Vertex-primitive graphs of valency 5

Gabriel Verret

The University of Western Australia

Symmetries and Covers of Discrete Objects, Queenstown
February 18th 2016

Primitive groups

G is a permutation group on a finite set Ω.

Primitive groups

G is a permutation group on a finite set Ω.

G is transitive if, for every two elements of Ω, there is an element of G mapping one to the other.

Primitive groups

G is a permutation group on a finite set Ω.

G is transitive if, for every two elements of Ω, there is an element of G mapping one to the other.

G is primitive if it is transitive and preserves no non-trivial partition of Ω.

Primitive groups

G is a permutation group on a finite set Ω.

G is transitive if, for every two elements of Ω, there is an element of G mapping one to the other.

G is primitive if it is transitive and preserves no non-trivial partition of Ω.

A graph is vertex-primitive if its automorphism group is primitive on vertices.

Primitive groups

G is a permutation group on a finite set Ω.

G is transitive if, for every two elements of Ω, there is an element of G mapping one to the other.

G is primitive if it is transitive and preserves no non-trivial partition of Ω.

A graph is vertex-primitive if its automorphism group is primitive on vertices.

(An automorphism of a graph is an adjacency-preserving permutation of the vertex-set.)

Vertex-primitive graphs

Vertex-primitive graphs are connected (or edgeless).

Vertex-primitive graphs

Vertex-primitive graphs are connected (or edgeless). They are not bipartite (except for K_2).

Vertex-primitive graphs

Vertex-primitive graphs are connected (or edgeless). They are not bipartite (except for K_2). They are vertex-transitive and thus regular.

Vertex-primitive graphs

Vertex-primitive graphs are connected (or edgeless). They are not bipartite (except for K_2). They are vertex-transitive and thus regular.

Valency $1 \implies K_2$.

Vertex-primitive graphs

Vertex-primitive graphs are connected (or edgeless). They are not bipartite (except for K_2). They are vertex-transitive and thus regular.

Valency 1 $\Longrightarrow K_2$.

Valency 2 $\Longrightarrow C_p$.

Vertex-primitive graphs

Vertex-primitive graphs are connected (or edgeless). They are not bipartite (except for K_2). They are vertex-transitive and thus regular.

Valency 1 $\Longrightarrow K_2$.

Valency 2 $\Longrightarrow C_p$.

Valency 3 $\Longrightarrow K_4$, Petersen, Coxeter, Wong, an infinite family on $\mathrm{PSL}(2, p)$ (Li, Lu, Marušič 2004).

Vertex-primitive graphs

Vertex-primitive graphs are connected (or edgeless). They are not bipartite (except for K_2). They are vertex-transitive and thus regular.

Valency 1 $\Longrightarrow K_2$.

Valency 2 $\Longrightarrow C_p$.

Valency 3 $\Longrightarrow K_4$, Petersen, Coxeter, Wong, an infinite family on $\mathrm{PSL}(2, p)$ (Li, Lu, Marušič 2004).

Valency 4 $\Longrightarrow$ 5 graphs, and 5 infinite families (Li, Lu, Marušič 2004).

Vertex-primitive graphs

Vertex-primitive graphs are connected (or edgeless). They are not bipartite (except for K_2). They are vertex-transitive and thus regular.

Valency 1 $\implies K_2$.

Valency 2 $\implies C_p$.

Valency 3 $\implies K_4$, Petersen, Coxeter, Wong, an infinite family on $\mathrm{PSL}(2, p)$ (Li, Lu, Marušič 2004).

Valency 4 $\implies$ 5 graphs, and 5 infinite families (Li, Lu, Marušič 2004).

Valency 5 $\implies$ 7 graphs and 5 infinite families (Fawcett, Giudici, Li, Praeger, Royle, V. 2016?).

Vertex-primitive graphs of valency 5

$\mathrm{Aut}(\Gamma)$	$\mathrm{Aut}(\Gamma)_v$	$\lvert \mathrm{V}(\Gamma) \rvert$
$\mathbb{Z}_2^4 \rtimes \mathrm{Sym}(5)$	$\mathrm{Sym}(5)$	16
$\mathrm{P\Gamma L}(2,9)$	$\mathrm{AGL}(1,5) \times \mathbb{Z}_2$	36
$\mathrm{PGL}(2,11)$	D_{10}	66
$\mathrm{Sym}(9)$	$\mathrm{Sym}(4) \times \mathrm{Sym}(5)$	126
$\mathrm{Suz}(8)$	$\mathrm{AGL}(1,5)$	1 456
$\mathrm{J}_3 \rtimes 2$	$\mathrm{A\Gamma L}(2,4)$	17 442
Th	$\mathrm{Sym}(5)$	756 216 199 065 600

