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Special Functions and Orthogonal Polynomials

Tight frames of multivariate orthogonal polynomials

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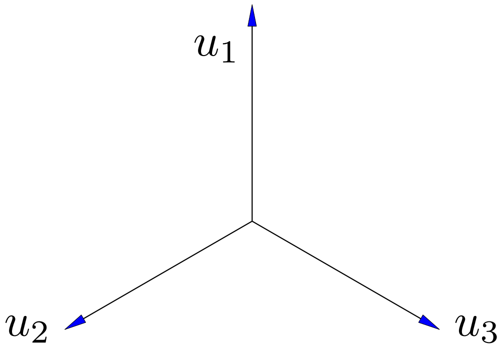
ABSTRACT

We show (by examples) that tight frame decompositions are useful and natural for finite dimensional Hilbert spaces which have symmetries, in particular for spaces of multivariate orthogonal polynomials.

A question

Let u_1, u_2, u_3 be three equally spaced unit vectors in $\mathbb{R}^2$.



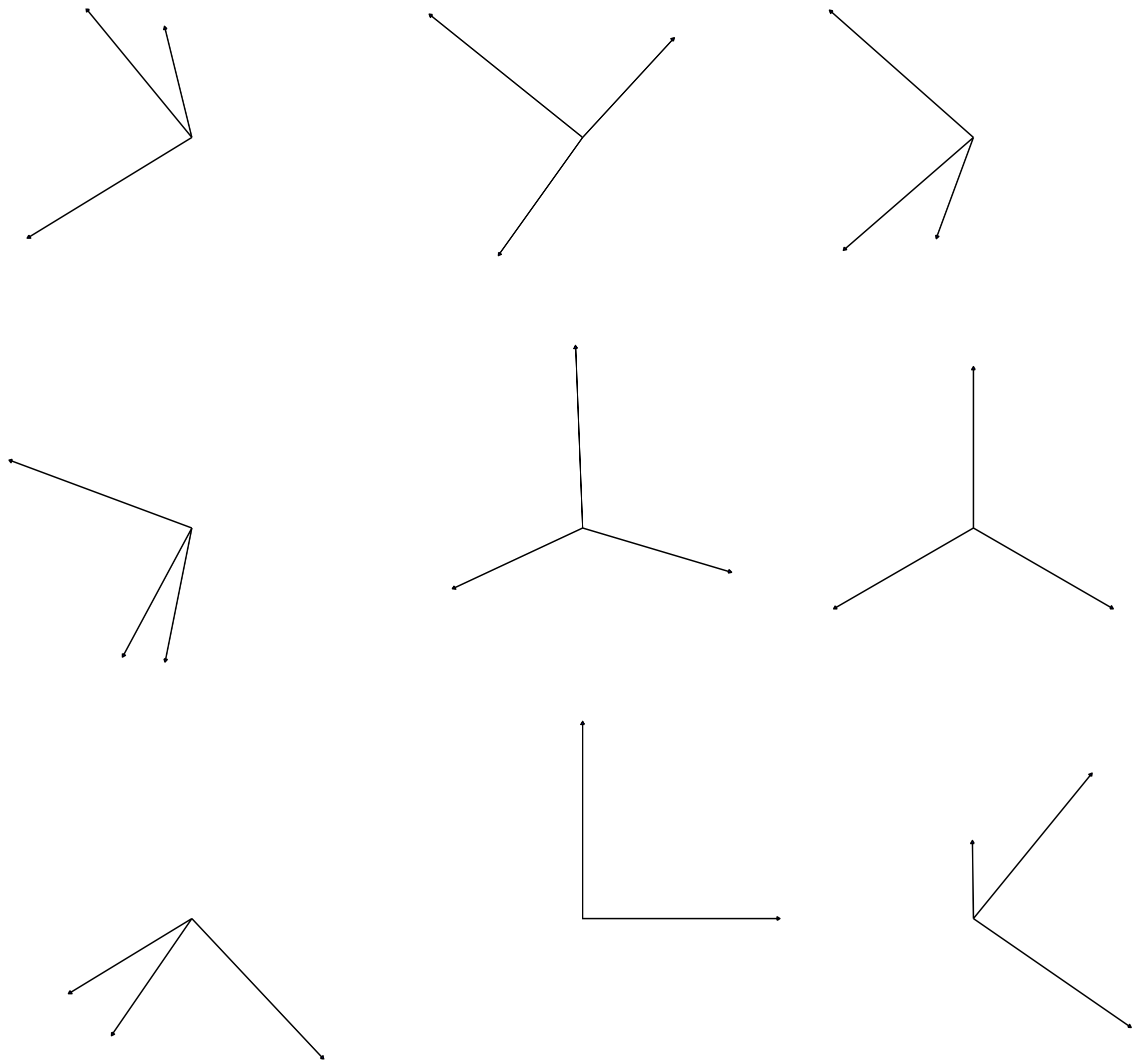
For a given nonzero vector $f \in \mathbb{R}^2$, what is the sum of its orthogonal projections onto these vectors?

