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Tight frames and their symmetries

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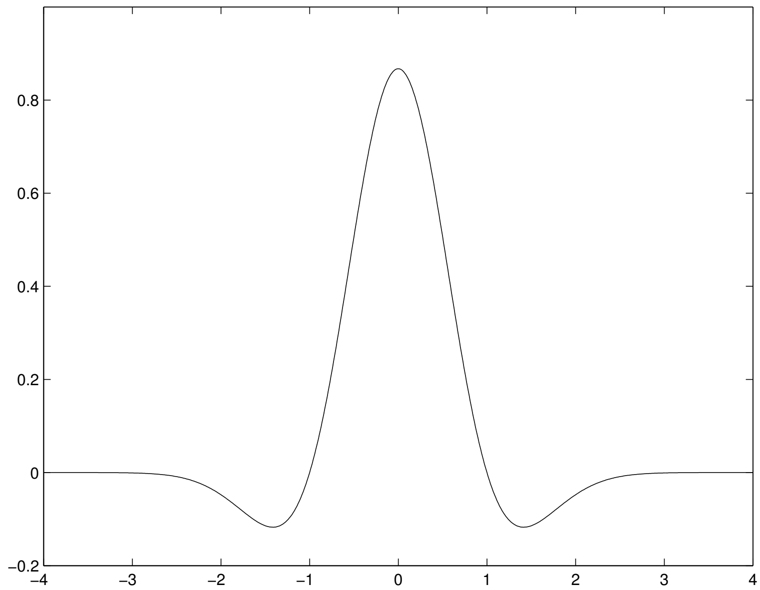
ABSTRACT

We show (by examples) that tight frame decompositions are useful and natural for finite dimensional Hilbert spaces which have symmetries, e.g., $\mathbb{R}^d$, $\mathbb{C}^d$ and spaces of multivariate orthogonal polynomials.

Frames in infinite dimensional spaces

The "Mexican hat" function

$$\psi(x) := \frac{2}{\sqrt{3}}\pi^{-\frac{1}{4}}(1-x^2)e^{-\frac{1}{2}x^2}$$



gives a *continuous* wavelet frame

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\Big(\frac{t-b}{a}\Big), \qquad a \in \mathbb{R}^+,\ b \in \mathbb{R},$$

and a (discrete) wavelet frame

$$\psi_{j,k}(t) := 2^{j/2}\psi(2^j t - kb), \qquad j,k \in \mathbb{Z},\ (b < 1.97).$$

