Flag graphs and symmetry type graphs

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Content

Aim.
The aim of this work is to give a classification on the possible different symmetry type of maniplexes.

I. Maniplexes and symmetry type graphs.

II. Map operations.

I(a). Maniplexes

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A *maniplex* $\mathcal{M}$ of rank $n-1$ (or $(n-1)$-maniplex) is defined by a connected graph $\mathcal{G}_{\mathcal{M}}$ which vertex set is $\mathcal{F}(\mathcal{M})$ and with edges of colour i corresponding to the matching s_i, to which we refer as the *flag graph* of the maniplex $\mathcal{M}$.

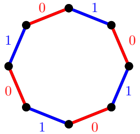
Examples of maniplexes

0-maniplex.

Graph with two vertices joined by an edge of colour 0.

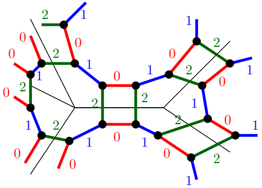
1-maniplex.

It is associated to an l-gon, which graph contains $2l$ vertices joined by a perfect matching of colours 0 and 1 and each of size l.



2-maniplex.

Can be considered as a map, as Lins and Vince defined a map (1982-1983), by a trivalent edge coloured graph.



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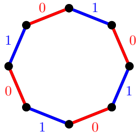

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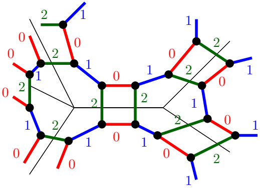

Thus, maniplexes generalize the notion of maps to higher rank.

Monodromy (or connection) group of $\mathcal{M}$

To each $(n-1)$-maniplex $\mathcal{M}$ we can associate a subgroup of the permutation group $Sym(\mathcal{F}(\mathcal{M}))$,

$$Mon(\mathcal{M}) := \langle s_0, s_1, \ldots, s_{n-1} \rangle$$

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And satisfy the following

(i) All $s_0, s_1, \ldots, s_{n-1}$ are fixed-point free involutions.

(ii) $s_i s_j = s_j s_i$ and $s_i s_j$ is fixed-point free, whenever $|i-j| \geq 2$.

(iii) The action of $\mathrm{Mon}(\mathcal{M})$ on $\mathcal{F}(\mathcal{M})$ is transitive.

Faces of rank $i = 0, 1, \ldots, n-1$ of $\mathcal{M}$

The set of *i*-faces of an $(n-1)$-maniplex corresponds to the orbit of the flags in $\mathcal{F}(\mathcal{M})$ under the action of the group generated by the set

$$F_i := \{s_j | i \neq j\}.$$

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- Every automorphism α of $\mathcal{M}$ induces a bijection on the flags.

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$\mathrm{Aut}(\mathcal{M})$ is isomorphic to the edge-colour preserving automorphism group of $\mathcal{G}_{\mathcal{M}}$.

- The action of the elements in $\mathrm{Aut}(\mathcal{M})$ commutes with the elements of $\mathrm{Mon}(\mathcal{M})$.

k-orbit maniplex

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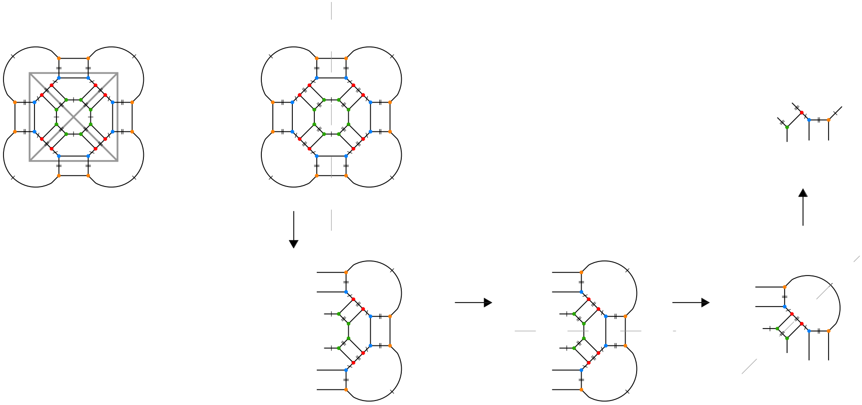
It can be seen that there are $2^n - 1$ different possible types of 2-orbit $(n-1)$-maniplexes.

