An infinite family of trivalent vertex-transitive Haar garphs that are not Cayley graphs

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Joint work with Marston Conder and Tomaž Pisanski

Voltage graphs and regular coverings

Definition

A voltage graph is a graph X (possibly with loops, multiple edges and semi-edges) together with a mapping $\gamma : A(X) \to G$, from the arcs of X to some group G such that inverse arcs are mapped to inverse group elements and semi-edges are mapped to involutions.

Definition

The regular covering graph Y of X has vertex set $V(Y) = V(X) \times G$ and edges of the form $\{(u, g), (v, \gamma_{(u,v)} g)\}$ for all edges $\{u, v\} \in E(X)$ and all $g \in G$.

Cayley graphs

Definition

Let G be a group, and $S \subset G$ with $1_G \notin S$. Then the Cayley graph $X = \mathrm{Cay}(G, S)$ is the graph with $V(X) = G$ and with edges of the form $\{g, sg\}$ for all $g \in G$ and $s \in S$.

Equivalently, since all edges can be written in the form $\{1, s\}g$, this is a covering graph over a single-vertex graph having loops and semi-edges, with voltages taken from S: the order of a voltage over a semi-edge is 2 (corresponding to an involution in S), while the order of voltage over a loop is greater than 2 (corresponding to a non-involution in S).

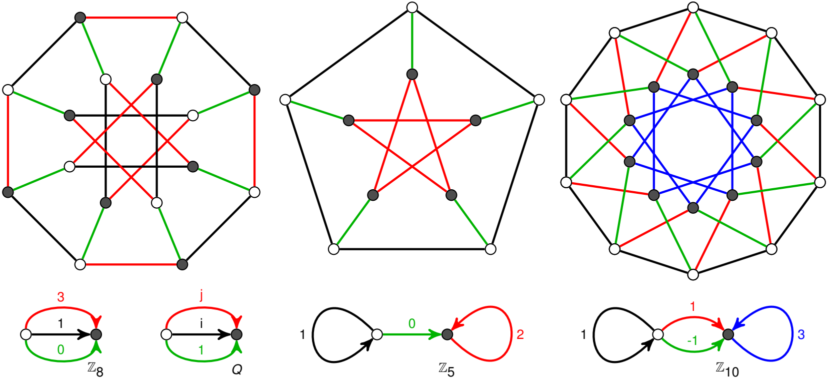
Theorem (Sabidussi)

A graph X is a Cayley graph over some group G if and only if $\mathrm{Aut}(X)$ *contains a regular subgroup isomorphic to G.*

Haar graphs, Bi-Cayley graphs

Definition

Given a group G and an arbitrary subset S of G, the Haar graph $H(G,S)$ is the regular G-cover of a dipole with $|S|$ parallel edges, labeled by elements of S. In other words, the vertex-set of $H(G,S)$ is $G \times \{0,1\}$, and the edges are of the form $\{(g,0),(sg,1)\}$ for all $g \in G$ and $s \in S$.



Haar graphs

The name 'Haar graph' comes from the fact that when G is an abelian group, the Schur norm of the corresponding adjacency matrix can be evaluated via computing a discrete Haar integral on G.

The group G acts on $H(G, S)$ as a group of automorphisms, by right multiplication, and moreover, G acts regularly on each of the two parts of $H(G, S)$, namely $\{(g, 0) : g \in G\}$ and $\{(g, 1) : g \in G\}$.

Conversely, if Γ is any bipartite graph and its automorphism group $\mathrm{Aut}\,\Gamma$ has a subgroup G that acts regularly on each part of Γ, then Γ is a Haar graph — indeed Γ is isomorphic to $H(G, S)$ where S is determined by the edges incident with a given vertex of Γ.

Bi-Cayley graphs

Haar graphs form a special subclass of *bi-Cayley graphs*, which are graphs that admit a semiregular group of automorphisms with two orbits of equal size. Every bi-Cayley graph can be realised as follows:

Definition

Let G be a group, and let S be arbitrary subset of G. The bi-Cayley graph of G with respect to the subsets L, R, S of G $\quad (1 \notin L \cup R, L = L^{-1}, R = R^{-1})$, denoted by $\mathrm{BCay}(G, L, R, S)$ is the simple graph with vertex set $G \times \{0, 1\}$ and with edge set

$$\begin{aligned} &\{(x,0)(lx,0)\}\ (x \in G, l \in L) && \text{left edges,} \\ &\{(x,1)(rx,1)\}\ (x \in G, r \in R) && \text{right edges,} \\ &\{(x,0)(sx,1)\}\ (x \in G, s \in S) && \text{middle edges.} \end{aligned}$$

For any $g \in G$ the map g_r defined by $(x, i)^{g_r} = (xg, i)$ $(x \in G, i \in \{0, 1\})$ is an automorphism of $\mathrm{BCay}(G, L, R, S)$. Hence $G_R = \{g_r \mid g \in G\} \cong G$ is a semiregular automorphism group with orbits.

The main questions

Connections between Haar-, Cayley-, VT graphs have been investigated recently by E., Pisanski.