(a) $$\sum_{j=1}^{3} \langle f, u_j \rangle u_j = 0 \quad (\text{since } u_1 + u_2 = u_3 = 0).$$

(b) $$\sum_{j=1}^{3} \langle f, u_j \rangle u_j = \frac{3}{2} f, \qquad \forall f \in \mathbb{R}^2.$$

Frames in finite dimensional spaces

The following sets of vectors $\{v_j\}_{j=1}^3$ form tight frames for $\mathbb{R}^2$



i.e., give decompositions of the form

$$f = \sum_{j=1}^{3} \langle f, v_j \rangle v_j, \qquad \forall f \in \mathbb{R}^2.$$

This is technically similar to an orthogonal expansion, except it has more terms (redundancy).

The start of a (long) story

The **Bernstein operator** $B_n : C([0,1]) \to \Pi_n$ is defined by

$$B_n f(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

In [Cooper,Waldron 2000] it was shown B_n has the diagonal form

$$B_n f = \sum_{k=0}^{n} \lambda_k^{(n)} \, p_k^{(n)} \, \mu_k^{(n)}(f),$$

where the eigenvalues $1 = \lambda_0^{(n)} = \lambda_1^{(n)} > \lambda_2^{(n)} > \cdots > \lambda_n^{(n)} > 0$ are

$$\lambda_k^{(n)} := 1 \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right)$$

and the corresponding eigenfunctions have the form

$$p_k^{(n)}(x) = x^k - \frac{k}{2} x^{k-1} + \text{lower order terms}.$$

The limiting eigenfunctions

The Bernstein operator converges as $n \to \infty$

$$\begin{array}{ccccccc} B_n f & = & \displaystyle\sum_{k=0}^{n} & \lambda_k^{(n)} & p_k^{(n)} & \mu_k^{(n)}(f) \\ \downarrow & & & \downarrow & \downarrow & \downarrow \\ f & = & \displaystyle\sum_{k=0}^{\infty} & 1 \;\cdot & p_k^* \;\cdot & \mu_k^*(f), \end{array}$$

where the "limit" eigenfunctions p_k^* are related to the Jacobi polynomials (similarly for the multivariate Bernstein operator).

