Automorphisms of Cayley Digraphs on 2-genetic p-groups

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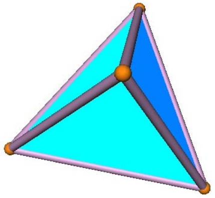
Outline

Automorphisms of a graph

- **A symmetry or an automorphism of a graph**:
  A permutation on its vertex set preserving adjacency.
- **Automorphism group of a graph Γ**: the permutation group of all symmetries of the graph under the composition of permutations, denoted by $\mathrm{Aut}(\Gamma)$.



Tetrahedron

- Automorphism group of the graph corresponding to the tetrahedron is S_4.

Automorphism group

- Computing automorphism group of a graph is a **basic and difficult problem** in algebraic graph theory. The problem is **NP-hard**, and there are a lot of works on this area.
- For "small" order up to **30000**, one may compute the automorphism group of a graph by MAGMA or GAP.
- There is no **general method** to compute automorphism group of a graph: combinatorics, group theory, covering...
- **Idea used often**: Let G be a vertex transitive group of a graph Γ. By Frattini argument, $A = GA_v$, and for stabilizers, there are many works relative to **Weiss Conjecture**.
- All vertex-transitive graphs are **coset graphs**, and among them, most are **Cayley graphs**.

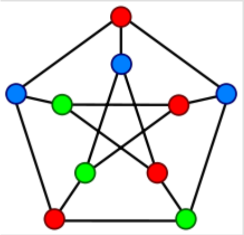
Cayley Digraphs

Let G be a finite group and $S \subset G$ with $1 \notin S$.

- **Cayley digraph** $X = \mathrm{Cay}(G, S)$**:** vertex set $V(X) = G$, directed edge set $E(X) = \{(g, sg) \mid g \in G, s \in S\}$.
- If $S = S^{-1}$, view (g, sg) and (sg, g) as an edge $\{g, sg\}$ and X is a undirected graph, called **Cayley graph**.
- For $g \in G$, define $\hat{g} : x \mapsto xg$, $x \in G$. Then $\hat{g} \in \mathrm{Aut}(X)$.
- $\hat{G} = \{\hat{g} : \mid g \in G\} \leq \mathrm{Aut}(X)$: transitive on $V(X)$.
- $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^{\alpha} = S\} \leq \mathrm{Aut}(X)$.
- $\hat{g}^{\alpha} = \alpha^{-1}\hat{g}\alpha = \hat{g^{\alpha}}$, $\hat{g} \in \hat{G}$, $\alpha \in \mathrm{Aut}(G, S)$. Then $\hat{G} \rtimes \mathrm{Aut}(G, S) \leq \mathrm{Aut}(X)$, $\hat{G} \cap \mathrm{Aut}(G, S) = 1$.
- **Characterization**: X is a Cayley digraph on $G \Leftrightarrow \mathrm{Aut}(X)$ has a regular subgroup isomorphic to G, acting regularly on vertices. $\mathrm{Cay}(G, S)$ is connected $\Leftrightarrow G = \langle S \rangle$.

Petersen graph, vertex-transitive but not Cayley

- A graph X is Cayley $\Leftrightarrow \mathrm{Aut}(X)$ has a regular subgroup.
- **Petersen graph P** is vertex-transitive and non-Cayley, the smallest vertex-transitive non-Cayley graph.
- **Check criterion**: $\mathrm{Aut}(P) = S_5$ and all involutions (elements of order 2) fix a vertex.



Any regular subgroup would have order 10 (even), so would contain an involution.

But, every involution fixes a vertex, contrary to the regularity.

Coset digraphs – Subidussi

G: a finite group; H a subgroup of G; D a union of several double-cosets of the form HgH with $g \notin H$.

- The **coset digraph** $X = \mathrm{Cos}(G, H, D)$ of G with respect to H and D: $V(X) = [G : H]$, the set of right cosets of H in G, $E(X) = \{(Hg, Hdg) \mid g \in G, d \in D\}$.
- Similarly to the Cayley case, if $D = D^{-1}$ we may view (Hg, Hdg) and (Hdg, Hg) as a undirected edge $\{Hg, Hdg\}$ and X is a undirected graph, called **coset graph**.
- If $H = 1$, $\mathrm{Cos}(G, H, D)$ is the Cayley digraph $\mathrm{Cay}(G, D)$. **Cayley digraph is a special case of coset digraph.**
- **Every G-vertex-transitive digraph X is isomorphic to a coset digraph** $\mathrm{Cos}(G, H, D)$, where H is the stabilizer of some $v \in V(X)$ and D consists of all elements of G which map v to one of its out-neighbors.