I(b). Symmetry type graph of $\mathcal{M}$, $T(\mathcal{M})$

Definition.

The *symmetry type graph* $T(\mathcal{M})$ of a maniplex $\mathcal{M}$ is a quotient graph of the flag graph $\mathcal{G}_{\mathcal{M}}$ obtained from the action of the group $\mathrm{Aut}(\mathcal{M})$ on the flags of $\mathcal{M}$.



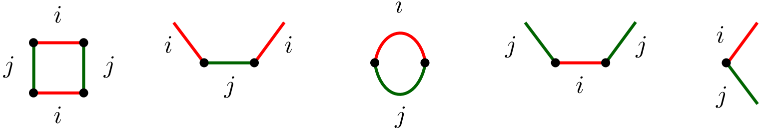
Symmetry type graph of $\mathcal{M}$, $T(\mathcal{M})$

Thus, the symmetry type graph of a k-orbit map has k-vertices

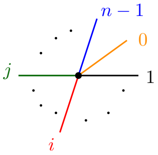
Given two flag orbits $\mathcal{O}_\Phi$ and $\mathcal{O}_\Psi$, as vertices of $T(\mathcal{M})$, there is an edge of colour $i = 0, 1, \ldots, n-1$ between them if and only if there exists flags $\Phi' \in \mathcal{O}_\Phi$ and $\Psi' \in \mathcal{O}_\Psi$ such that Φ' and Ψ' are i-adjacent in $\mathcal{M}$.

Counting symmetry types

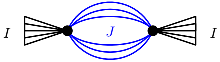
The number of types of k-orbit maniplexes depends on the number of n-valent pre-graphs on k vertices that can be properly edge coloured with n colours and that the connected components of the 2-factor with colours i and j, with $|i-j| \geq 2$ are always as the following.



The symmetry type graph of a reflexible maniplex consist of one vertex and n semi-edges.

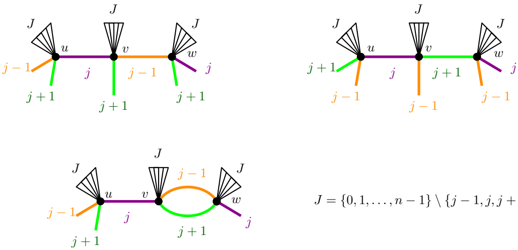


There are $2^n - 1$ different possible symmetry type graphs on 2 vertices.



$I \subset \{0, 1, \dots, n-1\}$, $J = \{0, 1, \dots, n-1\} \setminus I$

There are $2n - 3$ different possible symmetry type graphs on 3 vertices.



$J = \{0, 1, \dots, n-1\} \setminus \{j-1, j, j+1\}$

Face transitivity

Definition.

An $(n-1)$-maniplex $\mathcal{M}$ is *i-face-transitive* if $\mathrm{Aut}(\mathcal{M})$ is transitive on the faces of rank $i = 0, 1, \ldots, n-1$.

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An $(n-1)$-maniplex $\mathcal{M}$ is *fully-face-transitive* if it is i-face-transitive for every $i = 0, 1, \ldots, n-1$.

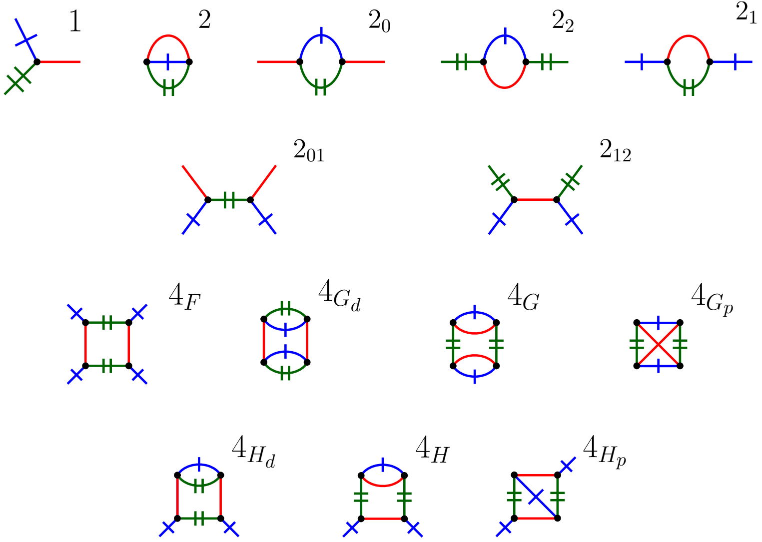
Highly symmetric maniplexes

Given the symmetry type graph of a maniplex one can read from the appropriate coloured subgraphs the different types of face-transitivities that the maniplex has.