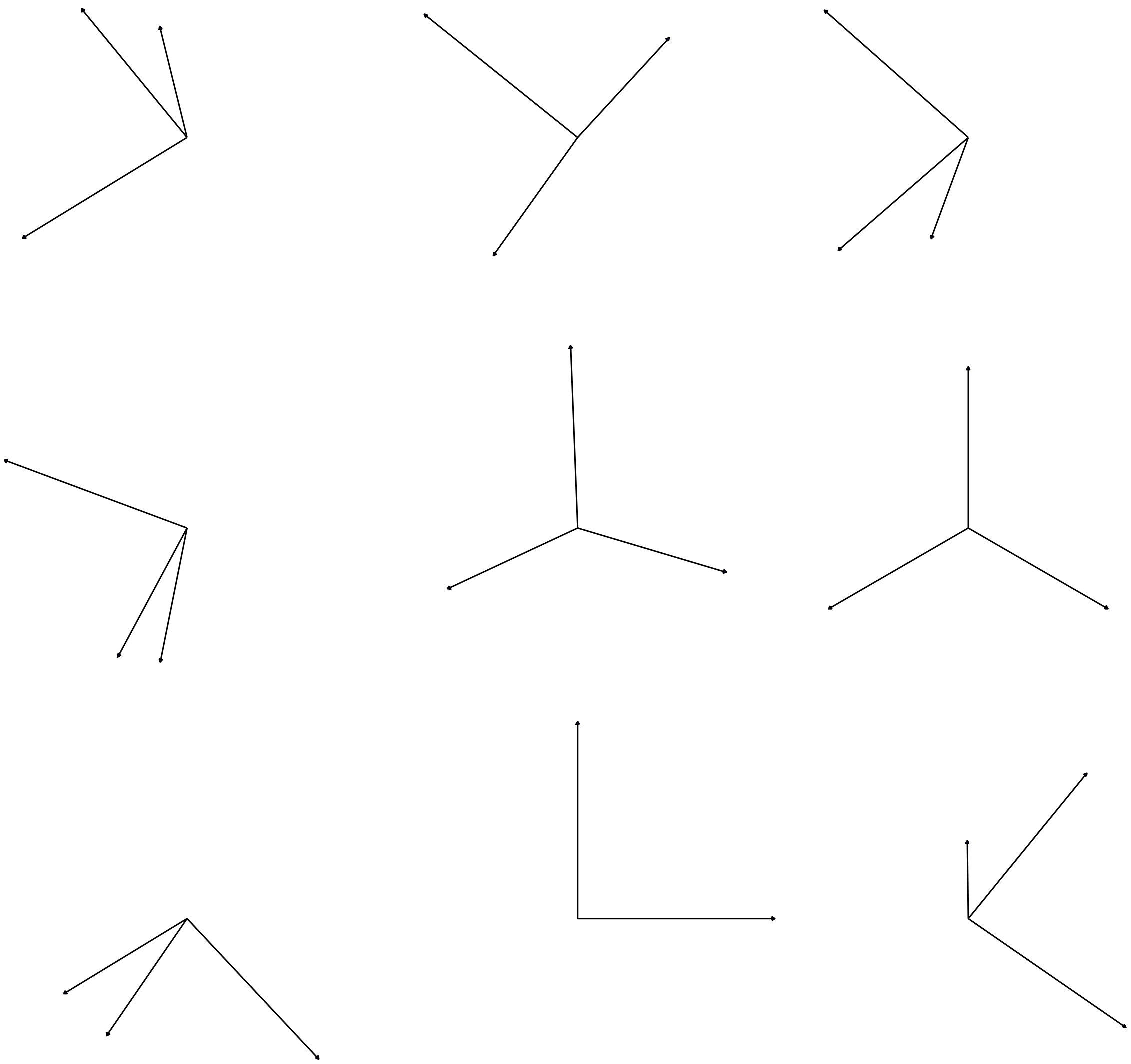
These lead to frame decompositions of the form

$$f = \sum_j \langle f, \psi_j \rangle \phi_j, \qquad \forall f \in L_2(\mathbb{R}),$$

where the ψ_j are obtained by applying "simple operations" to a single function ψ.

Frames in finite dimensional spaces

The following sets of vectors $\{v_j\}_{j=1}^3$ form tight frames for $\mathbb{R}^2$



i.e., give decompositions of the form

$$f = \sum_{j=1}^{3} \langle f, v_j \rangle v_j, \qquad \forall f \in \mathbb{R}^2.$$

This is technically similar to an orthogonal expansion, except it has more terms (redundancy).

The start of a (long) story

The **Bernstein operator** $B_n : C([0,1]) \to \Pi_n$

$$B_n f(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

has the diagonal form

$$B_n f = \sum_{k=0}^{n} \lambda_k^{(n)}\, p_k^{(n)}\, \mu_k^{(n)}(f),$$

where the eigenvalues $1 = \lambda_0^{(n)} = \lambda_1^{(n)} > \lambda_2^{(n)} > \cdots > \lambda_n^{(n)} > 0$ are

$$\lambda_k^{(n)} := 1 \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right)$$

and the corresponding eigenfunctions have the form

$$p_k^{(n)}(x) = x^k - \frac{k}{2} x^{k-1} + \text{lower order terms.}$$

The Bernstein operator converges as $n \to \infty$

$$\begin{array}{cccccccc} B_n f & = & \displaystyle\sum_{k=0}^{n} & \lambda_k^{(n)} & & p_k^{(n)} & & \mu_k^{(n)}(f) \\ \downarrow & & & \downarrow & & \downarrow & & \downarrow \\ f & = & \displaystyle\sum_{k=0}^{\infty} & 1 & \cdot & p_k^{*} & \cdot & \mu_k^{*}(f), \end{array}$$

