Classification of regular Cayley maps on dihedral groups

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Young Soo Kwon
Yeungnam University, Korea
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(Joint Work with Istvan Kovacs))

Outline

1. Introduction to maps, regular maps, Cayley maps and regular Cayley maps

2. Skew-morphisms and their properties

3. Skew-morphisms of dihedral groups

4. Classification of regular Cayley maps on dihedral groups

5. Future research

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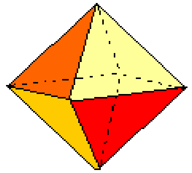
[Definition]

1. A (topological) map $\mathfrak{M} = G \to S$ is a 2-cell embedding of a graph G into a closed surface S.



Introduction to maps, regular maps, Cayley maps and regular Cayley maps

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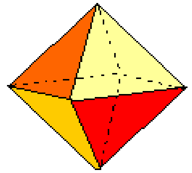

1. A (topological) map $\mathfrak{M}=G \to S$ is a 2-cell embedding of a graph G into a closed surface S.
2. For any map $\mathfrak{M}=G \to S$, a mutually incident vertex-edge pair is called an *arc* of $\mathfrak{M}$. The set of arcs of $\mathfrak{M}$ is denoted by D(G).

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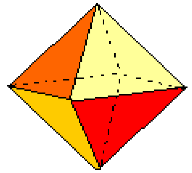

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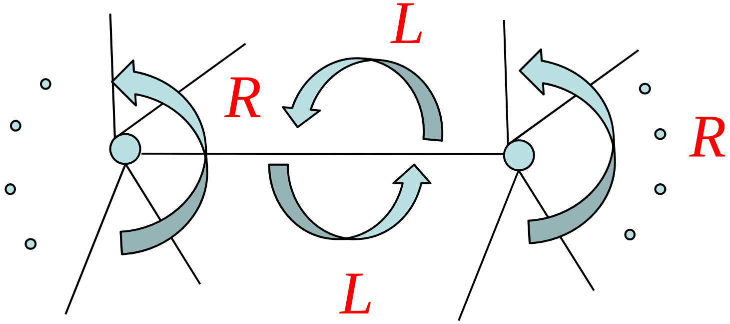
3. Any orientable map $\mathfrak{M}=G \to S$ can be descibed by a triple (D;R,L) such that

   (1) D is the set of arcs of the underlying graph G.

   (2) R is a permutation of D whose orbits coincide with the sets of arcs based at the same vertex.

   (3) L is an involution of D exchanging two arcs incident to the same edge.



4. A *map isomorphism* : graph iso. extended to a surface homeo.

5. A *map automorphism*: graph auto. extended to a surface homeo.

$\mathrm{Aut}(\mathfrak{M})$: the set of automorphisms of $\mathfrak{M}$.

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6. For any map $\mathfrak{M} = G \to S$, $\mathrm{Aut}^+(\mathfrak{M})$ acts semi-regularly on $D(G)$.

If the action is regular then we call $\mathfrak{M}$ a *regular map* or a *regular embedding* of G.

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Note that $\mathfrak{M} = G \to S$ is reflexible iff $\text{Aut}^-(\mathfrak{M}) \neq \emptyset$.

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[Definition]

1. For a group Γ and a set $X \subset \Gamma$ such that $X^{-1} = X$, a Cayley graph $\mathrm{Cay}(\Gamma : X)=(V, E)$ is a graph such that $V = \Gamma$ and $E = \{\{g, gx\} \mid x \in X\}$.

2. For any $g \in \Gamma$, let $L_g : \Gamma \to \Gamma$ such that $L_g(h) = gh$ for any $h \in \Gamma$. Let $L_\Gamma = \{L_g \mid g \in \Gamma\}$.

$$L_\Gamma \leq \mathrm{Aut}(Cay(\Gamma : X))$$

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3. For a Cayley graph $G = Cay(\Gamma : X)$ and cyclic permutation p of X,

a Cayley map CM(Γ:X,p) is a map $\mathfrak{M} = (D; R, L)$ such that

$D = \Gamma \times X$, $R(g, x) = (g, p(x))$ and $L(g, x) = (gx, x^{-1})$ for any $g \in \Gamma$ and $x \in \Gamma$.