Vertex-primitive graphs of valency 5

$\mathrm{Aut}(\Gamma)$	$\mathrm{Aut}(\Gamma)_v$	$\lvert\mathrm{V}(\Gamma)\rvert$
$\mathbb{Z}_2^4 \rtimes \mathrm{Sym}(5)$	$\mathrm{Sym}(5)$	16
$\mathrm{P\Gamma L}(2,9)$	$\mathrm{AGL}(1,5) \times \mathbb{Z}_2$	36
$\mathrm{PGL}(2,11)$	D_{10}	66
$\mathrm{Sym}(9)$	$\mathrm{Sym}(4) \times \mathrm{Sym}(5)$	126
$\mathrm{Suz}(8)$	$\mathrm{AGL}(1,5)$	1 456
$\mathrm{J}_3 \rtimes 2$	$\mathrm{A\Gamma L}(2,4)$	17 442
Th	$\mathrm{Sym}(5)$	756 216 199 065 600

$\mathrm{Aut}(\Gamma)$	$\mathrm{Aut}(\Gamma)_v$	$\lvert\mathrm{V}(\Gamma)\rvert$	Conditions
$\mathrm{PSL}(2,p)$	$\mathrm{Alt}(5)$	$\frac{p^3-p}{120}$	$p \equiv \pm 1, \pm 9 \pmod{40}$
$\mathrm{P\Sigma L}(2,p^2)$	$\mathrm{Sym}(5)$	$\frac{p^6-p^2}{120}$	$p \equiv \pm 3 \pmod{10}$
$\mathrm{PSp}(6,p)$	$\mathrm{Sym}(5)$	$\frac{p^9(p^6-1)(p^4-1)(p^2-1)}{240}$	$p \equiv \pm 1 \pmod{8}$
$\mathrm{PGSp}(6,p)$	$\mathrm{Sym}(5)$	$\frac{p^9(p^6-1)(p^4-1)(p^2-1)}{120}$	$p \equiv \pm 3 \pmod{8}$, $p \geq 11$

Vertex-primitive graphs of valency 5

$\mathrm{Aut}(\Gamma)$	$\mathrm{Aut}(\Gamma)_v$	$\lvert\mathrm{V}(\Gamma)\rvert$
$\mathbb{Z}_2^4 \rtimes \mathrm{Sym}(5)$	$\mathrm{Sym}(5)$	16
$\mathrm{P\Gamma L}(2,9)$	$\mathrm{AGL}(1,5) \times \mathbb{Z}_2$	36
$\mathrm{PGL}(2,11)$	D_{10}	66
$\mathrm{Sym}(9)$	$\mathrm{Sym}(4) \times \mathrm{Sym}(5)$	126
$\mathrm{Suz}(8)$	$\mathrm{AGL}(1,5)$	1 456
$\mathrm{J}_3 \rtimes 2$	$\mathrm{A\Gamma L}(2,4)$	17 442
Th	$\mathrm{Sym}(5)$	756 216 199 065 600

$\mathrm{Aut}(\Gamma)$	$\mathrm{Aut}(\Gamma)_v$	$\lvert\mathrm{V}(\Gamma)\rvert$	Conditions
$\mathrm{PSL}(2,p)$	$\mathrm{Alt}(5)$	$\frac{p^3-p}{120}$	$p \equiv \pm 1, \pm 9 \pmod{40}$
$\mathrm{P\Sigma L}(2,p^2)$	$\mathrm{Sym}(5)$	$\frac{p^6-p^2}{120}$	$p \equiv \pm 3 \pmod{10}$
$\mathrm{PSp}(6,p)$	$\mathrm{Sym}(5)$	$\frac{p^9(p^6-1)(p^4-1)(p^2-1)}{240}$	$p \equiv \pm 1 \pmod{8}$
$\mathrm{PGSp}(6,p)$	$\mathrm{Sym}(5)$	$\frac{p^9(p^6-1)(p^4-1)(p^2-1)}{120}$	$p \equiv \pm 3 \pmod{8}$, $p \geq 11$

(Note K_6 hiding sneakily...)

Idea of proof: suborbits

An orbit of the point-stabiliser G_ω is called a suborbit of G.

Idea of proof: suborbits

An orbit of the point-stabiliser G_ω is called a suborbit of G.

In an arc-transitive graph, the neighbourhood of a vertex is a suborbit of the automorphism group.

Idea of proof: suborbits

An orbit of the point-stabiliser G_ω is called a suborbit of G.

In an arc-transitive graph, the neighbourhood of a vertex is a suborbit of the automorphism group.