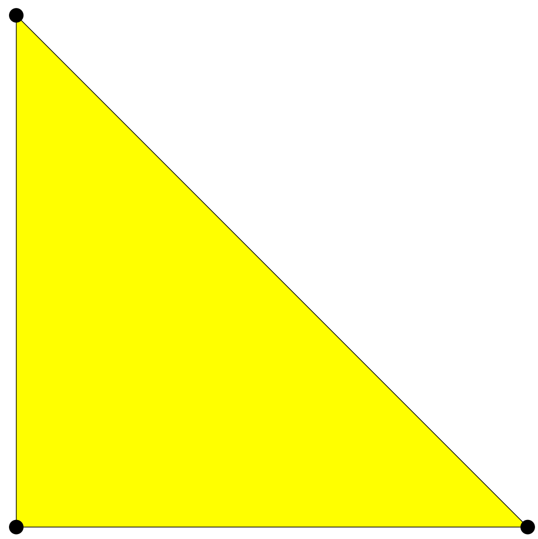
where the "limit" eigenfunctions p_k^* are related to the Jacobi polynomials (similarly for the multivariate Bernstein operator).

Jacobi polynomials on a simplex

Let $T = \operatorname{conv}(V)$ be a simplex in $\mathbb{R}^d$ with $d+1$ vertices V, with corresponding barycentric coordinates $\xi = (\xi_v)_{v\in V}$, and define the Jacobi inner product

$$\langle f, g\rangle_\nu := \int_T fg\,\xi^{\nu-1}, \qquad \nu = (\nu_v)_{v\in V} > 0.$$

e.g., for $d = 2$, $T = \operatorname{conv}\{e_1, e_2, 0\}$, $\nu - 1 = (\alpha, \beta, \gamma)$



$$\xi_{e_1}(x,y) = x$$

$$\xi_{e_2}(x,y) = y$$

$$\xi_0(x,y) = 1 - x - y$$

$$\langle f, g\rangle_\nu = \int_0^1 \int_0^{1-x} f(x,y)g(x,y)\, x^\alpha y^\beta (1-x-y)^\gamma\, dy\, dx$$

The **Jacobi polynomials** of degree k are

$$\mathcal{P}_k^\nu := \{f \in \Pi_k : \langle f, p\rangle_\nu = 0, \forall p \in \Pi_{k-1}\}.$$

This space has

$$\dim(\mathcal{P}_k^\nu) = \binom{k+d-1}{d-1}.$$

Each polynomial in $\mathcal{P}_k^\nu$ is uniquely determined by its leading term, e.g., for $\xi_0^2 +$ lower order terms, the leading term is

$$\{(1-x-y)^2\}_\downarrow = x^2 - 2xy + y^2.$$

Orthogonal and biorthogonal systems

We describe the known representations for $\mathcal{P}_k^\nu$ in terms of the leading terms (for the case $d = 2$, $k = 2$).

Biorthogonal system (Appell 1920's): partial symmetries

$$x^2, \quad xy, \quad y^2.$$

Orthogonal system (Prorial 1957, et al): no symmetries

$$x^2 + y^2 + 2xy, \quad x^2 - y^2, \quad x^2 - y^2 - 4xy.$$

For the three dimensional space of all quadratic Jacobi polynomials on the triangle, we want an orthonormal basis with leading terms determined by the six polynomials

$$x^2, \quad xy, \quad y^2, \quad x(1-x-y), \quad y(1-x-y), \quad (1-x-y)^2.$$

Let

$$\Phi := \{p_{\xi^\alpha} = \xi^\alpha + \text{l.o.t} \in \mathcal{P}_2 : |\alpha| = 2\}$$

be these six functions. Then Φ is a frame for $\mathcal{P}_2^\nu$ (i.e., it spans) but it is *not* tight. We would like to find contants $c_\alpha > 0$ with

$$f = \sum_{|\alpha|=2} c_\alpha \langle f, p_{\xi^\alpha}\rangle p_{\xi^\alpha} = \sum_{|\alpha|=2} \langle f, \tilde{p}_{\xi^\alpha}\rangle \tilde{p}_{\xi^\alpha}, \qquad \forall f \in \mathcal{P}_2^\nu,$$

where $\tilde{p}_{\xi^\alpha} := \sqrt{c_\alpha} p_{\xi^\alpha}$.

