HAMILTON CYCLES IN EMBEDDED GRAPHS

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Lovász problem

Problem (Lovász, 1969)

Does every vertex-transitive graph admit a Hamilton path?

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Critical case: cubic Cayley graphs

Hamilton cycles in cubic Cayley graphs

Theorem (Glover, Marušič, Kutnar, Malnič, 2007–2012)

Let $K = \mathrm{Cay}(H;\, r, r^{-1}, l)$ *be a cubic Cayley graph, where* $H = \langle r, l \mid r^s = l^2 = (rl)^3 = 1, \dots \rangle$ *is a finite quotient of the modular group* $PSL(2,\mathbb{Z})$. *Then* K *has a Hamilton path.*

Moreover, if $|H| \equiv 2 \pmod 4$ *or if* $|r| \equiv 0, \pm 1 \pmod 4$*, then* K *has a Hamilton cycle.*

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Question 1.
What about the missing case $|H| \equiv 0 \pmod 4$ and $|r| \equiv 2 \pmod 4$?

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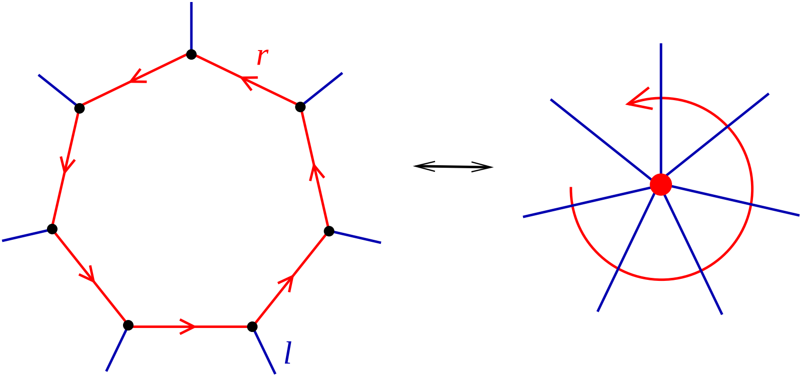
Question 2.
What about the finite quotients of the group

$$\langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^3 = (yz)^3 = 1, \dots \rangle \text{ ?}$$

Proof: topological background



Proof: topological background

Main idea

- Take the Cayley map $\mathcal{M}$ corresponding to $H = \langle r, l \mid r^s = l^2 = (rl)^3 = 1, \dots \rangle$
- Select a suitable set $\mathcal{F}$ of faces of $\mathcal{M}$ such that $\bigcup \mathcal{F}$ is connected and null-homologous, i.e., a 'tree' of faces.
- Construct a Hamilton cycle as the topological boundary $\partial(\bigcup \mathcal{F})$
- The result is a contractible Hamilton cycle in $\mathcal{CM}(H;\, r, r^{-1}, l)$.

The idea of constructing a Hamilton cycle as a boundary of a set of faces of map goes back to W. R. Hamilton (1858).

Do we need symmetry?

Do we need orientability?

Do we need contractible Hamilton cycles?

Bounding Hamilton cycles in embedded graphs

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Definition 2. Let $G \hookrightarrow S$ be an embedding of a graph forming polytopal map $\mathcal{M}$. A weak 2-face colouring of $\mathcal{M}$ is a colouring of faces of $\mathcal{M}$ with two colours s.t. at each vertex of there are precisely two edges separating differently coloured faces.

Bounding Hamilton cycles: characterisation

Theorem 1

Let $\mathcal{M}$ be a polytopal map on a closed surface of Euler genus g. The following statements are equivalent.

(i) *$\mathcal{M}$ has a bounding Hamilton cycle.*

(ii) *The vertices of $\mathcal{M}^*$ can be a partitioned into two subsets which induce one-sided subgraphs H and K such that $\beta(H) + \beta(K) = g$.*

(iii) *$\mathcal{M}$ has a weak 2-face-colouring such that the vertices of $\mathcal{M}^*$ receiving colour 1 induce a one-sided subgraph of $\mathcal{M}^*$.*

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(iii) $\mathcal{M}$ *has a weak 2-face-colouring such that the vertices of $\mathcal{M}^*$ receiving colour 1 induce a one-sided subgraph of $\mathcal{M}^*$.*

Theorem 2

A polytopal map $\mathcal{M}$ admits a contractible Hamilton cycle $\iff$ $\mathcal{M}$ *has a weak 2-face-colouring such that the vertices of $\mathcal{M}^*$ receiving colour 1 induce a tree.*

Remarks

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According to the Strong Embedding Conjecture, every 2-connected graph has a polytopal embedding.

