On the Unitary Equivalence between Cyclic Harmonic Frames

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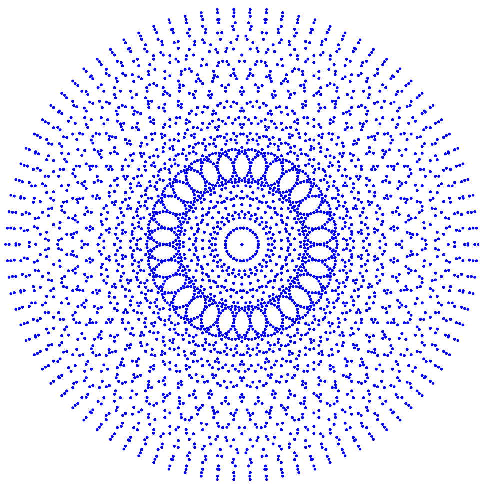

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Abstract

This thesis investigates unitary equivalence for a special class of finite tight frames called harmonic frames, which are constructed as the orbit of an unitary action of a finite abelian group. The notion of a multiplicative equivalence is defined and we show that in $\mathbb{C}^1$ and $\mathbb{C}^2$, unitary equivalence is completely characterised by multiplicative equivalence. It turns out that multiplicative equivalence is unitarily equivalence via an automorphism (permutation which respects group structure). In dimensions greater than two, a partial classification of unitary equivalences in terms of multiplicative equivalence is made. The unitary equivalence between any two frames is categorised into equivalences induced by permutations which preserve group structure and those which do not. Pertinent to this whole study of equivalences, the sums of roots of unity, their minimal vanishing sums and finding nice bases for cyclotomic fields reveal an intricate relationship due to their links with group characters.

Acknowledgements

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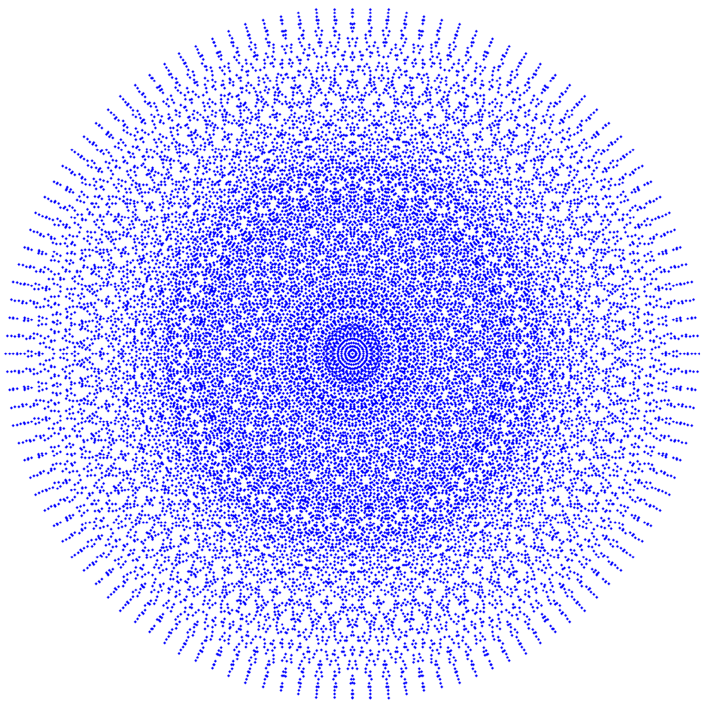

"The journey of a thousand miles begins with a single step."

(Lao Tzu)

Contents

Chapter 1

Introduction

The theory of frames has been in the literature for some time, dating back at least as far as 1937 [Sch37], but the modern development can be attributed to the seminal paper of Duffin and Schaeffer [DS52], where the notion of frame bounds were introduced. Since then, frame bounds have been assimilated into the modern definition of frames. Frames are essentially spanning sequences (when order is important) or sets, in a Hilbert space, satisfying the frame bound conditions. While much emphasis has been on the study of infinite dimensional frames, the study of finite tight frames should not neglected. As most real life applications often truncate dimensions at some point, it makes sense to explore the theory in finite dimensions. Some additional niceties are afforded to the finite context, from the results in functional analysis (e.g., the guaranteed existence of an inner product), to being able to draw upon the horde of well established results in finite group and character theory, as well as topics like the roots of unity. Tight frames in particular, give a nice generalisation to the notion of an orthonormal basis and all the bells and whistles of being able to encode vectors with an inner product expansion.