Q 1. For what finite non-abelian groups G are all Haar graphs $H(G, S)$ Cayley graphs?

Q 2. For what finite non-abelian groups G is there a Haar graph with Aut $H(G, S) \cong G$?

Q 3. Is there a Haar graph $H(G, S)$ which is vertex-transitive but non-Cayley?

In this talk we will answer Q 3.

Some motivating results

- Hladnik et al.: Haar graphs over $\mathbb{Z}_n$ are Cayley graphs over dihedral groups. If a A is belian, $H(A, S) \cong \text{Cay}(D(A), \overline{S})$.
- Lu et al.: three infinite families of cubic semi-symmetric (edge- but not vertex-transitive) graphs as Haar graphs over the alternating group A_n.
- Exoo, Jajcay: the smallest known approximate (3,30)-cage as a Haar graph over $SL(2, 83)$.
- Zhou, Feng: a family of VT (both Cayley and non-Cayley) cubic graphs as bi-Cayley graphs over abelian groups

Doubly generalized Petersen graphs

Named so by Zhou and Feng. Automorphism groups computed by Kutnar and Petecki. A direct construction:

Definition

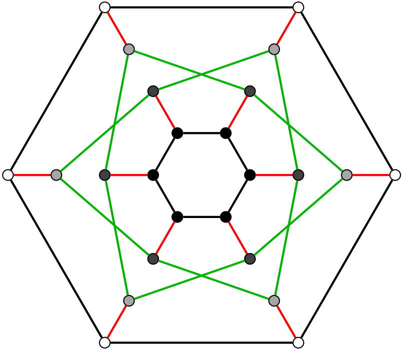
Let $D(n, r)$ be the simple graph with four types of vertices, called u_i, v_i, w_i and z_i (for $i \in \mathbb{Z}_n$), and three types of edges, given by the sets

$$\Omega = \{\{u_i, u_{i+1}\}, \{z_i, z_{i+1}\} : i \in \mathbb{Z}_n\} \qquad \text{(the 'outer' edges),}$$

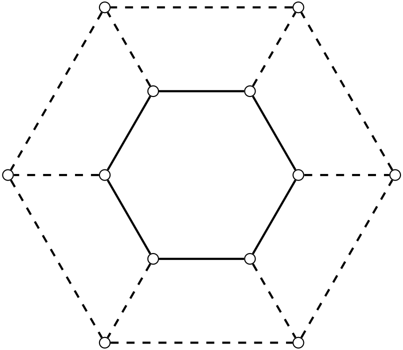
$$\Sigma = \{\{u_i, v_i\}, \{w_i, z_i\} : i \in \mathbb{Z}_n\} \qquad \text{(the spokes'), and}$$

$$I = \{\{v_i, w_{i+r}\}, \{v_i, w_{i-r}\} : i \in \mathbb{Z}_n\} \qquad \text{(the 'inner' edges).}$$

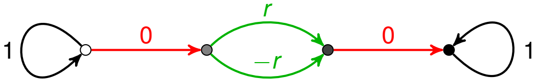
Generalized- and Doubly generalized Petersen graphs



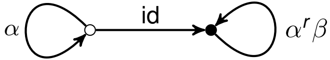
$D(6,1)$



$G(6,1)$



$\mathbb{Z}_n$



$\mathbb{Z}_n \times \mathbb{Z}_2$

Properties of the graphs $D(n,r)$

Proposition

Every $D(n,r)$ is connected. The graph $D(n,r)$ is bipartite if and only if n is even.

Proposition

For every n and r, the graph $D(n,r)$ is isomorphic to $D(n,n-r)$, and $D(2n,r)$ is isomorphic to $D(2n,n-r)$.

Proposition

For every r, the graph $D(2r+1,r)$ is planar, and isomorphic to the generalised Petersen graph $G(4r+2,2)$.

A word of caution: it can happen that $D(n,r) \not\cong D(n,s)$ even when $G(n,r) \cong G(n,s)$. For instance, $G(7,2) \cong G(7,3)$ but $D(7,2) \not\cong D(7,3)$, since $D(7,3)$ is planar but $D(7,2)$ is not.

Symmetries of $D(n, r)$

$$\begin{array}{llllll}
\alpha: & u_i \mapsto u_{i+1}, & v_i \mapsto v_{i+1}, & w_i \mapsto w_{i+1}, & z_i \mapsto z_{i+1} & \text{(rotation)},\\
\beta: & u_i \mapsto z_i, & v_i \mapsto w_i, & w_i \mapsto v_i, & z_i \mapsto u_i & \text{(flip symmetry)},\\
\gamma: & u_i \mapsto u_{-i}, & v_i \mapsto v_{-i}, & w_i \mapsto w_{-i}, & z_i \mapsto z_{-i} & \text{(reflection)}.
\end{array}$$

Note also that α and β commute with each other. In fact, Zhou and Feng proved that $D(n, r)$ is isomorphic to the bi-Cayley graph $\mathrm{BCay}(G, R, L, \{1\})$ over $G = \langle \alpha, \beta \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2$, with $R = \{\alpha, \alpha^{-1}\}$ and $L = \{\alpha^r \beta, \alpha^{-r} \beta\}$.