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5. For a Cayley map $G = CM(\Gamma : X, p)$ with $p = (x_0, x_1, \ldots, x_{d-1})$,

$\kappa : [d] \to [d]$ defined by $x_i^{-1} = x_{\kappa(i)}$ is called a distribution of inverses.

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[Lemma]

For a regular Cayley map $CM(\Gamma : X, p) = (D; R, L)$, $\exists$ a group homo. $\psi : \langle R, L \rangle \to \langle \rho, \kappa \rangle$

s. t. $\psi^{-1}(\langle \rho, \kappa \rangle_i) \simeq \Gamma$ for any $i \in [d]$, where $\rho = (0, 1, \ldots, d-1)$

Skew-morphisms and their properties

For a group Γ, a bijection $\phi:\Gamma\to\Gamma$ is called skew-morphism with power function $\pi:\Gamma\to\mathbb{Z}$ if $\phi(1_\Gamma)=1_\Gamma$ and $\phi(gh)=\phi(g)\phi^{\pi(g)}(h)$ for all $g,h\in\Gamma$.

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[Lemma]

1. A Cayley map $CM(\Gamma:X,p)$: regular $\Leftrightarrow$ $\exists$ a skew-mor. $\phi:\Gamma\to\Gamma$ s. t. $\phi(X)=X$ and $\phi|_X=p$.

   A skew-morphism of Γ containing an orbit O satisfying $O^{-1}=O$ and $\Gamma=\langle O\rangle$: admissible

   other skew-morphisms: nonadmissible.

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   ϕ: a skew-morphism of a group Γ w.r.t a power function π $\Rightarrow$

3. $Ker(\phi)=\{g\in\Gamma\mid\pi(g)=1\}\leq\Gamma$.

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6. $\pi(gh)=\pi(h)+\pi(\phi(h))+\cdots+\pi(\phi^{\pi(g)-1}(h))=\sum_{i=0}^{\pi(g)-1}\pi(\phi^i(h))$.

7. ϕ^j is a skew-morphism $\Leftrightarrow$ $\sum_{i=0}^{j-1}\pi(\phi^i(g))$ is a multiple of j module $|\langle\phi\rangle|$ for any $g\in\Gamma$.

Skew-morphisms of dihedral groups

$D_n = \langle a, b \mid a^n = b^2 = (ab)^2 = 1 \rangle$: dihedral group of order 2n.

Let $A_n = \langle a \rangle$ and $B_n = D_n - A$. a^i: A-type element, $a^i b$: B-type element

CM(D_n: X, p) is balanced $\Leftrightarrow$ all elements in X are B-type elements.

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[Theorem] (2013, I. Istvan, D. Marusic, M. Muzychuk 2015, Z.Y. Zhang)

$\mathfrak{M}$=CM(D_n: X, p) : regular Cayley map on D_n s.t. L_{A_n} is core-free in Aut($\mathfrak{M}$).

$\Rightarrow$ $\mathfrak{M}$ is equivalent to one of the following:

(1) n=1, CM(D_1, $\{b\}$, (b)) balanced

(2) n=2, CM(D_2, $\{a, b, ab\}$, (a, b, ab)) ABB-type

(3) n=3, CM(D_3, $\{a, a^{-1}, b, a^2 b\}$, $(a, a^{-1}, b, a^2 b)$) AABB-type

(4) n=4, CM(D_4, $\{a, a^{-1}, b\}$, (a, a^{-1}, b)) AAB-type

(5) n=2m with odd m, CM(D_n, $a\langle a^2\rangle \cup b\langle a^2\rangle$, $(b, a, a^2 b, a^3, \ldots, a^{n-2} b, a^{n-1})$) alternating

CM(D_n: X, p): regular $\Rightarrow$ $\exists$ H $\le$ A_n s.t. CM(D_n/H: X/H, $\bar{p}$): regular, core-free

[Lemma]

$\mathfrak{M} = \mathrm{CM}(D_n : X, p)$: regular Cayley map on D_n $\Rightarrow$

(1) $X \cap A_n \neq \emptyset \Rightarrow \exists\, x \in X \cap A_n$ s.t. $A_n = \langle x \rangle$

(2) The kernel of the corresponding skew-morphism is dihedral subgroup.