Frames have been used since the 1980's in wavelet theory to obtain desirable Fourier expansions and in modern times through the study of convex polytopes[CFK09], physics (quantum mechanics [Eld02]), and engineering (Gabor systems, filter banks and a plethora of other engineering applications). See [KC07] for a nice canvas of some areas of application.

Sometimes redundancy is desirable in an encoding of information. If one was to encode a vector using an orthonormal basis, to achieve redundancy in any of the basis vectors, you would need to encode your vector in two copies of the basis. Certain tight frames have the advantage of allowing for such redundancy across the board, without the need for full duplicates. The extra efficiency achieved is attributed by encoding the information in a uniform way.

A prototypical example is the famous "Mercedes-Benz" frame, consisting of the three equally spaced vectors in $\mathbb{R}^2$. We can also think of them as $\{i, e^{\pi i/6}, e^{-\pi i/12}\}$ in $\mathbb{C}$ obtained by taking the 3–th roots of unity and translating by an angle of $\frac{\pi}{2}$, so that one of the vectors is i. There are no cookies for realising the graphical representation of these three vectors in $\mathbb{C}$. This type of arrangement allows one to equally weight the information encoded in each vector so that we are now able to lose the coefficient in front of any given frame vector and still reconstruct the original vector.

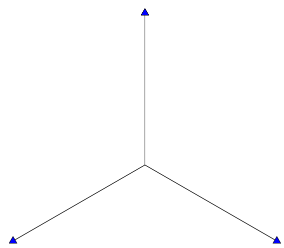

Figure 1.0.1: The Mercedes–Benz tight frame of three equally spaced vectors in $\mathbb{R}^2$.

The process can be generalised to provide as much redundancy as one requires and also into higher dimensions by taking n equally spaced vectors of dimension d. One notices that this symmetric arrangement has close associations with cyclic groups and the n–th roots of unity. Teasing out these connections, the thesis aims to shed light on when two cyclic harmonic frames are unitarily equivalent.

The main points from the rest of the thesis will now be outlined.

Chapter Two

This chapter is based mostly on [Wal10] with some reference to [Chr03]. It is a collection of well known basic results and concepts concerning finite tight frames, available in common frame theory books. We will have our first formal encounter with frames in the general sense early on in Definition 2.6, what it is for a frame to be tight (Definition 2.3) and why they are necessarily a spanning set (Proposition 2.2). In particular, every finite spanning set for a finite dimensional vector space is a frame. Next, the machinery of analysis, synthesis and frame operators are established by definitions 2.5, 2.6. These powerful tools are used to deduce some useful properties about tight frames. For example, Proposition 2.7, which establishes the equivalence between the inner product expansion and tight frames. The important concept of the Gramian (Definition 2.14) is introduced, as is the notion of unitary equivalence between two tight frames (Definition 2.15).

Gramians are shown to play a significant connection with unitary equivalences as two normalised tight frames are unitarily equivalent if and only if their Gramians are equal (Corollary 2.17). Theorem 2.20 then characterises normalised tight frames as orthonormal projections of an orthonormal basis, and that the Gramian can act as the orthonormal projection (Theorem 2.16). The important concept of an *angle multiset* (Definition 2.22) is introduced which will help us distinguish between unitarily inequivalent frames.

Then the connection between group theory and finite frame theory is made clear with G–frames (Definition 2.25) and a way to construct tight frames from irreducible representations of groups (Theorem 2.30). Special attention is directed to a type of frame called a harmonic frame (Definition 2.34), constructed using the character table of an abelian group. *Pontryagin duality* (Theorem 2.33) tells us that to generate harmonic frames, we can alternatively use subsets of the group, rather than subsets of characters. Theorem 2.35 tells us this construction gives us a G–frame with an abelian group G. These cyclic harmonic frames become the centre of interest for the rest of the thesis.

Chapter Three

This chapter is devoted to study of sums of roots of unity, when they vanish, and bases of cyclotomic fields. Vanishing sums are of interest as they allow us to understand the linear relations between roots of unity. These properties will help in chapter four when we try to give a classification for certain types of unitarily equivalent classes. Material from [Rom05], [CJ76] and [LL00] will be drawn upon, with some brief mention of other closely related areas.

