Locally triangular graphs and normal quotients of n-cubes

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The triangular graph T_n for $n \geqslant 2$:

Vertices 2-subsets of $\{1, \ldots, n\}$.

Adjacency $\{i,j\} \sim \{k,\ell\} \iff |\{i,j\} \cap \{k,\ell\}| = 1$.

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Lemma (Neumaier, 1985)

Let Γ be a graph. Let $n \geqslant 2$. The following are equivalent.

(i) Γ is a connected locally T_n graph.

(ii) Γ is a halved graph of an n-valent bipartite rectagraph with $c_3 = 3$.

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Goal: refine this result using groups!

Let $K \leqslant \mathrm{Aut}(Q_n)$. The normal quotient $(Q_n)_K$ has

Vertices $\{x^K : x \in \mathbb{F}_2^n\}$.

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Let $K \leqslant \mathrm{Aut}(Q_n)$. The minimum distance of K is

$$d_K := \begin{cases} \min\{d_{Q_n}(x, x^k) : x \in VQ_n, k \in K \setminus \{1\}\} & \text{if } K \neq 1, \\ \infty & \text{otherwise.} \end{cases}$$

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$$c \in C \implies d_{Q_n}(x, x^c) = d_{Q_n}(x, x + c) = |c|.$$

In a graph Γ, for $u, v \in V\Gamma$ such that $d_\Gamma(u, v) = i$, define

$$a_i(u, v) := |\Gamma_i(u) \cap \Gamma(v)|$$
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Theorem (F., 2016)

Let $K \leqslant \mathrm{Aut}(Q_n)$. Let $\ell \geqslant 1$. The following are equivalent.

(i) $(Q_n)_K$ is n-valent with $a_{i-1} = 0$ and $c_i = i$ for $1 \leqslant i \leqslant \ell$.

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In particular, the following are equivalent for a graph Π.

(i) Π is an n-valent rectagraph with $a_2 = 0$ and $c_3 = 3$.

(ii) $\Pi \simeq (Q_n)_K$ for some $K \leqslant \mathrm{Aut}(Q_n)$ such that $d_K \geqslant 7$.

Theorem (F., 2016)

Let Γ be a graph. Let $n \geqslant 2$. The following are equivalent.

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- K acts semiregularly on $\mathbb{F}_2^n$; in particular K is a 2-group.
- K is unique up to conjugacy in $\mathrm{Aut}(Q_n)$.
- $\mathrm{Aut}(\Gamma) = N_{E_n:S_n}(K)/K$ where $E_n = \{c \in \mathbb{F}_2^n : |c| \equiv 0 \bmod 2\}$.

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When $n = 8$, $\exists$ even $K \leqslant \mathrm{Aut}(Q_n)$ with $K \simeq Q_8$ and $d_K = 4$, but the halved graphs of $(Q_n)_K$ are regular with different valencies.

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Proposition

Let $K \leqslant \mathrm{Aut}(Q_n)$ be even where $d_K \geqslant 2$. If n is odd, then $(Q_n)_K$ has isomorphic halved graphs.

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What about n even? And if $d_K \geqslant 7$?