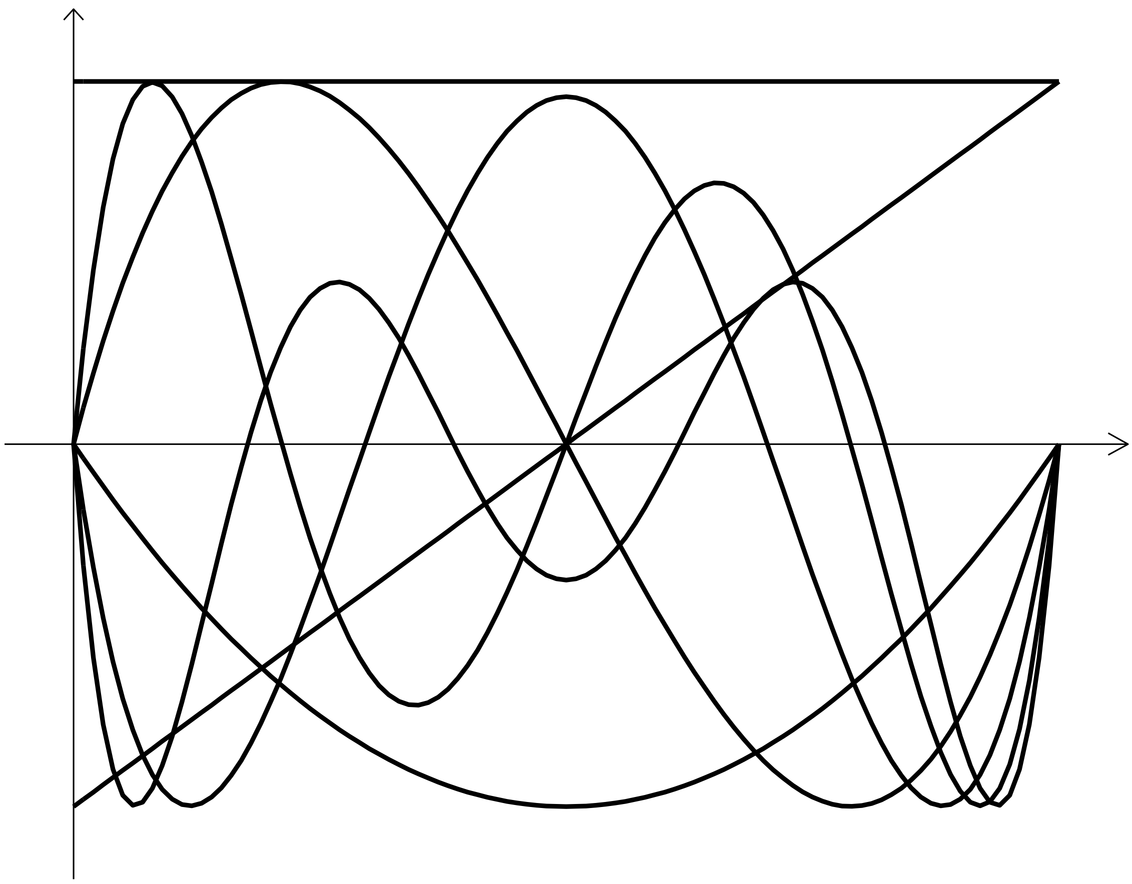


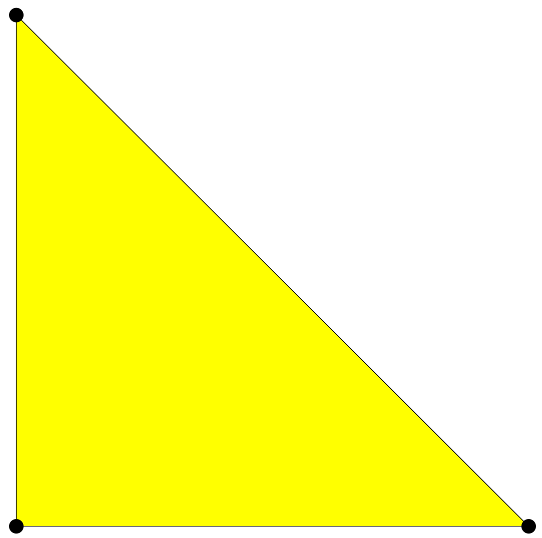
Fig. The first few limit eigenfunctions p_k^*.