Lemma 3.6 will show that any two different sums of roots of unity will give a unique complex number, unless the sums vanish. Theorems 3.12 and 3.15 tell us the minimal vanishing sums look like sums of all p–th roots of unity, for primes p, but note that an explicit classification of sums of roots of unity for given orders n has not been done for very large n.

The set of all primitive n–th roots of unity, with n square free, are shown to form a basis (Theorem 3.18) and a constructive proof will be presented in which the roots of unity are represented as sums of primitive roots of unity. The idea for this proof was hinted in [CJ76]. From this construction, Corollary 3.19 is deduced which allows us to infer that the coefficients in any particular basis representation take the same sign. This will be used in chapter four to prove a characterisation of the unitary equivalences of a particular family of cyclic harmonic frames.

Chapter Four

This chapter will present a majority of the new research on unitary equivalences between harmonic frames, jointly conducted with Shayne Waldron. Notions of unitary equivalence via an automorphism(Definition 4.2) and multiplicatively equivalence(Definition 4.4) are defined for harmonic frames in general. We prove that *unitary equivalence via an automorphism* is equivalent to *multiplicative equivalence* (Theorem 4.13). i.e., multiplicative equivalence implies it is possible to find a permutation inducing a unitary equivalence that respects group structure.

We show that cyclic harmonic frames for $\mathbb{C}^1$ and $\mathbb{C}^2$ are unitarily equivalent if and only if they are multiplicatively equivalent. This is a complete classification of equivalences for these types of frames in terms of multiplicative equivalence (Theorems 4.17, 4.20). For dimensions greater than two, Theorems 4.27, 4.30, 4.34 carve out families of cyclic harmonic frames which are unitarily equivalent but not multiplicatively equivalent. The idea of a σ–invariant number is defined for such families of frames (Definition 4.36) and shown to allow the construction of more families by freely adding or removing these elements (Proposition 4.37). This notion generalises the idea of lifted frames.

Finally, we construct a few cyclic harmonic frame families in dimensions greater than two, where unitary equivalence and multiplicative equivalence are the same. Subsets of $\mathbb{Z}_n^*$ with n square free (Theorem 4.40), and a more abstract class (utilising the situations with unique sums of roots of unity) are shown to exhibit this behaviour (Theorem 4.42).

The thesis is concluded with a few conjectures regarding some other identified families which may also exhibit this behaviour.

Chapter 2

General Theory of Finite Tight Frames

2.1 Introduction

In this chapter we will explore the basic concepts and definitions of finite tight frames necessary to understand the work of chapter four on unitary equivalences. Much can be said about the general theory of finite tight frames. A general exposition of the nature of finite frame theory can be found in [Wal10] and [Chr03]. The proofs of the theorems in this chapter are provided for the sake of completeness. Apart from the goal of setting up machinery for later use, the rest of the chapter is here to give us an appreciation of what it is to be a finite tight frame, their characteristics, and form, in order to solidify understanding.

2.2 Frame Bounds

Definition 2.1. *A countable sequence (f_j) of vectors in a Hilbert space $\mathcal{H}$ is a **frame** if and only if there exists $A, B > 0$ such that for all $v \in \mathcal{H}$,*

$$A\|v\|^2 \leq \sum_j |\langle v, f_j \rangle|^2 \leq B\|v\|^2. \tag{2.2.1}$$

A, B are the frame bounds given in [DS52]. One consequence of this definition is that the multiset of vectors constituting the frame must span the vector space as a set.

Proposition 2.2. *Let (f_j) be a frame, then $\overline{span\{f_j\}} = \mathcal{H}$.*

Proof. Suppose not. Let $V = \overline{\operatorname{span}\{f_j\}}$. Then $\mathcal{H} = V \bigoplus V^{\perp}$ and $V^{\perp} \neq 0$. Take a non trivial $v \in V^{\perp}$. Then,

$$0 < A\|v\|^2 \leq \sum_j |\langle v, f_j \rangle|^2 = 0,$$

a contradiction. Therefore $\overline{\operatorname{span}\{f_j\}} = \mathcal{H}$. □

While the formal definition of frames makes use of the concept of sequences, sometimes the ordering of the vectors involved will not affect whether a multiset constitutes a frame. It is convenient in particular cases to refer to frames as a (multi)set rather than a sequence.