Coset digraph – Subidussi

Let $X = \mathrm{Cos}(G, H, D)$ be a coset digraph.

- For $g \in G$, define $\hat{g}_H$: $Hx \mapsto Hxg$. Then $\hat{g}_H \in \mathrm{Aut}(\mathrm{Cos}(G, H, D))$. Set $\hat{G}_H = \{\hat{g}_H \mid g \in G\}$. Then $\hat{G}_H \leq \mathrm{Aut}(X)$ and X **is vertex-transitive**.
- By group theory, $\hat{G}_H \cong G/H_G$, where H_G is the largest normal subgroup of G contained in H.
- Let $\mathrm{Aut}(G, H, D) = \{\alpha \in \mathrm{Aut}(G) \mid H^\alpha = H, D^\alpha = D\}$. For $\alpha \in \mathrm{Aut}(G, H, D)$, define $\alpha_H : Hg \mapsto Hg^\alpha, g \in G$. Then $\mathrm{Aut}(G, H, D)_H = \{\alpha_H \mid \alpha \in \mathrm{Aut}(G, H, D)\} \leq \mathrm{Aut}(X)_H$.
- $\tilde{H} = \{\tilde{h} : g \mapsto g^h, g \in G \mid h \in H\}$. Then $\tilde{H} \leq \mathrm{Aut}(G, H, D)$ and $\tilde{H}_H \leq \mathrm{Aut}(G, H, D)_H$.

Automorphism subgroups

Let $X = \mathrm{Cos}(G, H, D)$ and $A = \mathrm{Aut}(X)$. If $H_G = 1$ then

- The above result can be reduced from:
  C. Godsil, On the full automorphism group of a graph, Combinatorica, 1 (1981), 243-256.
- $N_A(\hat{G}_H) = \hat{G}_H \mathrm{Aut}(G, H, D)_H$ with $\hat{G}_H \cap \mathrm{Aut}(G, H, D)_H = \tilde{H}$.
  And $\hat{G}_H \cong G$, $\mathrm{Aut}(G, H, D)_H \cong \mathrm{Aut}(G, H, D)$, $\tilde{H}_H \cong \tilde{H}$.
- In particular, if $X = \mathrm{Cay}(G, S)$ and $A = \mathrm{Aut}(X)$ then
  $N_A(\hat{G}) = \hat{G} \rtimes \mathrm{Aut}(G, S)$.

Normality of Cayley graphs

Let $X = \mathrm{Cay}(G, S)$ and $A = \mathrm{Aut}(X)$. The Cayley graph X is called ***Normal*** if $\hat{G} \trianglelefteq A$.

By Godsil [33], if X is normal then $\mathrm{Aut}(X) = \hat{G} \rtimes \mathrm{Aut}(G, S)$.

The normality of Cayley graph was **first proposed and systematically studied** by Mingyao Xu [63].

Xu Conjecture:

$$\frac{\text{Number of Normal Cayley graphs on } n \text{ vertices}}{\text{Number of Cayley graphs on } n \text{ vertices}} \to 1 \ (n \to \infty)$$

The conjecture is true only known for some special groups.

Motivation

- A group G is called *2-genetic* if each normal subgroup of G can be generated by two elements.
- A group G is called *metacyclic* if G has cyclic normal subgroup N such that G/N is cyclic.
- A metacyclic gorup is 2-genetic, but the reverse is not true.
- C.H. Li, H.S. Sim, Automorphisms ..., J. Austral. Math. Soc. 71(2001) 223-231.
  Let G be **a non-abelian metacyclic group of order an odd prime power p^n**, and let $\Gamma = \mathrm{Cay}(G, S)$ be **a connected Cayley graph on G**. If $\mathrm{Aut}(G, S)$ is a p'-group, then **either Γ is normal, or ...**
- This was used to classify half-arc-transitive metacirculant graphs of order p^n with valency less than $2p$ by C.H. Li, H.S. Sim, On ..., J. Combin. Theory B 81(2001) 45-57.