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[Some known results]

(1) balanced $\Rightarrow$ $p = \left(b, ab, a^{r+1}b, \ldots, a^{r^{d-2}+r^{d-3}+\cdots+1}b\right)$ with (n,r)=1 and

$r^{d-1} + \cdots + 1 \equiv 0$ (mod n). ('05, Y. Wang and R. Q. Feng)

(2) t-balanced $\Rightarrow$ $p = \left(b, a, a^{2k}b, a^{\ell}, a^{2k(\ell+1)}b, a^{\ell^2}, a^{2k(\ell^2+\ell+1)}b, \ldots, a^{\ell^{2j-1}}\right)$ with n : even,

$\ell^{j} \equiv -1$(mod n), $2k^2(\ell^{j} + \cdots + \ell) + \ell - 1 \equiv 0$ (mod n) ('06, J. H. Kwak, K and R. Q. Feng) AB-type

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(3) n:odd, nonbalanced $\Rightarrow$ $p = \left(a, a^{-\ell}, a^{\ell^2}b, a^{-\ell^3}b, a^{\ell^4}, a^{-\ell^5}, \ldots, a^{\ell^{4k-2}}b, a^{-\ell^{4k-1}}b\right)$ with

$n = 3k$, $\ell^j \equiv 1$(mod n), j:odd ('13, I. Kovacs, D. Marusic and M. Muzychuk) AABB-type

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(4) skew-type 3 $\Rightarrow$ $p=\left(a, a^{-\ell}, a^{\ell^2}b, a^{-\ell^3}b, a^{\ell^4}, a^{-\ell^5}, \ldots, a^{\ell^{4k-2}}b, a^{-\ell^{4k-1}}b\right)$ with

$n=3k$, $\ell^j \equiv 1$(mod n), j:odd ('15, Z. Y. Zhang)

(5) reflexible $\Rightarrow \ldots,$ $p=(b,a,a^{-1},a^{\frac{n}{2}+2}b,a^{\frac{n}{2}-1},a^{\frac{n}{2}+1})$ with $n=8k+4,\ldots.$

AAB-type ('16(?), I. Kovacs and K)

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(6) minimal kernel $\Rightarrow \ldots, p = (b, a, a^2 b, a^3, \ldots, a^{-2} b,\ a^{-1})$ with n : even or

$p = (b, a, a^{\frac{n}{2}+2} b, a^3, \ldots, a^{\frac{n}{2}-2} b,\ a^{-1})$ with $n = 8k, \ldots.$

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(7) skew-type 2 $\Rightarrow t-\text{balanced}$ or $p=(a,a^{\ell-\ell}b,a^{\ell^2-\ell+\frac{n}{2}}b,a^{\ell^3},a^{\ell^4-\ell}b,a^{\ell^5-\ell+\frac{n}{2}}b,\ldots)$

with $n=4k+2$, $\ell^{3j}\equiv$ -1 (mod n) for some j.

ABB-type ('16(?), K)

Classification of regular Cayley maps on dihedral groups

[Main Theorem]

Any regular Cayley map on dihedral group isomorphic to one of $CM(D_n : X,\ p)$ in the following list. (In fact, all maps in the list are regular)

1. core-free regular maps.