Definition 2.3. *A frame is* ***tight*** *if the two frame bounds are equal. i.e.,*

$$A\|v\|^2 = \sum_{j \in J} |\langle v, f_j \rangle|^2, \quad \forall v \in \mathcal{H}.$$

(f_j) *is a* ***finite*** *frame if J is a finite set.*

A frame is a **normalised tight frame** if $A = 1$. In literature, this is sometimes referred to as a *Parseval frame*. Since the bound A is just a normalising constant, it suffices to study normalised tight frames.

Proposition 2.4. *Any finite spanning multiset is a frame.*

Proof. Take a finite spanning multiset $\{f_1, f_2, \ldots, f_m\}$. The case where $f = 0$ is trivial, so suppose $f \neq 0 \in \mathcal{H}$. The Cauchy-Schwarz inequality implies

$$\sum_{j=1}^{m} |\langle f, f_j \rangle|^2 \leq \sum_{j=1}^{m} \|f_j\|^2 \|f\|^2, \qquad \forall f \neq 0 \in \mathcal{H}.$$

Hence $\sum_{j=1}^{m} \|f_j\|^2$ is an upper frame bound (not necessarily the smallest). The lower frame bound can be defined as

$$A := \inf\{\sum_{j=1}^{m} |\langle f, f_j \rangle|^2 : \|f\| = 1\}.$$

This is well defined as the infimum is attained via a compactness argument. Since for all $f \in \mathcal{H}$,

$$\sum_{j=1}^{m} |\langle f, f_j \rangle|^2 = \sum_{j=1}^{m} |\langle \frac{f}{\|f\|}, f_j \rangle|^2 \|f\|^2 \geq A\|f\|^2,$$

A is suitable as the lower frame bound. It follows that any finite spanning multiset in a finite dimensional vector space will be a frame. □

The same is not necessary true if we take a countably infinite set. For example, the multiset of $\{(1,0),(0,1),(0,1),\ldots\}$ is not a frame for $\mathbb{R}^2$ as no upper frame bound is possible.

2.3 Analysis and Synthesis Operators

For a given finite frame (not necessarily tight), its analysis and synthesis operators give much insight into many properties of the frame, including whether or not it is tight, as well as a direct connection to the key concept of the Gramian matrix of a frame.

Definition 2.5. *For a finite sequence $(f_j)_{j\in J} \in \mathcal{H}$, the **synthesis** or **pre-frame** operator is the linear map*

$$V : \ell_2(J) \to \mathcal{H}, \quad V(a) := \sum_{j\in J} a_j f_j, \quad a := (a_1, a_2, \ldots) \in \ell_2(J).$$

*The dual of this operator, the **analysis** or **frame-transform** operator is defined as*

$$V^* : \mathcal{H} \to \ell_2(J), \quad V^*(f) := (\langle f, f_j\rangle)_{j\in J}.$$

Definition 2.6. *Let*

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf := VV^*f = \sum_{j\in J} \langle f, f_j\rangle f_j, \quad f \in \mathcal{H}.$$

*Then S is called the **frame operator**.*

Note that S is a self-adjoint operator.

Proposition 2.7. *A frame is tight if and only if*

$$f = \frac{1}{A} \sum_{j\in J} \langle f, f_j\rangle f_j, \quad \forall f \in \mathcal{H}. \tag{2.3.1}$$

In particular, a finite frame is a tight frame if and only if $S = A\mathcal{I}$ for some frame bound A.

Proof. Let $(f_j)_{j\in J}$ denote a finite tight frame, $\mathcal{I}$ the identity operator. Consider the analysis operator S of the frame. Observe that by properties of the inner product,

$$A\|f\|^2 = \sum_{j\in J} |\langle f, f_j\rangle|^2 = \langle Sf, f\rangle, \quad \forall f \in \mathcal{H}.$$

Then,

$$\langle (S - A\mathcal{I})f, f\rangle = \langle Sf, f\rangle - A\langle f, f\rangle = A\|f\|^2 - A\|f\|^2 = 0, \quad \forall f \in \mathcal{H}.$$

Since $S - A\mathcal{I}$ is a self-adjoint operator, and $\langle (S - A\mathcal{I})f, f\rangle = 0$, for all $f \in \mathcal{H}$, we have that $S - A\mathcal{I} = 0$. Therefore a tight frame gives rise to the frame expansion in (2.3.1). Conversely, if (2.3.1) is true, then

$$\sum_{j\in J} |\langle f, f_j\rangle|^2 = \langle Sf, f\rangle = \langle Af, f\rangle = A\|f\|^2.$$

□

This type of expansion is sometimes referred to as a *Parseval* type expansion.