Signed frames

Theorem [PW]. *Let $\mathcal{H}$ be Hilbert space of dimension d, and*

$$n = \begin{cases} \frac{1}{2}d(d+1), & H \text{ real;} \\ d^2, & H \text{ complex.} \end{cases}$$

Then for almost every choice of unit vectors $u_1, \ldots, u_n$ in $\mathcal{H}$ there are unique scalars $c_1, \ldots, c_n$ for which

$$f = \sum_{j=1}^{n} c_j \langle f, u_j \rangle u_j, \qquad \forall f \in \mathcal{H}.$$

The c_j can be computed explicitly, some may nonnegative, and

$$\sum_{j=1}^{n} c_j = d = \dim(\mathcal{H}).$$

Example. For any three vectors in $\mathbb{R}^2$ for which none is a multiple of another, there is a unique such scaling as above.

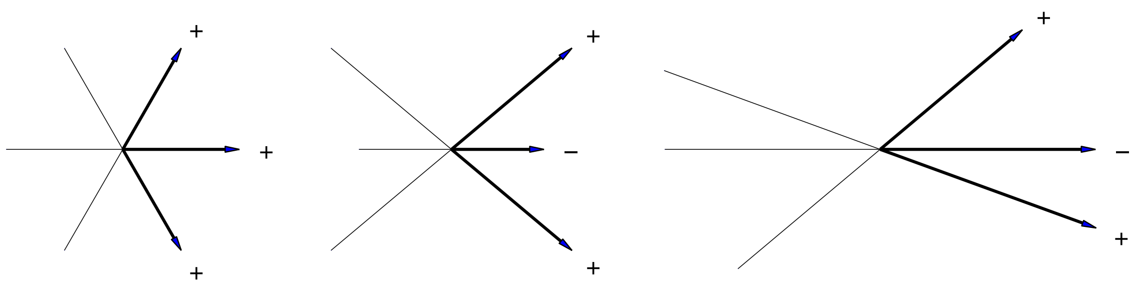


Fig. 1. Tight signed frames of three vectors in $\mathbb{R}^2$ with the signature indicated.

Example. For our six functions Φ, $d = \dim(\mathcal{P}_2^\nu) = 3$, and

$$n = \frac{1}{2}d(d+1) = 6.$$

A tight frame for the Jacobi polynomials

Let ϕ_α^ν be the orthogonal projection of

$$\xi^\alpha/(\nu)_\alpha, \qquad |\alpha| = n$$

onto $\mathcal{P}_n^\nu$, which is given by

$$\begin{aligned}\phi_\alpha^\nu &:= \frac{(-1)^n}{(n+|\nu|-1)_n} F_A\left(\begin{matrix}|\alpha|+|\nu|-1, -\alpha \\ \nu\end{matrix}; \xi\right) \\ &= \frac{(-1)^n}{(n+|\nu|-1)_n} \sum_{\beta\le\alpha} \frac{(n+|\nu|-1)_{|\beta|}(-\alpha)_\beta}{(\nu)_\beta} \frac{\xi^\beta}{\beta!},\end{aligned}$$

with F_A the Lauricella function of type A.

Theorem [WXR]. *The Jacobi polynomials on a simplex have the tight frame representation*

$$f = (|\nu|)_{2n} \sum_{|\alpha|=n} \frac{(\nu)_\alpha}{\alpha!} \langle f, \phi_\alpha^\nu\rangle_\nu \phi_\alpha^\nu, \qquad \forall f \in \mathcal{P}_n^\nu,$$

where the normalisation is $\langle 1, 1\rangle_\nu = 1$.