$\Longrightarrow$ Theorem 1 can potentially be applied to all 2-connected graphs.

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Remark 2

Part (ii) of Theorem 1 implies that a Hamilton cycle in a planar map $\mathcal{M}$ corresponds to a vertex partition of $\mathcal{M}^*$ into two induced trees.

Hamilton cycles in 2-face-coloured cubic polytopal maps

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Let $\mathcal{M}$ be a cubic polytopal map with a fixed weak 2-face colouring. If the vertices of $\mathcal{M}^*$ receiving colour 1 can be partitioned into an induced tree and an independent set, then $\mathcal{M}$ admits a contractible Hamilton cycle.

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Theorem

A connected cubic graph G admits a partition of its vertex-set into an induced tree and an independent set $\iff$ G has cellular embedding into an orientable surface with a single face.

Contractible Hamilton cycles in truncated triangulations

Every truncated triangulation $t(\mathcal{T})$ has a natural weak 2-face-colouring

- vertex-faces $\mapsto$ colour 0
- face-faces $\mapsto$ colour 1

Theorem

Let $\mathcal{T}$ be a triangulation of a closed surface and let $t(\mathcal{T})$ be the truncation of $\mathcal{T}$. The following statements are equivalent.

(i) *$t(\mathcal{T})$ has a contractible Hamilton cycle.*

(ii) *The vertex set of $\mathcal{T}^*$ admits a partition $\{A, J\}$ where A induces a tree in the underlying graph of $\mathcal{T}^*$ and J is independent.*

END OF PART I

Thank you!

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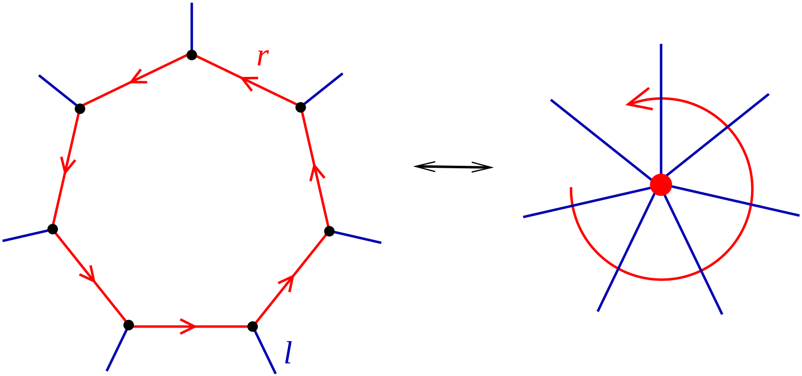
PART II

Truncated triangulations



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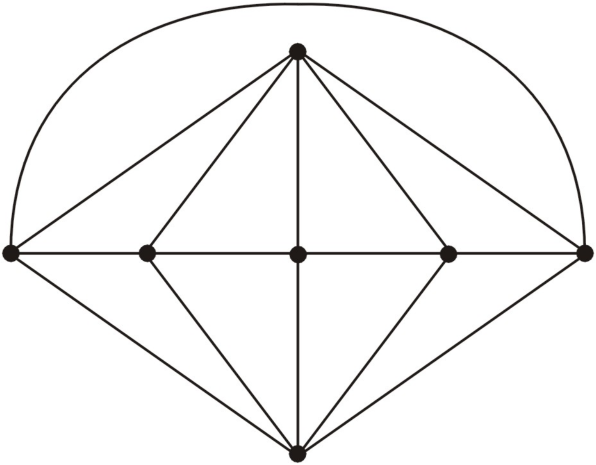
Example: Construction of a Hamilton cycle in $t(\mathcal{T})$