Main Result

Main Theorem

G: nonabelian 2-genetic group of order p^n for an odd prime p.
$\Gamma = \mathrm{Cay}(G, S)$: a connected Cayley digraph.
If $\mathrm{Aut}(G, S)$ is a p'-group then either **Γ is normal**, or $p = 3, 5, 7, 11$, **and** $\mathrm{ASL}(2, p) \leq \mathrm{Aut}(\Gamma)/\Phi(\mathrm{O}_p(A)) \leq \mathrm{AGL}(2, p)$, where the kernel of $A := \mathrm{Aut}(\Gamma)$ on $\Gamma_{\Phi(O_p(A))}$ is $\Phi(O_p(A))$:

1. $p = 3$, $n \geq 5$, and $\Gamma_{\Phi(\mathrm{O}_p(A))}$ has **out-valency at least** 8;
2. $p = 5$, $n \geq 3$ and $\Gamma_{\Phi(\mathrm{O}_p(A))}$ has **out-valency at least** 24;
3. $p = 7$, $n \geq 3$ and $\Gamma_{\Phi(\mathrm{O}_p(A))}$ has **out-valency at least** 48;
4. $p = 11$, $n \geq 3$ and $\Gamma_{\Phi(\mathrm{O}_p(A))}$ has **out-valency at least** 120.

Non-normal examples exist for each case in (1)-(4).

Remark on the main result

- There are only a few constructions of half-arc-transitive non-normal Cayley graphs on p-groups.
- In the main theorem, the underlying graphs of **non-normal Cayley digraphs** for $p = 7, 11$ are half-arc-transitive. Recently, Jin-Xin Zhou constructed **an infinite family of such graphs for valency 4**.
- Since a Sylow p-subgroup of $\mathrm{ASL}(2, p)$ is not metacyclic, the Theorem implies that if G **is metacyclic then Γ is normal**, which generalizes the main theorem in [C.H. Li, H.S. Sim, Automorphisms ..., J. Austral. Math. Soc. 71(2001) 223-231].

Proof of the main theorem

- $A = \mathrm{Aut}(\Gamma)$, $\mathrm{Aut}(G, S)$ p'-group $\mapsto \hat{G} \in \mathrm{Syl}_p(A)$.
- Let $H = O_p(A)$, $\overline{H} = H/\Phi(H)$ and $\overline{A} = A/\Phi(H)$.
- **Lemma 1:** $C_A(H) \leq H$.
- **Lemma 2:** H **is the kernel of** A **acting on** $\overline{H}$ **by conjugate, that is,** $A/H \leq \mathrm{Aut}(\overline{H})$.
- G is 2-genetic $\Rightarrow \overline{H} = \mathbb{Z}_p$ or $\mathbb{Z}_p \times \mathbb{Z}_p$.
- $\overline{H} = \mathbb{Z}_p \Rightarrow H = \hat{G} \trianglelefteq A$, **the normal case.**
- $\overline{H} = \mathbb{Z}_p \times \mathbb{Z}_p$ and $A/H \leq \mathrm{GL}(2, p)$.
- [47, Theorem 6.17], $(A/H) \cap \mathrm{SL}(2, p)$ contains $\mathrm{SL}(2, p) \Rightarrow \mathrm{SL}(2, p) \leq A/H \leq \mathrm{GL}(2, p)$, **the non-normal case.**

Proof of the main theorem

- Let L be the kernel of A on $V(\Gamma_{\Phi(H)})$. Then $L = \Phi(H)L_\alpha$, L_α p'-Hall, Frattini arg. $\Rightarrow A = \Phi(H)N_A(L_\alpha) \Rightarrow$ $H = H \cap N_A(L_\alpha) \Rightarrow L_\alpha \trianglelefteq A \Rightarrow L_\alpha = 1 \Rightarrow \Phi(H) = L$.
- $\mathrm{SL}(2, p) \le \overline{A}/\overline{H} \le \mathrm{GL}(2, p)$. $\overline{U}/\overline{H} := Z(\overline{A}/\overline{H})$ is p'-group $\mapsto$ $\overline{U} = \overline{H}\overline{V}$, Frattini argument $\Rightarrow \mathrm{ASL}(2, p) \le \overline{A} \le \mathrm{AGL}(2, p)$.
- **B/H:=SL(2,p)** $\le A/H \le \mathrm{GL}(2, p)$, $\hat{G} \le B$, $F/H := Z(B/H)$, $B/F = \mathrm{PSL}(2, p)$, $K =$ the kernel of B on Γ_H, $|\Gamma_H| = p$.
- $p \ne 3 \mapsto K = F \mapsto \mathrm{PSL}(2, p) = B/K \le \mathrm{Aut}(\Gamma_H)$ (degree $\le p + 1$), Galois $\mapsto p = 5, 7, 11$. Thus, $p = 3, 5, 7, 11$.
- **Lemma 4: $\Gamma_{\Phi(H)}$ has out-valency at least $p^2 - 1$.**
- For $p = 5, 7, 11$, $n \ge 3$ and out-valency $\ge 24, 48, 120$ ✓
- For $p = 3$, if $n = 3, 4$ then Γ is normal $\times \Rightarrow n \ge 5$ ✓