Jacobi polynomials on a simplex

Let $T = \mathrm{conv}(V)$ be a simplex in $\mathbb{R}^d$ with $d+1$ vertices V, with corresponding barycentric coordinates $\xi = (\xi_v)_{v\in V}$, and define the Jacobi inner product

$$\langle f, g\rangle_\nu := \int_T f g\, \xi^{\nu-1}, \qquad \nu = (\nu_v)_{v\in V} > 0.$$

e.g., for $d = 2$, $T = \mathrm{conv}\{e_1, e_2, 0\}$, $\nu - 1 = (\alpha, \beta, \gamma)$



$$\xi_{e_1}(x,y) = x$$

$$\xi_{e_2}(x,y) = y$$

$$\xi_0(x,y) = 1 - x - y$$

$$\langle f, g\rangle_\nu = \int_0^1 \int_0^{1-x} f(x,y) g(x,y)\, x^\alpha y^\beta (1-x-y)^\gamma \, dy\, dx$$

The **Jacobi polynomials** of degree k are

$$\mathcal{P}_k^\nu := \{f \in \Pi_k : \langle f, p\rangle_\nu = 0, \forall p \in \Pi_{k-1}\}.$$

This space has

$$\dim(\mathcal{P}_k^\nu) = \binom{k+d-1}{d-1}.$$

Each polynomial in $\mathcal{P}_k^\nu$ is uniquely determined by its leading term, e.g., for $\xi_0^2 +$ lower order terms, the leading term is

$$\{(1-x-y)^2\}_\downarrow = x^2 - 2xy + y^2.$$

Orthogonal and biorthogonal systems

We describe the known representations for $\mathcal{P}_k^\nu$ in terms of the leading terms (for the case $d = 2$, $k = 2$).

Biorthogonal system (Appell 1920's): partial symmetries

$$x^2, \quad xy, \quad y^2.$$

Orthogonal system (Prorial 1957, et al): no symmetries

$$x^2 + y^2 + 2xy, \quad x^2 - y^2, \quad x^2 - y^2 - 4xy.$$

For the three dimensional space of all quadratic Jacobi polynomials on the triangle, we want an orthonormal basis with leading terms determined by the six polynomials

$$x^2, \quad xy, \quad y^2, \quad x(1-x-y), \quad y(1-x-y), \quad (1-x-y)^2.$$

Let

$$\Phi := \{p_{\xi^\alpha} = \xi^\alpha + \text{l.o.t} \in \mathcal{P}_2 : |\alpha| = 2\}$$

be these six functions. Then Φ is a frame for $\mathcal{P}_2^\nu$ (i.e., it spans) but it is *not* tight. We would like to find contants $c_\alpha > 0$ with

$$f = \sum_{|\alpha|=2} c_\alpha \langle f, p_{\xi^\alpha} \rangle p_{\xi^\alpha} = \sum_{|\alpha|=2} \langle f, \tilde{p}_{\xi^\alpha} \rangle \tilde{p}_{\xi^\alpha}, \qquad \forall f \in \mathcal{P}_2^\nu,$$

where $\tilde{p}_{\xi^\alpha} := \sqrt{c_\alpha} p_{\xi^\alpha}$.

Signed frames

Theorem [PW]. *Let $\mathcal{H}$ be Hilbert space of dimension d, and*

$$n = \begin{cases} \frac{1}{2}d(d+1), & H \text{ real;} \\ d^2, & H \text{ complex.} \end{cases}$$

Then for almost every choice of unit vectors $u_1, \ldots, u_n$ in $\mathcal{H}$ there are unique scalars $c_1, \ldots, c_n$ for which

$$f = \sum_{j=1}^{n} c_j \langle f, u_j \rangle u_j, \qquad \forall f \in \mathcal{H}.$$

The c_j can be computed explicitly, some may nonnegative, and

$$\sum_{j=1}^{n} c_j = d = \dim(\mathcal{H}).$$

Example. For any three vectors in $\mathbb{R}^2$ for which none is a multiple of another, there is a unique such scaling as above.

