp. 398-404.

(2) Z.M. Chen, Interior and Outer Σ groups and minimal non-Σ groups, Southwest University Publishing House, 1988.

(3) M.Y. Xu, Introduction to Group Theory I, Science Publishing House, 1999.

The regular balanced Cayley maps of $M_{p,q}(m, r)$

Theorem

If q is odd, then the group $M_{p,q}(m, r)$ doesn't have regular balanced Cayley maps.

The regular balanced Cayley maps of $M_{p,q}(m, r)$

Theorem

Let $G = M_{p,2}(m, r)$, where $m \geq 2$, p is an odd prime and $r \equiv -1 \pmod{p}$. If $p - 1 = 2^e s$, where s is odd, then G has s non-isomorphic regular balanced Cayley maps. Especially, if p is a Fermat prime, then G has only one regular balanced Cayley map in the sense of isomorphism.

Regular balanced Cayley maps of $M_p(n, m)$

Theorem

Let $G = M_p(n, n)$ for some integer $n \geq 2$ and odd prime number p. Then, the group G doesn't have regular balanced Cayley maps.

Theorem

Let $G = M_p(n, m)$ for integers $n \geq 2, m \geq 1, m \neq n$ and for some odd prime number p. Then, the group G doesn't have regular balanced Cayley maps.

Continued

Corollary

For any odd prime number p, the metacyclic p-group doesn't have regular balanced Cayley maps.

Theorem

Let $G = M_2(n, m)$, *where* m *and* n *are positive integers and* $m > n \geq 2$. *Then,* G *doesn't have regular balanced Cayley maps.*

Continued

Theorem

Let $G = M_2(n, m)$, where m and n are positive integers, $n > m + 1$ and $m \geq 2$. Then, G doesn't have regular balanced Cayley maps.

Theorem

For positive integers $n > 2$, $M_2(n, 1)$ doesn't have regular balanced Cayley maps.

Continued

Theorem

Let $G = M_2(n, n)$, $n \geq 2$. *Then,* G *has only one regular balanced Cayley map of valency* 4 *in the sense of isomorphism.*

Theorem

Let $G = M_2(n+1, n)$, $n > 1$. *Then,* G *has only one regular balanced Cayley map of valency* 4 *in the sense of isomorphism.*