„When is $D(n, r)$ a Haar graph?"

Clue: Since the orbits of $G = \langle \alpha, \beta \rangle$ do not form the bipartition of $D(n, r)$, it follows that if $D(n, r)$ is a Haar graph, then it must be vertex-transitive.

Vertex-transitive $D(n, r)$

Theorem (Zhou, Feng)

The graph $D(n, r)$ is vertex-transitive if and only if $n = 5$ and $r = 2$, or n is even and $r^2 \equiv \pm 1 \mod \frac{n}{2}$. In the former case, $D(n, r)$ is isomorphic to the dodecahedral graph $G(10, 2)$, which is non-Cayley, and in the latter case, if $r^2 \equiv 1 \mod \frac{n}{2}$ then $D(n, r)$ is a Cayley graph, while if $r^2 \equiv -1 \mod \frac{n}{2}$ then $D(n, r)$ is non-Cayley.

Proposition

A Cayley graph $\mathrm{Cay}(G, S)$ is a Haar graph if and only if it is bipartite.

Theorem (Conder, E., Pisanski)

$D(n, r)$ is a Haar graph if and only if it is vertex-transitive and n is even.

Cubic VT non-Cayley Haar graphs

Combining the previous theorems we get the following:

Theorem (Conder, E., Pisanski)

(a) *If n is odd, or if n is even and $r^2 \not\equiv \pm 1$ mod $\frac{n}{2}$, then $D(n,r)$ is not a Haar graph, and is vertex-transitive only when $(n,r) = (5,2)$;*

(b) *If n is even and $r^2 \equiv 1$ mod $\frac{n}{2}$, then $D(n,r)$ is a Haar graph and a Cayley graph;*

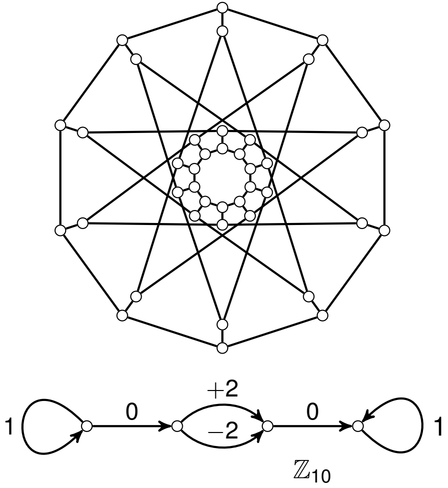
(c) *If n is even and $r^2 \equiv -1$ mod $\frac{n}{2}$, then $D(n,r)$ is a Haar graph and is vertex-transitive but not a Cayley graph.*

In particular, the graphs $D(n,r)$ of case (c) give infinitely many Haar graphs that are vertex-transitive but non-Cayley, in answer to the original question:

Corollary

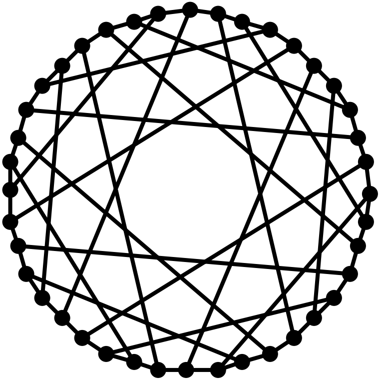
Each graph $D(2m,r)$ with $m > 2$ and $r^2 \equiv -1$ mod m is a Haar graph that is vertex-transitive but non-Cayley.

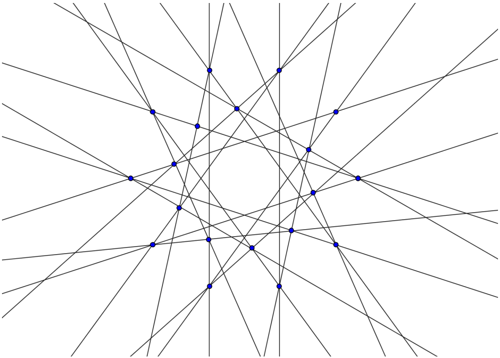
$D(10,2)$ or $F040A$, the smallest arc-transitive example

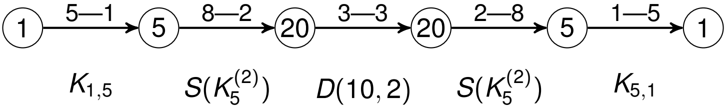


- $D(10,2)$, is the smallest arc-transitive non-Cayley Haar graph
- $|\operatorname{Aut} D(10,2)| = 480$
- $F040A$ in the Foster census
- in LCF notation $[15, 9, -9, -15]^{10}$
- Ivić Weiss used it as the middle layer graph of the rank 4 self-dual regular polytope ${}_4\{3,6,3\}_4$

$D(10,2)$ or $F040A$, the smallest arc-transitive example







THANK YOU!