Theorem. (Number of face-orbits of $\mathcal{M}$)

Let $\mathcal{M}$ be an $(n-1)$-maniplex with symmetry type graph $T(\mathcal{M})$. Then, the number of connected components in the $(n-1)$-factor of $T(\mathcal{M})$ of colours $\{0, 1, \ldots, n-1\} \setminus \{i\}$, determine the number of orbits of the i-faces of $\mathcal{M}$, where $i \in \{0, 1, \ldots, n-1\}$.

Edge-transitive maps



Fully-transitivity on k-orbit maniplexes ($k = 2, 3, 4$)

Hubard showed that there are $2^n - n - 3$ classes of fully-transitive **2-orbit** $(n-1)$-maniplexes.

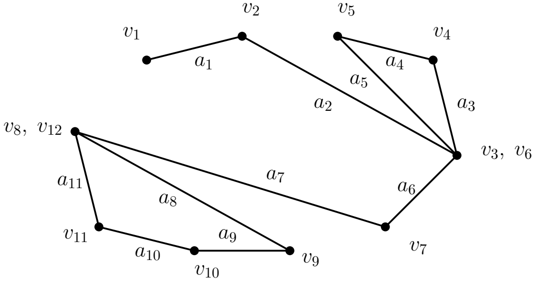
We showed that **3-orbit** maniplexes are never fully-transitive, but they are i-face-transitive.

Also, that if a **4-orbit** maniplex is not fully-transitive then it is i-face-transitive for all i but at most three ranks.

Generators of $\mathrm{Aut}(\mathcal{M})$ given $T(\mathcal{M})$

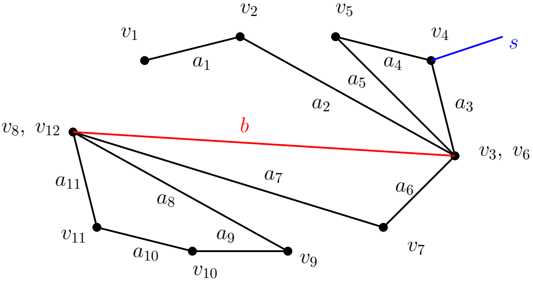
Let $\mathcal{M}$ be a k-orbit $(n-1)$-maniplex and let $T(\mathcal{M})$ its symmetry type graph.

- Suppose that $v_1, e_1, v_2, e_2 \ldots, e_{q-1}, v_q$ is a distinguished walk that visits every vertex of $T(\mathcal{M})$, with the edge e_i having colour a_i, for each $i = 1, \ldots q-1$.



Generators of $\mathrm{Aut}(\mathcal{M})$ given $T(\mathcal{M})$

- Let $S_i \subset \{0, \ldots, n-1\}$ be such that v_i has a semi-edge of colour s if and only if $s \in S_i$.
- Let $B_{i,j} \subset \{0, \ldots, n-1\}$ be the set of colours of the edges between the vertices v_i and v_j (with $i < j$) that are not in the distinguished walk



- Let $\Phi \in \mathcal{F}(\mathcal{M})$ be a base flag of $\mathcal{M}$ such that Φ projects to v_1 in $T(\mathcal{M})$.

Generators of $\mathrm{Aut}(\mathcal{M})$ given $T(\mathcal{M})$

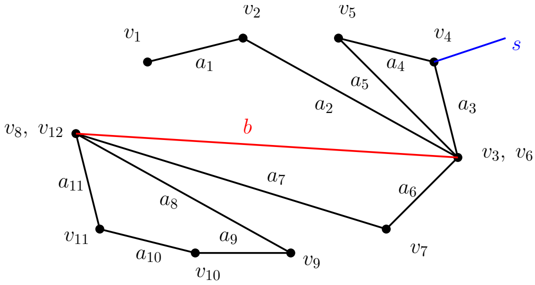
Theorem.