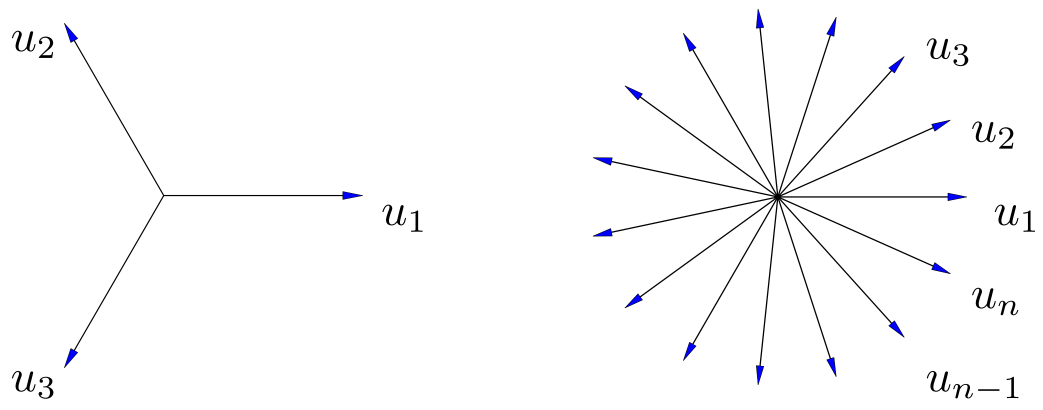
Remark. It can be shown that the polynomials

$$p_\alpha^\nu := (\nu)_\alpha \phi_\alpha^\nu = \xi^\alpha + \text{lower order terms}, \qquad |\alpha| = n$$

have a limit p_α^* as $\nu \to 0^+$, and that p_α^* is a limit eigenfunction for the Bernstein operator B_n on the simplex T.

Isometric tight frames

Any $n \geq 3$ equally spaced unit vectors $u_1, \ldots, u_n$ in $\mathbb{R}^2$



provide the following tight frame

$$f = \frac{2}{n} \sum_{j=1}^{n} \langle f, u_j \rangle u_j, \qquad \forall f \in \mathbb{R}^2.$$

Only five years ago, it wasn't generally known whether a tight frame of $n \geq d$ vectors existed for $\mathbb{R}^d$ (or $\mathbb{C}^d$), $d \geq 3$. At one of the problem sessions at Bommerholz 2000 it was asked what are the best frame bounds for a frame of $n \geq 3$ vectors in $\mathbb{R}^3$.

Independently, a number of people considered this question, e.g., Goyal, et al (signal processing), Zimmermann (in answer to the Bommerholz question), Waldron and Fickus (for the equidistribution of points). The field of construction and application of what are usually called *finite normalised tight frames* was born. Major advocates include Pete Casazza, John Benedetto and Jelena Kovačević.

Harmonic frames

An isometric frame which is generated by an abelian group of symmetries is called an **harmonic frame**.

Example. The character table of the cyclic group of order 3

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \qquad \omega := e^{\frac{2\pi i}{3}},$$

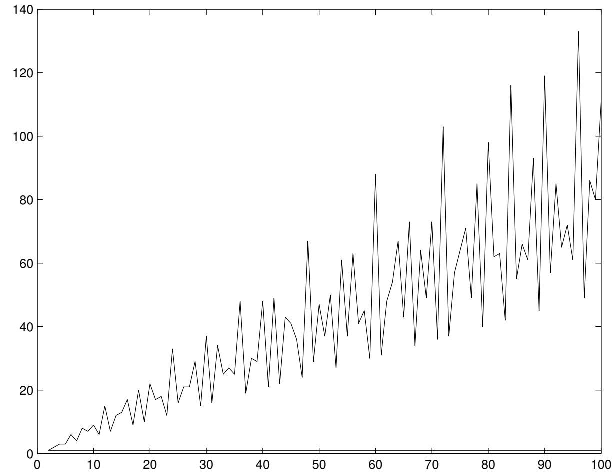
has orthogonal columns, and so the projection of them onto two coordinates gives isometric frames

$$\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \omega \\ \omega^2 \end{bmatrix}, \begin{bmatrix} \omega^2 \\ \omega \end{bmatrix}\} \text{ (real)} \qquad \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \omega \end{bmatrix}, \begin{bmatrix} 1 \\ \omega^2 \end{bmatrix}\} \text{ (complex)}$$

and these are harmonic.