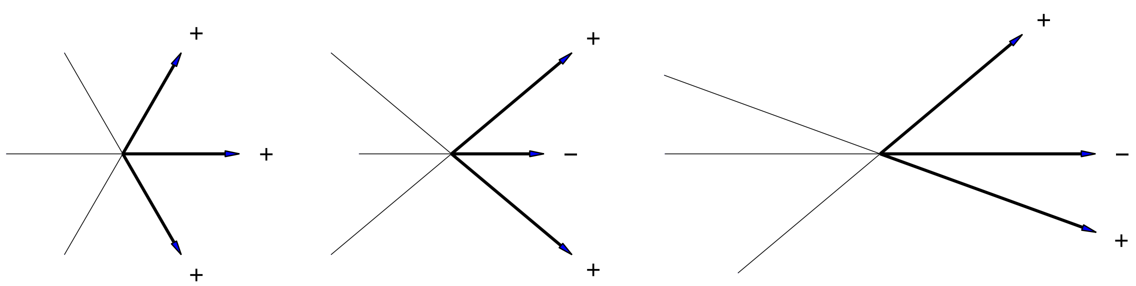


Fig. Tight signed frames of three vectors in $\mathbb{R}^2$ with the signature indicated.

Example. For our six functions Φ, $d = \dim(\mathcal{P}_2^\nu) = 3$, and

$$n = \frac{1}{2}d(d+1) = 6.$$

A tight frame for the Jacobi polynomials

Let ϕ^ν_α be the orthogonal projection of

$$\xi^\alpha/(\nu)_\alpha, \qquad |\alpha| = n$$

onto $\mathcal{P}^\nu_n$, which is given by

$$\begin{aligned}\phi^\nu_\alpha &:= \frac{(-1)^n}{(n+|\nu|-1)_n} F_A\left(\begin{matrix}|\alpha|+|\nu|-1,-\alpha \\ \nu\end{matrix};\xi\right) \\ &= \frac{(-1)^n}{(n+|\nu|-1)_n} \sum_{\beta\le\alpha} \frac{(n+|\nu|-1)_{|\beta|}(-\alpha)_\beta}{(\nu)_\beta} \frac{\xi^\beta}{\beta!},\end{aligned}$$

with F_A the Lauricella function of type A.

Theorem [WXR]. *The Jacobi polynomials on a simplex have the tight frame representation*

$$f = (|\nu|)_{2n} \sum_{|\alpha|=n} \frac{(\nu)_\alpha}{\alpha!} \langle f, \phi^\nu_\alpha\rangle_\nu \phi^\nu_\alpha, \qquad \forall f \in \mathcal{P}^\nu_n,$$

where the normalisation is $\langle 1, 1\rangle_\nu = 1$.

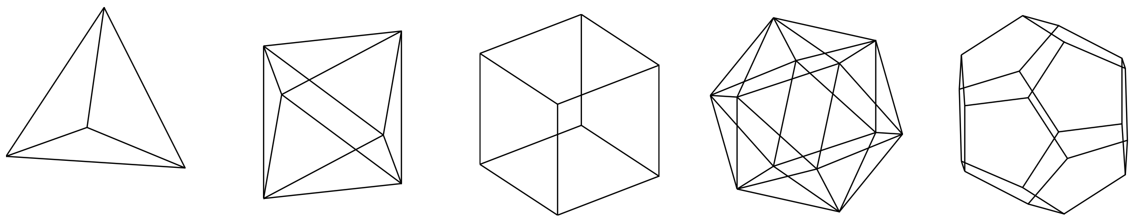
Remark. It can be shown that the polynomials

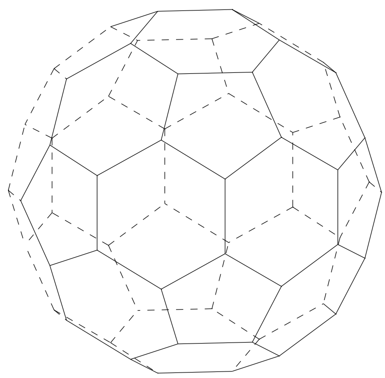
$$p^\nu_\alpha := (\nu)_\alpha \phi^\nu_\alpha = \xi^\alpha + \text{lower order terms}, \qquad |\alpha| = n$$

have a limit p^*_α as $\nu \to 0^+$, and that p^*_α is a limit eigenfunction for the Bernstein operator B_n on the simplex T.