The automorphism group of $\mathcal{M}$ is generated by the union of the sets

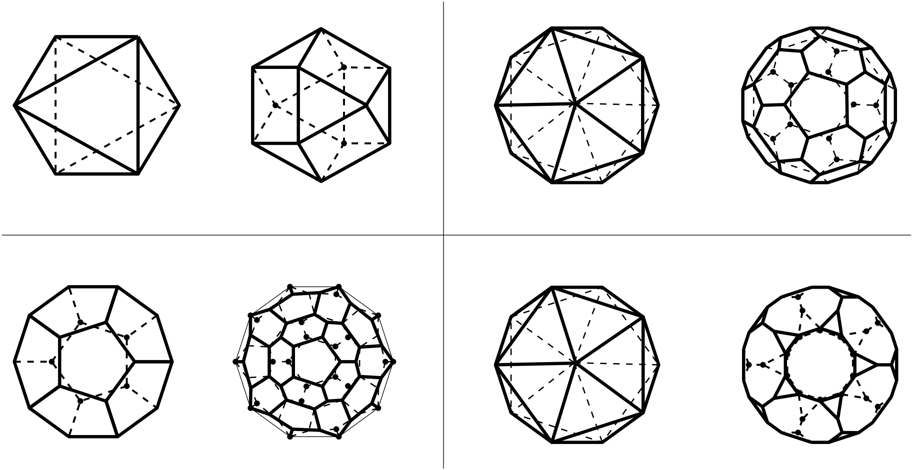
$$\{\alpha_{a_1,a_2,\ldots,a_i,s,a_i,a_{i-1},\ldots,a_1} \mid i = 1,\ldots,k-1, s \in S_i\},$$

and

$$\{\alpha_{a_1,a_2,\ldots,a_i,b,a_j,a_{j-1},\ldots,a_1} \mid i,j \in \{1,\ldots,k-1\}, i < j, b \in B_{i,j}\}.$$



II. Map operations



Theorem. [Orbanić, Pellicer, Weiss]

Let $\mathcal{M}$ be a k-orbit map. Then the medial map $\mathrm{Me}(\mathcal{M})$ is a k-orbit or a 2k-orbit map, depending on whether or not $\mathcal{M}$ is a self-dual map.

Theorem. [Orbanić, Pellicer, Weiss]

Let $\mathcal{M}$ be a k-orbit map. Then the truncation map $\mathrm{Tr}(\mathcal{M})$ is a k-orbit, $\frac{3k}{2}$-orbit or a 3k-orbit map.

Theorem.

Each of the 14 edge-transitive symmetry type graphs is the symmetry type graph of a medial map.

Proposition.

Let $\mathcal{M}$ be a k-orbit map. Then $\mathrm{Me}(\mathrm{Me}(\mathcal{M}))$ is a k-orbit map if $\mathcal{M}$ is a map on the torus of type $\{4,4\}$, or is a map on the Klein Bottle of type $\{4,4\}_{|m,n|}$, where n is odd.

Theorem.

Let $\mathcal{M}$ be a k-orbit map and $\mathrm{Cham}_t(\mathcal{M})$ the t-times chamfering map of $\mathcal{M}$ having s flag-orbits. Then one of the following holds.

1. $s = 4^t k, 2^t k$ or k.
2. *If* $s \neq 4^t k$, *then* $\chi(\mathcal{M}) = 0$ *($\mathcal{M}$ is on the torus or on the Klein bottle) and $\mathcal{M}$ is of type* $\{6, 3\}$.
3. *If $\mathcal{M}$ is a the torus of type $\{6, 3\}$ then $s = k$ and $k = 1, 2, 3, 4$.*
4. *If $\mathcal{M}$ is on the Klein bottle of type $\{6, 3\}$ then $s = 2^t k$ and $3|k$.*

Conclusion

We extended the classification of all possible symmetry types of k-orbit 2-maniplexes

- self-dual, properly and improperly, k-orbit maps with $k \leq 7$.
- with the operations medial and truncation on maps, up to $k \leq 6$.

Also, we determined all possible symmetry types of maps that result from other maps after applying the chamfering operation and give the number of possible flag-orbits that has the chamfering map of a k-orbit map.

Thank you

Remarks

In order to characterize the symmetry types of k-orbit maniplexes, as well it was done in this thesis for 2-maniplexes, we lead to the open problem of study different operations on maniplexes and the symmetry types of maniplexes that are obtained from applying such operations on a maniplex.