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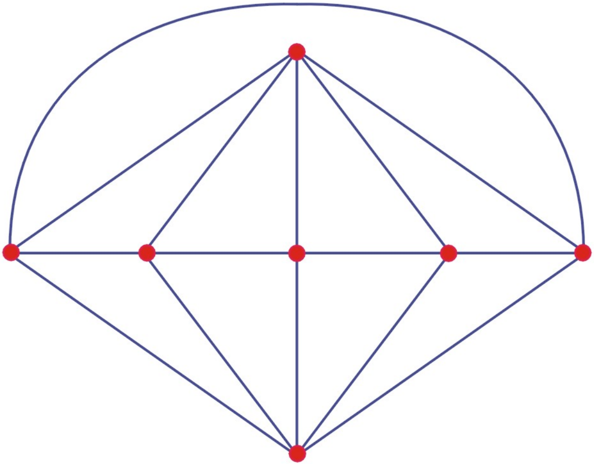

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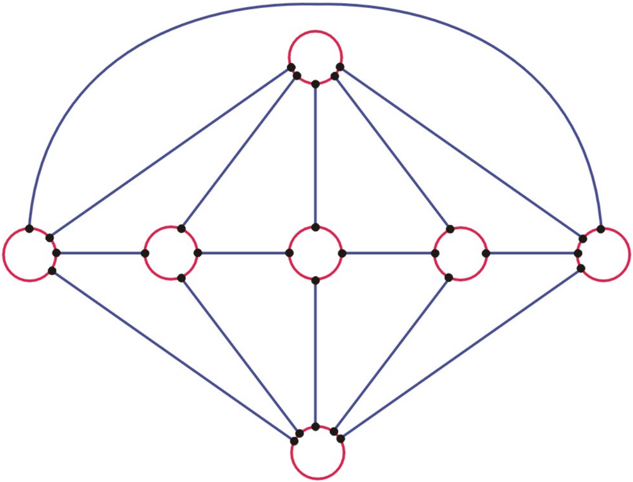

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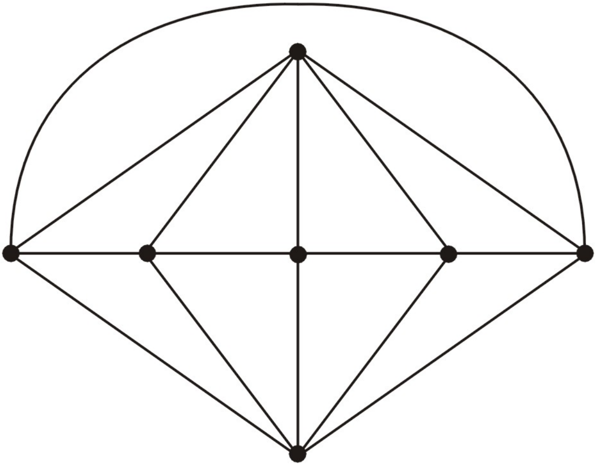

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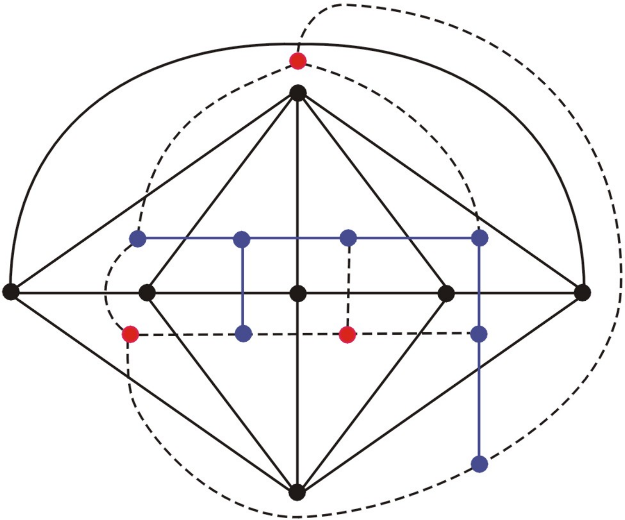