2. $p = \left(b, ab, a^{r+1}b, \ldots, a^{r^{d-2}+r^{d-3}+\cdots+1}b\right)$ with $r^{d-1} + \cdots + r + 1 = 0$ (mod n). (balanced)

3. $p = (a, a^{\ell-\ell}b, a^{\ell^2-\ell+\frac{n}{2}}b, a^{\ell^3}, a^{\ell^4-\ell}b, a^{\ell^5-\ell+\frac{n}{2}}b, \ldots)$

with $n = 4k+2$, $\ell^{3j} \equiv -1$ (mod n) for some j. ABB-type

4. $p = (b, a, a^{\ell+\frac{n}{2}}, a^{\ell^2+\ell^{3j}+\frac{n}{2}}b, a^{\ell^3}, a^{\ell^4+\frac{n}{2}}, a^{\ell^5+\ell^{3j}+\frac{n}{2}}b, \ldots)$ with $n = 8k+4$, $\ell^{3j+1} \equiv \frac{n}{2}-1$ (mod n) or

$p = (b, a, a^{\ell+\frac{n}{2}}, a^{\ell^2+\ell^{6j+3}}b, a^{\ell^3}, a^{\ell^4+\frac{n}{2}}, a^{\ell^5+\ell^{6j+3}}b, \ldots)$ with $n = 8k+4$, $\ell^{3j+2} \equiv \frac{n}{2}-1$ (mod n)

AAB-type

5. $p=\left(a, a^{-\ell}, a^{\ell^2}b, a^{-\ell^3}b, a^{\ell^4}, a^{-\ell^5}, \ldots, a^{\ell^{4k-2}}b, a^{-\ell^{4k-1}}b\right)$

with $n=3k$, $\ell^j \equiv 1 (\text{mod n})$, j:odd AABB-type .

6. (1) $p=\left(b, a, a^{2k}b, a^{\ell}, a^{2k(\ell+1)}b, a^{\ell^2}, a^{2k(\ell^2+\ell+1)}b, \ldots, a^{\ell^{2j-1}}\right)$ with …((2j+1)-balanced) or

(2) $p=(b, a, a^2b, a^3, \ldots, a^{-2}b,\ a^{-1})$ with n : even or

$p=(b, a, a^{\frac{n}{2}+2}b, a^3, \ldots, a^{\frac{n}{2}-2}b,\ a^{-1})$ with $n=8k$ (minimal kernel) or

(3) $n=2^{\alpha}n_1n_2$ with $\alpha \geq 1$, n_1 and n_2 are coprime odd numbers satisfying the follows:

(i) $p\ (\text{mod}\ \ 2^{\alpha}n_1)=(b, a, a^2b, a^3, \ldots, a^{-2}b,\ a^{-1})$ or

$(b, a, a^{2^{\alpha-1}n_1+2}b, a^3, \ldots, \text{a}^{2^{\alpha-1}n_1-2}\text{b},\ a^{-1})$ with $\alpha \geq 3$ minimal kernel

$p\ (\text{mod}\ \ 2n_2)=\left(b, a, a^{2k}b, a^{\ell}, a^{2k(\ell+1)}b, a^{\ell^2}, a^{2k(\ell^2+\ell+1)}b, \ldots, a^{\ell^{2j-1}}\right)$ t-balanced

(ii) $\gcd(2^{\alpha-1}n_1,\ 2j)$ divides j -1 namely

$\alpha=1 \Rightarrow \gcd(n_1,\ j)=1,\quad \alpha \geq 2 \Rightarrow \gcd(2n_1,\ j)=1.$

AB-type

[Sketch of proof]

3. ABB-type $\Rightarrow$ $\exists$ a group homo. $\psi : \langle R, L \rangle \to \langle \rho, \kappa \rangle$ s. t. $\psi^{-1}(\langle \rho, \kappa \rangle_0) \simeq D_n \Rightarrow$

$\rho\kappa\rho^{-1}\kappa\rho^{-\kappa(0)} = \rho^{\kappa(0)}\kappa\rho\kappa\rho^{-1} \Rightarrow \kappa(0) = \dfrac{d}{2} \Rightarrow \kappa(3i) = 3i + \dfrac{d}{2} \Rightarrow$