Ideas of the proof of Lemma 1

Lemma 1: Let $A = \mathrm{Aut}(\Gamma)$ and $H = O_p(A)$. Then $C_A(H) \leq H$.

- Let B be a **component** of A, that is, a **subnormal** quasisimple subgroup: $B = B'$ and $B/Z(B) \cong T$ (NS).
- [38, Lemma 2.5] $\Rightarrow$ B **has a proper subgroup** C **of** p**-power index and** $O_{p'}(B) = 1$ $\Rightarrow$ $Z(B)$ p-group, $B/Z(B)$ has a proper subgroup $BZ(B)/Z(B)$ of p-power index.
- [38, Lemma 2.3] $\Rightarrow p \nmid |M(B/Z(B))| \mapsto p \nmid |Z(B)| \Rightarrow$ $Z(B) = 1$ **and** $B \cong T$.
- [48, 6.9(iv), p. 450]$\mapsto$ any two distinct components of G commute elementwise.
- $E(A) =$ **product of all components of** A $\Rightarrow B \leq E(A)$.

Ideas of the proof of Lemma 1

- $B \cong T \Rightarrow B$ is a direct factor of $E(A) \Rightarrow B^a$ is also a direct factor of $E(A)$, $\forall a \in A$.
- B contains a normal subgroup of A isomorphic to T^n, but:
- **Lemma 3: Any m.n.s of A is abelian.**
- A has no component $\mapsto E(A) = 1$.
- $F(A) = O_{p_1}(A) \times \cdots \times O_{p_t}(A)$, $\pi(A) = \{p_1, \cdots, p_t\}$.
- Generalized Fitting subgroup: $F^*(A) = E(A)F(A) = F(A)$.
- [38, Lemma 2.5] $\Rightarrow O_{p'}(A) = 1 \Rightarrow F^*(A) = O_p(A) = H$.
- [48, Theorem 6.11] $\Rightarrow C_A(F^*(A)) \leq F^*(A) \Rightarrow C_A(H) \leq H$.

Ideas of the proof of Lemma 2

Lemma 2: Set $H = O_p(A)$ and $\overline{H} = H/\Phi(H)$. Let $C_A(H) \leq H$. Then H is the kernel of A acting on $\overline{H}$ by conjugate.

- $\overline{H} \cong \mathbb{Z}_p^n$ is a vector space of dimension n over the field $\mathbb{Z}_p$. Let $\mathrm{Aut}(\overline{H}) = \mathrm{GL}(n, V)$.
- $\sigma : A \to \mathrm{Aut}(H)$, $g \mapsto \sigma_g$, where $\sigma_g : h \mapsto h^g$, $h \in H$.
- $\tau : \mathrm{Aut}(H) \to \mathrm{GL}(n, V)$, $\alpha \mapsto \tau_\alpha : h\Phi(H) \mapsto h^\alpha\Phi(H)$, $h \in H$.
- $C_A(H) \leq H \Rightarrow \mathrm{Ker}(\sigma) = C_A(H) = Z(H)$. Set $S := \mathrm{Ker}(\tau)$ and $K := \mathrm{Ker}(\sigma\tau) \Rightarrow K/Z(H) \cong K^\sigma \leq S$.
- Clearly, $H \leq K$. It suffice to show K is a p-group.
- $\Omega = \{(h_1t_1, h_2t_2, \ldots, h_nt_n) \mid t_i \in \Phi(H)\}$, $|\Omega| = |\Phi(H)|^n$ p-power. S is semiregular on $\Omega \mapsto S$ is p-group.
- K is a p-group $\Rightarrow K \leq H \Rightarrow H = K$.

Ideas of the proof of Lemma 3

Lemma 3: Any minimal normal subgroups of A is abelian.