Theorem [VW04]. *All harmonic frames of n vectors can be obtained by taking rows of the character table of an abelian group of order n.*

The list of all harmonic frames



The number of inequivalent harmonic frames in $\mathbb{C}^2$

```
St 2 2 (0.009) Ex 1 Fl 1 EV 0 EP 0 FS 0 FC 1
Gp 1 0.009 *
Fr 1 R L 1 1.4142 1.4142 1.4142 0 0.0000 0.0000 1 <2, 1> 1 {1} 0.00 {1,2}

St 4 2 (0.031) Ex 12 Fl 8 EV 3 EP 1 FS 1 FC 3
Gp 1 0.011 *-e**p
Gp 2 0.02 ---Fee
Fr 1 C L 1 1.0000 1.1380 1.4142 2 0.0000 0.7071 1 <4, 1> 1 {1} 0.00 {1,2}
Fr 2 C U 1 1.4142 1.6261 1.7320 2 0.0000 0.7071 1 <4, 1> 1 {1} 0.00 {2,3}
Fr 3 R U 2 1.4142 1.6094 2.0000 1 0.0000 1.0000 2 <8, 3> 2 {1,2} 0.00 {2,

St 5 2 (0.02) Ex 10 Fl 10 EV 5 EP 2 FS 0 FC 3
Gp 1 0.02 *eee*p*eep
Fr 1 C L 1 0.8312 1.0881 1.3449 3 0.3090 0.8090 1 <5, 1> 1 {1} 0.00 {1,2}
Fr 2 C U 1 1.5811 1.5811 1.5811 3 0.3090 0.8090 1 <5, 1> 1 {1} 0.5 {2,3}
Fr 3 R U 1 1.1755 1.5388 1.9021 3 0.3090 0.8090 1 <10, 1> 2 {1} 0.00 {2,5

St 6 2 (0.029) Ex 15 Fl 11 EV 1 EP 4 FS 0 FC 6
Gp 2 0.029 *---e*****-pppp
Fr 1 C L 1 0.7071 1.0555 1.4142 4 0.0000 0.8660 1 <6, 2> 1 {2} 0.00 {1,2}
Fr 2 C U 1 1.4142 1.5413 1.7320 4 0.0000 0.8660 1 <6, 2> 1 {2} 0.00 {2,3}
Fr 3 C U 1 1.2247 1.5223 2.0000 3 0.5000 1.0000 2 <6, 2> 1 {2} 0.00 {2,4}
Fr 4 C U 1 1.4142 1.5413 1.7320 2 0.0000 1.0000 3 <18, 3> 1 {2} 0.00 {2,5
Fr 5 R U 1 1.0000 1.4928 2.0000 3 0.5000 1.0000 2 <12, 4> 2 {2} 0.00 {2,6
Fr 6 C U 1 1.2247 1.5210 1.8708 4 0.0000 0.8660 1 <6, 2> 1 {2} 0.00 {3,4}
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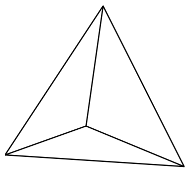