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$$(\pi(x_0), \pi(x_1), \pi(x_2), \pi(x_3), \pi(x_4), \ldots) = \left(\frac{d}{2}+1,\ 1,\ \frac{d}{2}+1,\ \frac{d}{2}+1,\ 1,\ \frac{d}{2}+1,\ \ldots\right)$$

$\Rightarrow$ skew-type 2

$$\Rightarrow\ p = (a, a^{\ell-\ell}b, a^{\ell^2-\ell+\frac{n}{2}}b, a^{\ell^3}, a^{\ell^4-\ell}b, a^{\ell^5-\ell+\frac{n}{2}}b, \ldots)$$

with $n = 4k+2$, $\ell^{3j} \equiv$ -1 (mod n) for some j.

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with $n = 4k+2$, $\ell^{3j} \equiv -1 \pmod{n}$ for some j.

$$Ker(\phi) = \langle a^2, b\rangle$$

$$\phi(a^{2i}) = a^{2i\ell}, \qquad \phi(a^{2i+1}) = a^{2i\ell}b,$$

$$\phi(a^{2i}b) = a^{2i\ell+\ell^2-\ell+\frac{n}{2}}b, \qquad \phi(a^{2i+1}b) = a^{2i\ell+\ell^2+\ell+\frac{n}{2}}$$

Conversely, such ϕ is a well-defined skew-morphism.

4. AAB-type $\Rightarrow$ Assume that $p = (b,\ a,\ \ldots\ )$

The orbit of a^2 has the same length with that of a and the type is ABB $\Rightarrow$

skew-type 4 $\Rightarrow$ $\kappa(3i+1) = 3i + \kappa(1),\ \kappa(3i+2) = 3i + \kappa(2)$ and

$$\kappa(1) + \kappa(2) \equiv 3(\text{mod d}),\ 3\kappa(1) \equiv \frac{d}{2} + 3\ (\text{mod d}) \Rightarrow \ldots$$

4. AAB-type $\Rightarrow$ Assume that $p = (b,\ a,\ \ldots\ )$

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$$\kappa(1) + \kappa(2) \equiv 3(\text{mod d}),\ 3\kappa(1) \equiv \frac{d}{2} + 3\ (\text{mod d}) \Rightarrow \ \ldots$$

$p = (b, a, a^{\ell + \frac{n}{2}}, a^{\ell^2 + \ell^{3j} + \frac{n}{2}} b, a^{\ell^3}, a^{\ell^4 + \frac{n}{2}}, a^{\ell^5 + \ell^{3j} + \frac{n}{2}} b, \ldots)$ with $n = 8k+4,\ \ell^{3j+1} \equiv \frac{n}{2}$-1 (mod n) or

$(b, a, a^{\ell + \frac{n}{2}}, a^{\ell^2 + \ell^{6j+3}} b, a^{\ell^3}, a^{\ell^4 + \frac{n}{2}}, a^{\ell^5 + \ell^{6j+3}} b, \ldots)$ with $n = 8k+4,\ \ell^{3j+2} \equiv \frac{n}{2}$-1 (mod n)

depending on $\kappa(1) = \frac{d}{6} + 1$ or $\frac{5d}{6} + 1$.

$$Ker(\phi) = \langle a^4, a^3 b \rangle$$

$\phi(a^{4i}) = a^{4i\ell}$,	$\phi(a^{4i+3}b) = a^{4i\ell + 2\ell + \ell^{3j} + 1} b$,	1
$\phi(a^{4i+1}) = a^{4i\ell + \ell + \frac{n}{2}}$,	$\phi(a^{4i+2}b) = a^{4i\ell + 2\ell + \ell^{3j} + 1 + \frac{n}{2}} b$,	$12j+5$
$\phi(a^{4i+2}) = a^{4i\ell + 2\ell + \ell^{3j}} b$,	$\phi(a^{4i+1}b) = a^{4i\ell + \ell + 1}$,	$9j+4$
$\phi(a^{4i+3}) = a^{4i\ell + 3\ell + \ell^{3j}} b$,	$\phi(a^{4i}b) = a^{4i\ell + 1}$,	$3j+2$