- Suppose $N \cong T_1 \times T_2 \times \cdots \times T_k$, where $T_i \cong T$ is n.a.s.
- Let $\Omega = \{T_1, \ldots, T_k\}$. $\hat{G}$ is 2-genetic $\Rightarrow$ $k \leq 2$.
- Let $B = N_A(T_1)$. Then $B \trianglelefteq A$ and $A/B \lesssim S_2 \Rightarrow$ **B is transitive on $V(\Gamma)$ and $\hat{G} \leq B$**. Consider B instead of A
- Let $\Delta_i \in V(\Gamma_{T_1})$ and $|\Delta_i| = p^m$.
- $p \nmid (T_1)_u$, [38, Corollary 2] $\Rightarrow T_1$ is 2-transitive on each Δ_i $\Rightarrow$ **$[\Delta_i]$ is complete digraph K_{p^m} or a null graph**.
- We may assume that T_1 has at least two orbits.

Ideas of the proof of Lemma 3

- T_1 equivalent 2-transitive action on Δ_i and $\Delta_j \Rightarrow (\Delta_i, \Delta_j) = \{(\alpha_{il}, \alpha_{jl}) \mid 1 \leq l \leq p^m\}, \{(\alpha_{ik}, \alpha_{jl}) \mid 1 \leq k, l \leq p^m\}$ or $\{(\alpha_{ik}, \alpha_{jl}) \mid 1 \leq k, l \leq p^m, k \neq \ell\}$.
- $\forall g \in \mathrm{S}_{p^m}$, $\sigma_g : \alpha_{il} \mapsto \alpha_{il^g}$, $\sigma_g \in \mathrm{Aut}(\Gamma)$.
- Let $\mathrm{S}_{p^m} = \{\sigma_g \mid g \in \mathrm{S}_{p^m}\} \leq \mathrm{Aut}(\Gamma) \Rightarrow A_{p^m} \leq K \leq B$, **$K$ is the kernel of B acting on Γ_{T_1}.**
- **If $m > 1$ then** $p^{m+1} || K| \Rightarrow p^{n+1} || A| \times \Rightarrow m = 1$.
- $m = 1$, [38, Lemma 2.3] $\Rightarrow p \nmid |\mathrm{Out}(T_1)|$.
- $p \nmid |B/T_1C_B(T_1)| \Rightarrow \hat{G} \leq T_1C_B(T_1) = T_1 \times C_B(T_1)$, 2-genetic $\Rightarrow \hat{G}$ **is abelian,** a contradiction.

Ideas of the proof of Lemma 4

Lemma 4: $\Gamma_{\Phi(H)}$ has out-valency at least $p^2 - 1$ $(p = 3, 5, 7, 11)$.

- $\Phi(H)$ is the kernel of A on $V(\Gamma_{\Phi(H)})$. Let $\alpha \in V(\Gamma_{\Phi(H)})$.
- Let $\Omega = \{\Delta_1, \cdots, \Delta_p\}$ be the orbits of $\overline{H}$ on $V(\Gamma_{\Phi(H)})$.
- $\overline{B} = B/\Phi(H) = \mathrm{ASL}(2, p) \leq \overline{A} \leq \mathrm{Aut}(\Gamma_{\Phi(H)}) \Rightarrow |\Gamma_{\Phi(H)}| = p^3$, $|\overline{B}| = p^3(p^2 - 1)$, $|\overline{B}_\alpha| = p^2 - 1$.
- $\Delta \in \Omega, \alpha \in \Delta \Rightarrow |\Delta| = p^2$, $\overline{B}_\Delta = \overline{H} \cdot \overline{B}_\alpha$, $|\overline{B}_\Delta| = p^2(p^2 - 1)$.
- $\overline{B}_\Delta$ is sharply 2-transitive on Δ and any p'-subgroup W of $\overline{B}_\Delta$ fixe a vertex and has all other orbits of length $|W|$.
- $[\Delta] = K^*_{p^2}$ $(Out[\Delta] = p^2 - 1)$ or the null digraph of order p^2.
- **One may assume $[\Delta]$ is the null digraph of order p^2.**