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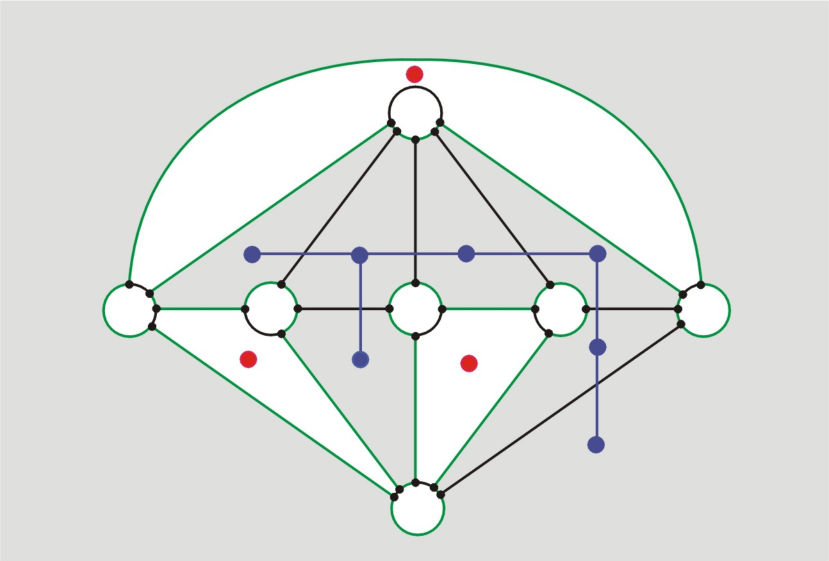
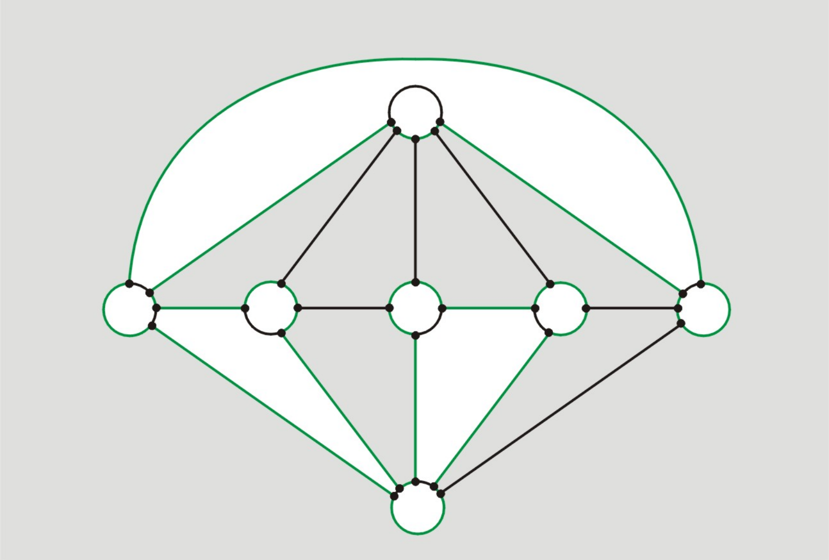
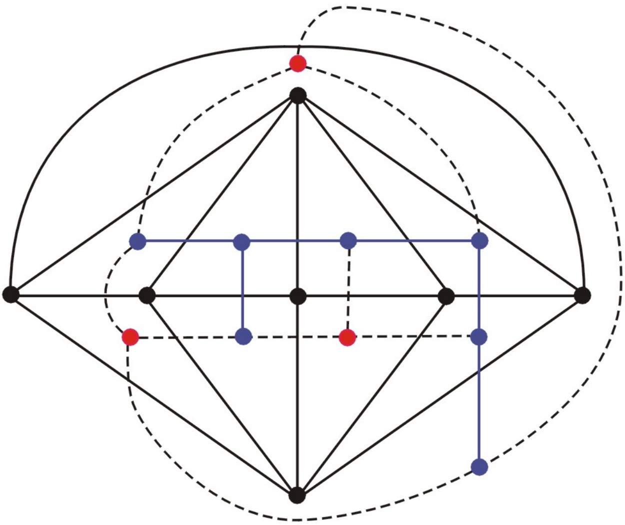


Example: The required Hamilton cycle



When does such a structure exist?