Well distributed points on the sphere

A number of nice configurations of points on the sphere give isometric (equal length vector) tight frames, e.g.,





These turn out to be examples of the orbit of a single vector $v \in \mathbb{C}^d$ under a finite group G of unitary matrices which form an *irreducible representation*, i.e.,

$$\operatorname{span}\{gw : g \in G\} = \mathbb{C}^d, \qquad \forall w \neq 0.$$

Theorem ([VW04]). *If* $\operatorname{span}\{gw\}_{g\in G} = \mathbb{C}^d$ *for some vector* w*, then one can construct a vector* v *for which*

$$Gv := \{gv : g \in G\}$$

is a tight frame for $\mathbb{C}^d$*.*

A nice example

The group of symmetries of the triangle ($G = D_3 \approx S_3$) induces a representation on the quadratic Legendre polynomials $\mathcal{P}_2$ on the triangle. Since there is a polynomial whose orbit spans $\mathcal{P}_2$, we can construct a single polynomial

$$f = (2\sqrt{5}-5\sqrt{2})\left(\xi_v^2+\xi_w^2+\xi_u^2-\frac{1}{2}\right)+15\sqrt{2}\left(\xi_v^2-\frac{4}{5}\xi_v+\frac{1}{10}\right) \in \mathcal{P}_2$$

whose orbit under G consists of *three* polynomials which form an orthonormal basis for $\mathcal{P}_2$.

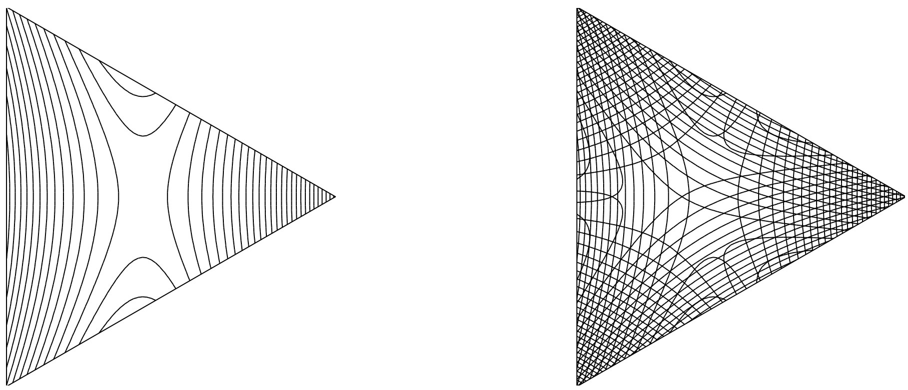


Fig. Contour plots of f and those of its orbit showing the triangular symmetry.

Orthogonal polynomials on the disc

Let $\mathcal{P}_n = \mathcal{P}_n^w$ be the $n+1$ dimensional space of orthogonal polynomials of degree n on the unit disc

$$D := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

given by the radially symmetric inner product

$$\langle f, g\rangle := \int_D f\,g\,w = \int_0^{2\pi}\int_0^1 (fg)(r\cos\theta, r\sin\theta)\,w(r)\,rdr\,d\theta.$$

The **Gegenbauer polynomials** are given by the weight

$$w(r) := (1-r^2)^\alpha \qquad \alpha > -1.$$

These polynomials have long been used to analyse the optical properties of a circular lens, and to reconstruct images from Radon projections, etc.

Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denote rotation through the angle θ, i.e.,

$$\begin{aligned} R_\theta(x,y) &:= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}. \end{aligned}$$

Let the group of rotations of the disc (which are symmetries of the weight)

$$\mathrm{SO}(2) = \{R_\theta : 0 \leq \theta < 2\pi\}$$

act on functions defined on the disc in the natural way, i.e.,

$$R_\theta f := f \circ R_\theta^{-1}.$$

The Logan Shepp polynomials

[Logan, Shepp 1975] showed the **Legendre polynomials** on the disc (constant weight $w = 1$) have an orthonormal basis given by the $n+1$ polynomials

$$p_j(x, y) := \frac{1}{\sqrt{\pi}} U_n\big(x \cos \frac{j\pi}{n+1} + y \sin \frac{j\pi}{n+1}\big), \qquad j = 0, \ldots, n,$$