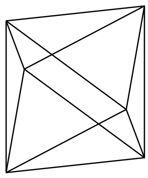
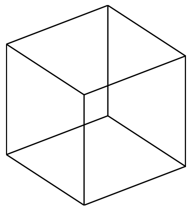
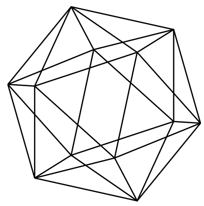
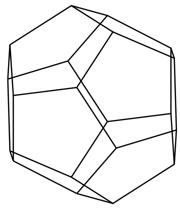
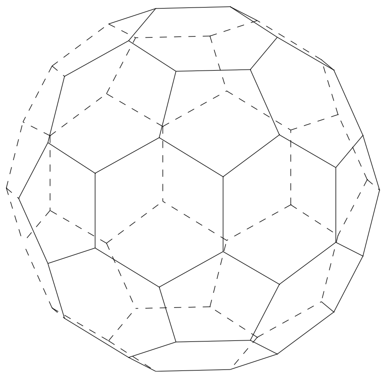
Well distributed points on the sphere

A number of nice configurations of points on the sphere give isometric tight frames, e.g.,



These turn out to be examples of the orbit of a single vector $v \in \mathbb{C}^d$ under a finite group G of unitary matrices which form an *irreducible representation*, i.e.,

$$\mathrm{span}\{gw : g \in G\} = \mathbb{C}^d, \qquad \forall w \neq 0.$$

Theorem ([VW04]). *If* $\mathrm{span}\{gw\}_{g\in G} = \mathbb{C}^d$ *for some vector* w*, then one can construct a vector* v *for which*

$$Gv := \{gv : g \in G\}$$

is a tight frame for $\mathbb{C}^d$*.*

A nice example

Since the group of symmetries of the triangle (the dihedral group $G = D_3 \approx S_3$) induces an irreducible representation on the quadratic Legendre polynomials $\mathcal{P}_2$ on the triangle, we can construct a single polynomial

$$f = (2\sqrt{5}-5\sqrt{2})\left(\xi_v^2+\xi_w^2+\xi_u^2-\frac{1}{2}\right)+15\sqrt{2}\left(\xi_v^2-\frac{4}{5}\xi_v+\frac{1}{10}\right) \in \mathcal{P}_2$$

whose orbit under G consists of *three* polynomials which form an orthonormal basis for $\mathcal{P}_2$.

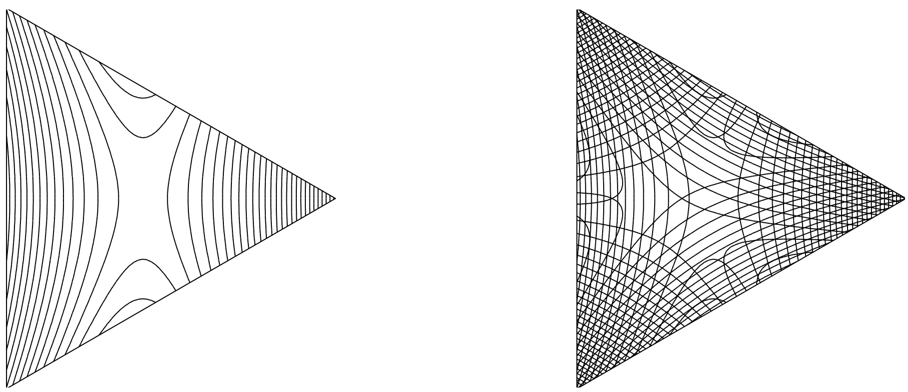


Fig. 1. Contour plots of f and those of its orbit showing the triangular symmetry.

Heisenberg frames