In a vertex-transitive graph, the neighbourhood is a union of suborbits.

Idea of proof: suborbits

An orbit of the point-stabiliser G_ω is called a suborbit of G.

In an arc-transitive graph, the neighbourhood of a vertex is a suborbit of the automorphism group.

In a vertex-transitive graph, the neighbourhood is a union of suborbits.

To find all vertex-primitive graphs of small valency, we first find all primitive groups with small suborbits...

Idea of proof: suborbits

An orbit of the point-stabiliser G_ω is called a suborbit of G.

In an arc-transitive graph, the neighbourhood of a vertex is a suborbit of the automorphism group.

In a vertex-transitive graph, the neighbourhood is a union of suborbits.

To find all vertex-primitive graphs of small valency, we first find all primitive groups with small suborbits...

...and then do a little more work.

Small suborbits

If G is primitive and has a suborbit of length…

Small suborbits

If G is primitive and has a suborbit of length...

$1 \implies G \cong \mathbb{Z}_p$ (non-trivial suborbit).

Small suborbits

If G is primitive and has a suborbit of length...

$1 \implies G \cong \mathbb{Z}_p$ (non-trivial suborbit).

$2 \implies G \cong \mathrm{D}_p$.

Small suborbits

If G is primitive and has a suborbit of length...

$1 \Longrightarrow G \cong \mathbb{Z}_p$ (non-trivial suborbit).

$2 \Longrightarrow G \cong \mathrm{D}_p$.

$3 \Longrightarrow G \cong$ ... (Wong 1966).

Small suborbits

If G is primitive and has a suborbit of length...

$1 \implies G \cong \mathbb{Z}_p$ (non-trivial suborbit).

$2 \implies G \cong \mathrm{D}_p$.

$3 \implies G \cong$ ... (Wong 1966).

$4 \implies G \cong$ ... (Sims 1967, Quirin 1971, Wang 1992 (CFSG)).

Small suborbits

If G is primitive and has a suborbit of length...

$1 \implies G \cong \mathbb{Z}_p$ (non-trivial suborbit).

$2 \implies G \cong \mathrm{D}_p$.

$3 \implies G \cong$ ... (Wong 1966).

$4 \implies G \cong$ ... (Sims 1967, Quirin 1971, Wang 1992 (CFSG)).

$5 \implies G \cong$ ... (Fawcett, Giudici, Li, Praeger, Royle, V. 2016 (CFSG!)).

Idea of proof

Let Δ be a suborbit of length 5 for G_ω.

Idea of proof

Let Δ be a suborbit of length 5 for G_ω.

Wang (1995,1996) dealt with the case when G_ω^Δ is soluble, or when G_ω^Δ is insoluble and unfaithful.

Idea of proof

Let Δ be a suborbit of length 5 for G_ω.

Wang (1995,1996) dealt with the case when G_ω^Δ is soluble, or when G_ω^Δ is insoluble and unfaithful. (With a few omissions.)

Idea of proof

Let Δ be a suborbit of length 5 for G_ω.

Wang (1995,1996) dealt with the case when G_ω^Δ is soluble, or when G_ω^Δ is insoluble and unfaithful. (With a few omissions.)

It remains to deal with the case when G_ω is isomorphic to $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$.

Idea of proof

Let Δ be a suborbit of length 5 for G_ω.

Wang (1995,1996) dealt with the case when G_ω^Δ is soluble, or when G_ω^Δ is insoluble and unfaithful. (With a few omissions.)

It remains to deal with the case when G_ω is isomorphic to $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$.

In other words, we are looking for all groups with a core-free maximal subgroup isomorphic to $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$.

Idea of proof

Let Δ be a suborbit of length 5 for G_ω.

Wang (1995,1996) dealt with the case when G_ω^Δ is soluble, or when G_ω^Δ is insoluble and unfaithful. (With a few omissions.)

It remains to deal with the case when G_ω is isomorphic to $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$.

In other words, we are looking for all groups with a core-free maximal subgroup isomorphic to $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$.

There are a few affine examples but we quickly reduce to the almost simple case.