where U_n is the n–th *Chebyshev polynomial of the second kind.*

This says that an orthonormal basis can be constructed from a single simple polynomial p_0 (a ridge function obtained from a univariate polynomial) by rotating it through the angles

$$\frac{j\pi}{n+1}, \qquad 0 \leq j \leq n.$$

It turns out, that for *any* weight w such an orthogonal expansion always exists, though the 'simple' polynomial p_0 is not in general a ridge function. Moreover, such an expansion reflects the rotational symmetry of the weight in a deeper way, e.g., for the Legendre polynomials there exists the tight frame decompositions

$$\begin{aligned} f &= \frac{n+1}{k} \sum_{j=0}^{k-1} \langle f, R^j_{\frac{2\pi}{k}} p_0 \rangle R^j_{\frac{2\pi}{k}} p_0 \\ &= \frac{n+1}{2\pi} \int_0^{2\pi} \langle f, R_\theta p_0 \rangle R_\theta p_0 \, d\theta, \qquad \forall f \in \mathcal{P}_n, \end{aligned}$$

where $k \geq n+1$ with k not even if $k \leq 2n$.

A tight frame

For the weight function $w : [0,1] \to \mathbb{R}^+$ and a fixed n, let

$$P_j \neq 0, \qquad 0 \leq j \leq \frac{n}{2}$$

be an orthogonal polynomial of degree j for the univariate weight $(1+x)^{n-2j} w(\sqrt{\frac{1+x}{2}})$ on $[-1,1]$, and

$$h_j := \frac{\pi}{2^{n-2j+1}} \int_{-1}^{1} P_j^2(x)(1+x)^{n-2j} w\Bigl(\sqrt{\frac{1+x}{2}}\Bigr)\, dx.$$

Theorem [W07]. *Let $v \in \mathcal{P}_n$ be the polynomial with real coefficients defined by*

$$v(x,y) := \frac{1}{\sqrt{n+1}} \sum_{0 \leq j \leq \frac{n}{2}} \frac{2}{1+\delta_{j,\frac{n}{2}}} \frac{1}{\sqrt{h_j}} \operatorname{Re}(\xi_j z^{n-2j}) P_j(2|z|^2 - 1),$$

where $z := x + iy$, $\xi_j \in \mathbb{C}$, $|\xi_j| = 1$, with $\xi_{\frac{n}{2}} \in \{-1, 1\}$. Then $\{R^j_{\frac{\pi}{n+1}} v\}_{j=0}^n$ is an orthonormal basis for $\mathcal{P}_n$, and

$$\begin{aligned} f &= \frac{n+1}{k} \sum_{j=0}^{k-1} \langle f, R^j_{\frac{2\pi}{k}} v\rangle R^j_{\frac{2\pi}{k}} v \\ &= \frac{n+1}{2\pi} \int_0^{2\pi} \langle f, R_\theta v\rangle R_\theta v\, d\theta, \qquad \forall f \in \mathcal{P}_n, \end{aligned}$$

whenever $k \geq n+1$ and k is odd, or $k \geq 2(n+1)$.