Let S be the shift and Ω the modulation operator on $\mathbb{C}^d$, i.e., with $\omega := e^{\frac{2\pi i}{d}}$ a primitive d–th root of unity

$$S := \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & 0 & & 1 & 0 \end{bmatrix}, \quad \Omega := \begin{bmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{d-1} \end{bmatrix}.$$

These generate the (discrete) Heisenberg group, with

$$\Omega^k S^j = \omega^{jk} S^j \Omega^k.$$

Numerically there exists a $v \in \mathbb{C}^d$ for which $\Phi := (S^j \Omega^k)_{j,k=0}^{d-1}$ is an isometric tight frame with equal cross correlation, i.e.,

$$\langle \phi_j, \phi_k \rangle = \frac{1}{\sqrt{d+1}}, \qquad j \neq k.$$

These are known as **Heisenberg frames**, **SICPOVM's** (rank one quantum measurements) and as sets of **equiangular lines**.

Explicit solutions for v are known only for $d \leq 7$ and $d = 19$, e.g., for $d = 7$ one has $v = (a, b, b, c, b, c, c)^T \in \mathbb{C}^7$, where

$$a = -\frac{\sqrt{8 - 5\sqrt{2}}(2\sqrt{2} + 1 \pm 7i)}{2\sqrt{7}(3\sqrt{2} - 2)},$$

$$b = \frac{\sqrt{8 - 5\sqrt{2}}}{4\sqrt{7}} + \frac{\sqrt[4]{2}}{4}, \quad c = \frac{\sqrt{8 - 5\sqrt{2}}}{4\sqrt{7}} - \frac{\sqrt[4]{2}}{4}.$$

More on harmonic frames

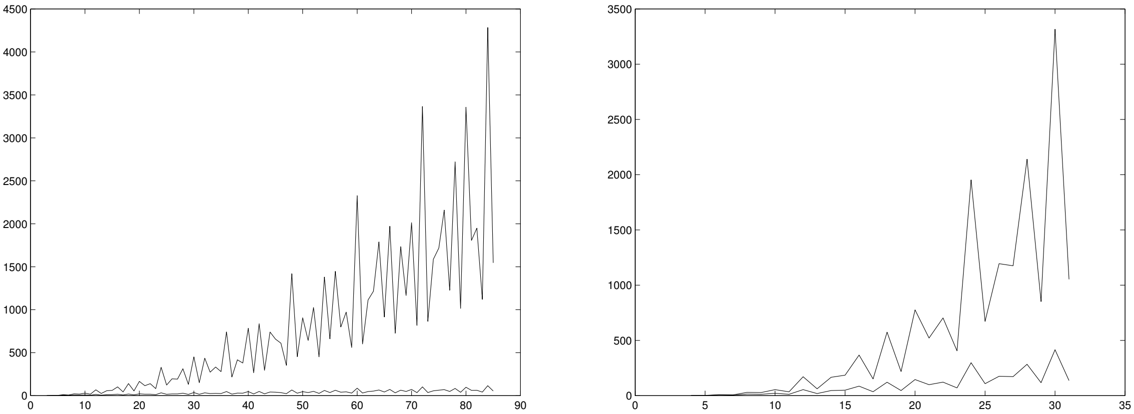


Fig. The number of inequivalent harmonic frames of n vectors in $\mathbb{C}^3$ and $\mathbb{C}^4$. The lower graph shows how many of them are lifted.

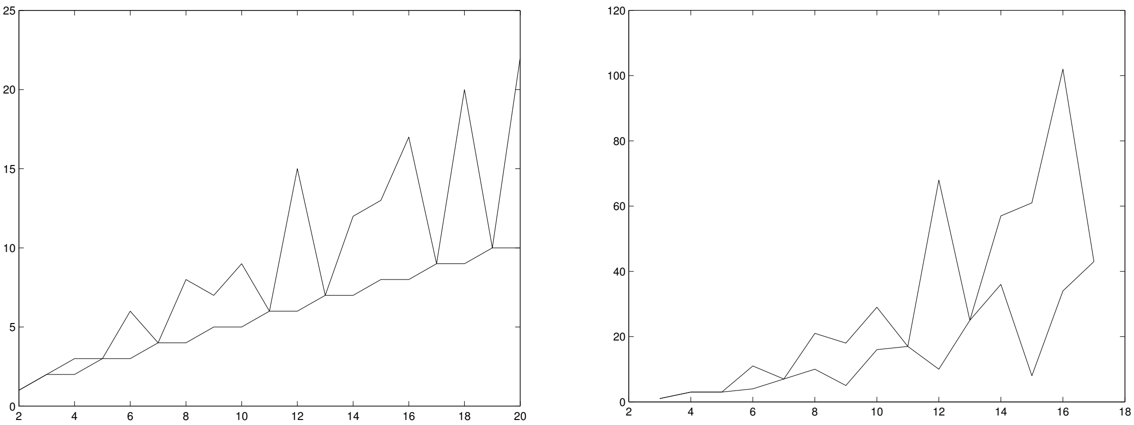


Fig. The number of harmonic frames of n vectors in $\mathbb{C}^2$ and $\mathbb{C}^3$, and those with $n-d$ erasures (lower graph).