Almost simple groups with a maximal $\mathrm{Sym}(5)$

G	m	Conditions
$\mathrm{Alt}(7)$	1	
M_{11}	1	
$\mathrm{M}_{12} \rtimes \mathbb{Z}_2$	1	
$\mathrm{J}_2 \rtimes \mathbb{Z}_2$	1	
Th	2	
$\mathrm{PSL}(2, 5^2)$	1	
$\mathrm{P\Sigma L}(2, p^2)$	2	$p \equiv \pm 3 \pmod{10}$
$\mathrm{PSL}(2, 2^{2r}) \rtimes \mathbb{Z}_2$	1	r odd prime
$\mathrm{PGL}(2, 5^r)$	1	r odd prime
$\mathrm{PSL}(3, 4) \rtimes \langle \sigma \rangle$	1	σ a graph-field aut.
$\mathrm{PSL}(3, 5)$	1	
$\mathrm{PSp}(6, p)$	2	$p \equiv \pm 1 \pmod{8}$
$\mathrm{PGSp}(6, 3)$	1	
$\mathrm{PGSp}(6, p)$	2	$p \equiv \pm 3 \pmod{8}$, $p \geq 11$

$m := |\mathrm{N}_G(\mathrm{Sym}(4)) : \mathrm{Sym}(4)|$

Almost simple groups with a maximal $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$

1. Alternating groups

Almost simple groups with a maximal $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$

1. Alternating groups
2. Classical groups

Almost simple groups with a maximal $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$

1. Alternating groups
2. Classical groups
3. Lie groups of exceptional type

Almost simple groups with a maximal $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$

1. Alternating groups
2. Classical groups
3. Lie groups of exceptional type
4. Sporadic groups

Almost simple groups with a maximal $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$

1. Alternating groups
2. Classical groups
3. Lie groups of exceptional type
4. Sporadic groups (Thompson sporadic group: degree $\approx 7 \times 10^{14}$, order $\approx 9 \times 10^{16}$)

Consequence: half-arc-transitive graphs

A graph is half-arc-transitive if its automorphism group is transitive on edges and vertices but not on arcs.

Consequence: half-arc-transitive graphs

A graph is half-arc-transitive if its automorphism group is transitive on edges and vertices but not on arcs.

No vertex-primitive half-arc-transitive graph of valency at most 10.

Consequence: half-arc-transitive graphs

A graph is half-arc-transitive if its automorphism group is transitive on edges and vertices but not on arcs.

No vertex-primitive half-arc-transitive graph of valency at most 10.

There are examples of valency 14.

Consequence: half-arc-transitive graphs

A graph is half-arc-transitive if its automorphism group is transitive on edges and vertices but not on arcs.

No vertex-primitive half-arc-transitive graph of valency at most 10.

There are examples of valency 14.

This leaves valency 12...

Motivation: vertices with almost the same neighbourhood

Easy exercise: a vertex-primitive graph with two vertices having the same neighbourhood must be edgeless.

Motivation: vertices with almost the same neighbourhood

Easy exercise: a vertex-primitive graph with two vertices having the same neighbourhood must be edgeless.

Theorem (Spiga, V. 2016?)

A vertex-primitive graph with two vertices having neighbourhood differing by one is either complete or has prime order.

Motivation: vertices with almost the same neighbourhood

Easy exercise: a vertex-primitive graph with two vertices having the same neighbourhood must be edgeless.

Theorem (Spiga, V. 2016?)

A vertex-primitive graph with two vertices having neighbourhood differing by one is either complete or has prime order.

To do the next case (neighbourhood differing by two) using our approach, one would need to know the vertex-primitive graphs of valency at most 6.

Motivation: vertices with almost the same neighbourhood

Easy exercise: a vertex-primitive graph with two vertices having the same neighbourhood must be edgeless.

Theorem (Spiga, V. 2016?)

A vertex-primitive graph with two vertices having neighbourhood differing by one is either complete or has prime order.

To do the next case (neighbourhood differing by two) using our approach, one would need to know the vertex-primitive graphs of valency at most 6.

This looks quite difficult at the moment:

Motivation: vertices with almost the same neighbourhood

Easy exercise: a vertex-primitive graph with two vertices having the same neighbourhood must be edgeless.

Theorem (Spiga, V. 2016?)

A vertex-primitive graph with two vertices having neighbourhood differing by one is either complete or has prime order.

To do the next case (neighbourhood differing by two) using our approach, one would need to know the vertex-primitive graphs of valency at most 6.

This looks quite difficult at the moment:

1. Finding the vertex-primitive graphs of valency 6 does not seem easy (especially for the exceptional groups of Lie type).

Motivation: vertices with almost the same neighbourhood

Easy exercise: a vertex-primitive graph with two vertices having the same neighbourhood must be edgeless.

Theorem (Spiga, V. 2016?)

A vertex-primitive graph with two vertices having neighbourhood differing by one is either complete or has prime order.

To do the next case (neighbourhood differing by two) using our approach, one would need to know the vertex-primitive graphs of valency at most 6.

This looks quite difficult at the moment:

1. Finding the vertex-primitive graphs of valency 6 does not seem easy (especially for the exceptional groups of Lie type).
2. Once we have the graphs, we still have to do a little extra work.