Zonal functions

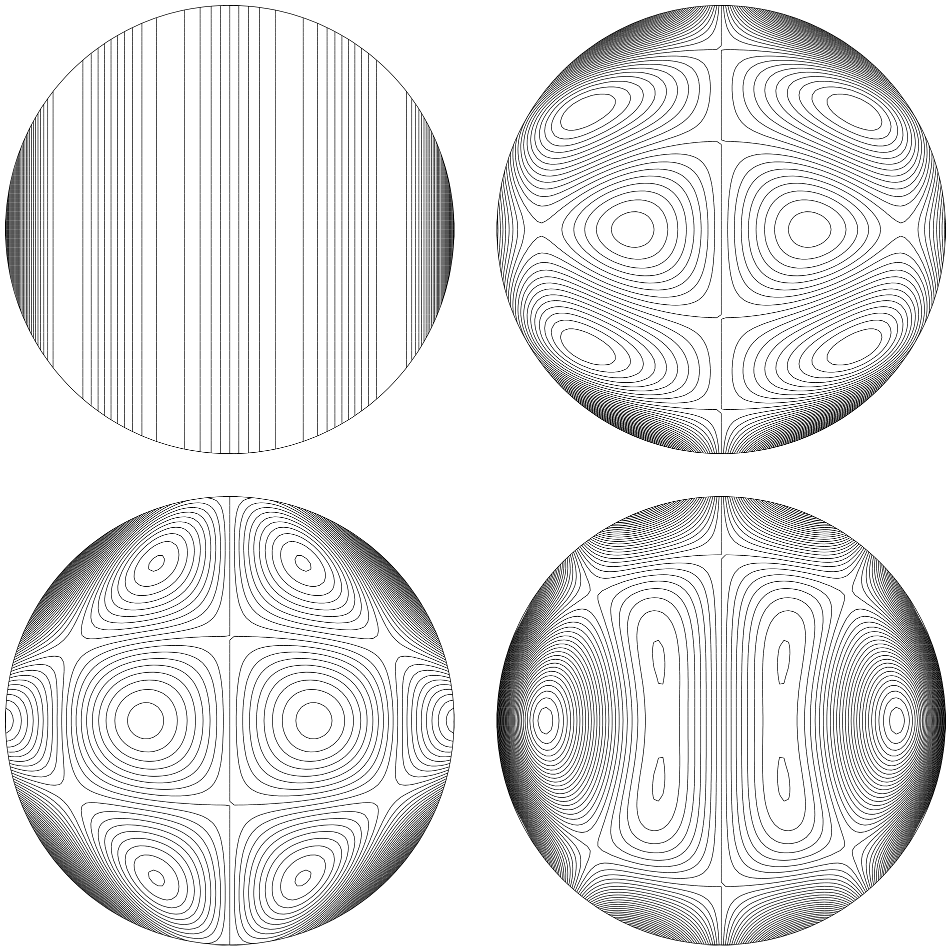


Fig. Contour plots of the Legendre polynomial $v \in \mathcal{P}_5$ for the choices $\xi_0 = 1$ and $\xi_1, \xi_2 \in \{-1, 1\}$. The first is the Logan-Shepp polynomial.

A function f on the ball or $\mathbb{R}^d$ is **zonal** if it can be written in the form

$$f(x) = g(\langle x, \xi \rangle, |x|).$$

Compare this with

$$f(x) = g(\langle x, \xi \rangle) \quad \text{(ridge function with direction } \xi\text{)},$$
$$f(x) = g(|x|) \quad \text{(radial function)}.$$

Orthogonal polynomials on a ball

Let $\mathcal{P}_n$ be the orthogonal polynomials on a ball in $\mathbb{R}^d$.

Theorem. *Let $p = p_\xi$ be the zonal function*

$$p_\xi := \sqrt{\frac{\operatorname{area}(S)}{\dim(\mathcal{P}_n)}} \sum_{0 \le j \le \frac{n}{2}} Z_\xi^{(n-2j)} \frac{P_j(|\cdot|^2)}{\|P_j\|_w} \in \mathcal{P}_n.$$

Then

$$\begin{aligned} f &= \dim(\mathcal{P}_n) \int_{\mathrm{SO}(d)} \langle f, gp \rangle gp \, d\mu(g) \\ &= \frac{\dim(\mathcal{P}_n)}{\operatorname{area}(S)} \int_S \langle f, p_\xi \rangle p_\xi \, d\xi, \qquad \forall f \in \mathcal{P}_n, \end{aligned}$$

where μ denotes the normalised Haar measure on $\mathrm{SO}(d)$.

Here $Z_\xi^{(k)}$ is the zonal harmonic of degree k, and P_j is a univariate orthogonal polynomial of degree j.

Corollary (Legendre polynomials). *For the weight $w = 1$ on the unit ball p_ξ is is the ridge polynomial given by*

$$p_\xi(x) = \frac{\sqrt{2n+d}}{\sqrt{\operatorname{area}(S)}\sqrt{\dim(\mathcal{P}_n)}} C_n^{d/2}(\langle x, \xi \rangle).$$

Here C_n^λ are Gegenbauer polynomials.