5. AABB-type $\Rightarrow$ $\exists$ a group homo. $\psi:\langle R,L\rangle \to \langle \rho,\kappa\rangle$ s. t. $\psi^{-1}(\langle \rho,\kappa\rangle_0) \simeq D_n \Rightarrow$

$\rho^2\kappa\rho^{-2}\kappa\rho^{-\kappa(0)} = \rho^{\kappa(0)}\kappa\rho^2\kappa\rho^{-2} \Rightarrow$

$\kappa(4i) = 4i + \kappa(0),\ \kappa(4i+1) = 4i+1-\kappa(0)$ and $4\kappa(0) = 0 \Rightarrow$

$(\pi(x_0), \pi(x_1), \pi(x_2), \pi(x_3), \pi(x_4), \ldots) = (2\kappa(0)+1, \kappa(0)+1, 1, \kappa(0)+1, 2\kappa(0)+1, \ldots)$ and

$\pi(x_0^2) = \kappa(0)+1 = \pi(x_1),$ namely $\pi(a^2) = \pi(a) \Rightarrow$ skew-type 3 $\Rightarrow$

$$p = \left(a, a^{-\ell}, a^{\ell^2}b, a^{-\ell^3}b, a^{\ell^4}, a^{-\ell^5}, \ldots, a^{\ell^{4k-2}}b, a^{-\ell^{4k-1}}b\right)$$

with $n = 3k$, $\ell^j \equiv 1(\text{mod n})$, j:odd.

6. AB-type $\Rightarrow$ ϕ^2 : skew-morphism of D_n, furthermore

$\phi^2|_{\langle a^2, b\rangle}$: group auto. $\Rightarrow$ $\phi^2|_{\langle a\rangle}$ corresponds to balanced or t-balanced

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$\phi^2|_{\langle a\rangle}$ corresponds to balanced $\Rightarrow$ our map is t-balanced

$\phi^2|_{\langle a\rangle}$ contains an orbit of length $\frac{n}{2}$ like $(a, a^3, a^5, \ldots)$ $\Rightarrow$ our map is minimal kernel

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$\phi^2|_{\langle a^2,b\rangle}$: group auto. $\Rightarrow$ $\phi^2|_{\langle a\rangle}$ corresponds to balanced or t-balanced

$\phi^2|_{\langle a\rangle}$ corresponds to balanced $\Rightarrow$ our map is t-balanced

$\phi^2|_{\langle a\rangle}$ contains an orbit of length $\dfrac{n}{2}$ like $(a,\ a^3, a^5, \ldots)$ $\Rightarrow$ our map is minimal kernel

$\phi^2|_{\langle a\rangle}$: other t-balanced $\Rightarrow$

$n = 2^\alpha n_1 n_2$ with $\alpha \geq 1$, n_1 and n_2 are coprime odd numbers satisfying

$$p \pmod{2^\alpha n_1} = (b, a, a^2 b, a^3, \ldots, a^{-2} b,\ a^{-1}) \text{ or}$$

$$(b, a, a^{2^{\alpha-1} n_1 + 2} b, a^3, \ldots, a^{2^{\alpha-1} n_1 - 2} b,\ a^{-1}) \text{ with } \alpha \geq 3 \text{ minimal kernel}$$

$$p \pmod{2n_2} = \left(b, a, a^{2k} b, a^{\ell}, a^{2k(\ell+1)} b, a^{\ell^2}, a^{2k(\ell^2+\ell+1)} b, \ldots, a^{\ell^{2j-1}}\right) \text{ t-balanced}$$

closed under inverse $\Leftrightarrow$ $\gcd(2^{\alpha-1} n_1,\ 2j)$ divides j -1

Future Research

1. Group structure of each regular Cayley map on dihedral group

2. Classification of regular Cayley maps on dihedral group
   using group theoretic method.

3. Classification of full (admissible and nonadmissible)
   skew-morphisms of dihedral group.

Thank you!!!!