Ideas of the proof of Lemma 4

- **Let K be the kernel of B on $V(\Gamma_H)$.** Set $\overline{K} = K/\Phi(H)$.
- $B/H = \mathrm{SL}(2,p) \Rightarrow F/H := Z(\mathrm{SL}(2,p)) \cong \mathbb{Z}_2$, $\overline{F} = F/\Phi(H)$ $\Rightarrow \overline{F}/\overline{H} \cong \mathbb{Z}_2$, $\overline{F} \leq \overline{B}_\Delta$, **and** $|\overline{F}_\alpha| = 2 \Rightarrow$ There exist some $i \neq j$ such that $Out((\Delta_i, \Delta_j)) \geq 2$.
- **For $p = 3$, $B/K \cong \mathbb{Z}_3$ and $\overline{B}/\overline{K} \cong \mathbb{Z}_3$** $\Rightarrow \overline{K}$ fixes each Δ_i and is 2-transitive on each $\Delta_i \Rightarrow Out(\Delta_i, \Delta_j) \geq p^2 - 1$.
- **For $p = 5, 7$ or 11. $\overline{B}/\overline{K} \cong B/K \cong \mathrm{PSL}(2,p) \Rightarrow \overline{B}$ is 2-transitive on $\Omega \Rightarrow \overline{B}_{\Delta_i}$ is transitive on $\Omega \backslash \{\Delta_i\}$.**
- $\overline{B}_{\Delta_i} = \overline{H} \cdot \overline{B}_{\alpha_i}$ and $|\overline{B}_{\alpha_i}| = p^2 - 1 \Rightarrow \overline{B}_{\alpha_i}$ is transitive on $\Omega \backslash \{\Delta_i\}$ and $|(\overline{B}_{\alpha_i})_{\Delta_j}| = (p^2-1)/(p-1) = p+1 \Rightarrow$ $Out((\Delta_i, \Delta_j)) \geq p+1$.
- $\overline{B}$ 2-transitive on $\Omega \Rightarrow Out(\Gamma_{\Phi(H)}) \geq (p+1)(p-1) = p^2 - 1$.

Further work

Based on the main results, we propose the following problem:

- **Classify half arc-transitive graphs on a 2-genetic group of odd-prime power order p^n**. In particular, **do it for valency less than $2p$**.

There are only two non-isomorphic non-abelian groups of order p^3, of which both are 2-genetic.

- **Classify edge-transitive or half-arc-transitive graphs of prime-cube order**.

In 1992, Xu [66] classified tetravalent half-arc-transitive graphs of prime-cube order. Based on the main theorem, a similar classification can be done for valencies 6 and 8.

Definition

Let p be an odd prime. Denote

$$G_1(p) = \langle a, b \mid a^{p^2} = 1, b^p = 1, b^{-1}ab = a^{1+p} \rangle$$

$$G_2(p) = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle.$$

Let e be an element of order $j < p$ in $\mathbb{Z}^*_{p^2}$ and set

$$T^{j,k} = \{b^k a, b^k a^e, \ldots, b^k a^{e^{j-1}}, (b^k a)^{-1}, (b^k a^e)^{-1}, \ldots, (b^k a^{e^{j-1}})^{-1}\}$$

for each $1 \leq k \leq p-1$. Define

$$\Gamma^{j,k} = \mathrm{Cay}(G_1(p), T^{j,k}).$$

Let λ be an element of order 4 in $\mathbb{Z}^*_p$. Then $4 \mid (p-1)$. For each $0 \leq k \leq p-1$ with $k \neq 2^{-1}(1+\lambda)$, let $S_{4,k} = R \cup R^{-1}$, where $R = \{a, b, a^\lambda b^{\lambda-1} c^k, a^{-\lambda-1} b^{-\lambda} c^{1-k}\}$ and define

$$\Gamma_{4,k} = \mathrm{Cay}(G_2(p), S_{4,k}).$$

Half-arc-transitive graphs of order p^3 of small valency

Let Γ be a graph of order p^3 for an odd prime p. Then

(1) If Γ has valency 6 then Γ is half-arc-transitive if and only if $3 \mid (p-1)$ and $\Gamma \cong \Gamma^{3,k}$. There are exactly $(p-1)/2$ nonisomorphic half-arc-transitive graphs in $\Gamma^{3,k}$;

(2) If Γ has valency 8 then Γ is half-arc-transitive if and only if $4 \mid (p-1)$ and $\Gamma \cong \Gamma^{4,k}$ or $\Gamma_{4,k}$. There are exactly $(p-1)/2$ nonisomorphic half-arc-transitive graphs in $\Gamma^{4,k}$ and $\Gamma_{4,k}$, respectively.

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Thank you!