Vertex-partitions in cubic graphs

Theorem

The following are equivalent for every connected cubic graph G.

(i) $V(G)$ *has a partition* $\{A, J\}$ *where* A *induces a tree and* J *is independent.*

(ii) G *has an orientable cellular embedding with a single face.*

Contractible Hamilton cycles in truncated triangulations

Theorem

Let $\mathcal{T}$ be a triangulation of a closed surface and let $t(\mathcal{T})$ be the truncation of $\mathcal{T}$. The following statements are equivalent.

(i) *$t(\mathcal{T})$ has a contractible Hamilton cycle.*

(ii) *The underlying graph of $\mathcal{T}^*$ admits an orientable cellular embedding with a single face.*

Corollaries

Corollary

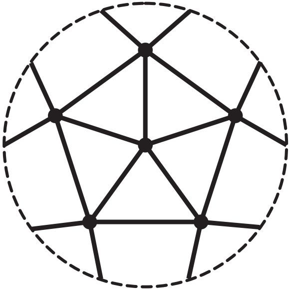
Let $\mathcal{T}$ be a triangulation of a closed surface with f faces. If $\mathcal{T}$ has no separating 3-cycles, then its trucation admits a Hamilton path. Moreover, $t(\mathcal{T})$ has a contractible Hamilton cycle in each of the following cases:

(i) $f \equiv 2 \pmod 4$

(ii) *$\mathcal{T}^*$ is cyclically 5-connected and $\mathcal{T}$ has a vertex of degree $0 \pmod 4$.*

(iii) *$\mathcal{T}^*$ is cyclically 6-connected and $\mathcal{T}$ has two adjacent vertices with degrees $deg(u) \equiv deg(v) \equiv \pm 1 \pmod 4$.*

Interesting example



A unified approach to results of Glover, Marušič et al.

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(i) *K has a Hamilton path.*

(ii) *K has a bounding Hamilton cycle with respect to its natural embedding as a Cayley map $CM(H; r, l)$ $\iff$ $|H| \equiv 2 \pmod 4$ or if $|r| \equiv 0, \pm 1 \pmod 4$.*

Furthemore, if $CM(H; r, l)$ has a bounding Hamilton cycle, then it has a contractible one.

Proof of (ii).

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Theorem

Let $\mathcal{M}$ be a polytopal map with n vertices and let Q be a one-sided subgraph of $\mathcal{M}^*$ determining a bounding Hamilton cycle in $\mathcal{M}$.
If $\beta(Q) = b$, then

$$\sum_{v \in V(Q)} (deg(v) - 2) - 2b + 2 = n.$$

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In our case, $\mathcal{M} = CM(H; r, l)$ is orientable, so Q must have an even Betti number. Hence $n = |H| = 2 \pmod 4$, a contradiction.

Truncations of Coxeter triangulations of the torus

Coxeter and Moser classified regular toroidal triangulations as $\{3,6\}_{b,c}$ where b and c are non-negative integer parameters. The size of the orientation-preserving automorphism group is $6(b^2+bc+c^2)$.

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Corollary

The truncation of $\{3,6\}_{b,c}$ has a bounding Hamilton cycle $\iff$ at least one of b and c is odd.

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The truncation of $\{3, 6\}_{b,c}$ has a bounding Hamilton cycle $\iff$ at least one of b and c is odd.

Altshuler (1972) proved that all these graphs are hamiltonian.

$\Longrightarrow$ If b and c are even, then all Hamilton cycles are non-bounding.

Hamiltonicity of three-involution cubic Cayley graphs

Theorem

Let $K = \mathrm{Cay}(H; x, y, z)$ be a cubic Cayley graph, where $H = \langle x, y, z \mid x^2 = y^2 = z^2 = 1, (xy)^3 = (yz)^3 = 1, \dots \rangle$. Then K admits a bounding Hamilton cycle with respect to the natural associated embedding $\iff$ $|H| \equiv 2 \pmod 4$ or $|xz|$ is even.

Furthemore, if K has a bounding Hamilton cycle (with respect to the natural embedding), then it has a contractible one.

Thank you!