Corollary 2.8. *The image of a tight frame under a unitary transformation is still a tight frame.*

Proof. Let V be the pre-frame operator for a tight frame, A the frame bound, and U be an unitary operator.

$$(UV)(UV)^* = UVV^*U^* = UA\mathcal{I}U^* = AU\mathcal{I}U^* = A\mathcal{I}.$$

□

Remark 2.9. *The only normalised tight frames which are a basis are the orthonormal bases. As a basis, they admit a unique expansion. For a normalised tight frame with redundancy, the coefficients used in the frame expansion may not be unique, but if we use the inner product expansion, then the coefficients associated with that are.*

Example 2.10. *The sequence of vectors* $((1, 0), (0, 1), (1, 0), (0, 1))$ *forms a tight frame while* $((1, 0), (1, 0), (0, 1))$ *does not. The frame operators of the two frames are (respectively),*

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Example 2.11. *The classic example of a tight frame is three equally spaced vectors in* $\mathbb{R}^2$ *which give the following frame decomposition:*

$$f = \frac{2}{3}\sum_{j=1}^{3} \langle f, u_j\rangle u_j, \quad \forall f \in \mathbb{R}^2.$$

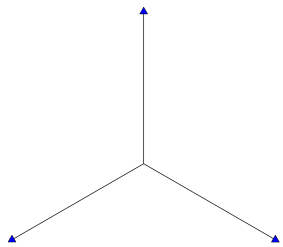

Figure 2.3.1: The Mercedes–Benz tight frame of three equally spaced vectors in $\mathbb{R}^2$.

This idea of using equally spaced vectors was generalised to n vectors at least as early as 1937 in [Sch37], i.e., n equally spaced vectors $u_1, u_2, \ldots, u_n \in \mathbb{R}^2$ give rise to a tight frame with decomposition,

$$f = \frac{2}{n} \sum_{j=1}^{n} \langle f, u_j \rangle u_j, \quad \forall f \in \mathbb{R}^2.$$

Example 2.12. *A continuous analogue of Example 2.11 is possible by taking a continuous set of rotations around the circle given by*

$$f = \frac{1}{\pi} \int_0^{2\pi} \langle f, f_\theta \rangle f_\theta \, d\theta, \qquad f_\theta := \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad \forall f \in \mathbb{R}^2.$$

Proposition 2.13. *If $(f_j)_{j \in J}$ is a finite tight frame and $d := dim(\mathcal{H})$, then*

$$\operatorname{trace}(S) = \sum_{j \in J} \|f_j\|^2 = dA. \tag{2.3.2}$$

Proof. A simple calculation shows that $\operatorname{trace}(S) = \sum_{j \in J} \|f_j\|^2$, but since we also have $S = VV^*$, it is a $d \times d$ matrix. As $(f_j)_{j \in J}$ is also a tight frame, $S = A\mathcal{I}$, so $\operatorname{trace}(S) = dA$. Hence (2.3.2) is satisfied. $\square$

Definition 2.14. *For a finite sequence of vectors $(f_j)_{j \in J} \in \mathcal{H}$, the* ***Gramian*** *or* ***Gram matrix*** *is the $n \times n$ matrix*

$$\operatorname{Gram}((f_j)_{j \in J}) := [\langle f_k, f_j \rangle]_{j,k \in J}.$$

This is just the composition of the analysis and synthesis operators in opposite order to the frame operator, i.e., the Gramian is the matrix representing the operator $V^*V : \ell_2(J) \to \ell_2(J)$ with respect to the standard orthonormal basis $\{e_j\}_{j \in J}$.

It is now appropriate to introduce the notion of equivalence between two tight frames, a fundamental concept explored in this thesis.

Definition 2.15. *Two tight frames $(\phi_j)_{j\in J}$, $(\psi_k)_{k\in K}$ are* ***unitarily equivalent*** *if there exists a bijection $\sigma : J \to K$, a unitary map U and a $c > 0$ such that*

$$\phi_j := cU\psi_{\sigma j}, \qquad \forall j \in J,$$

i.e., there exists a map U that *takes one frame to another* and preserves inner products under some permutation σ.

When we do not wish to allow for a reordering of frame vectors, we set σ to be the trivial permutation. The scaling factor c is set to 1 when dealing with tight frames of equal norm (i.e., $\|f_i\| = \|f_j\|$, $\forall i, j$). Harmonic frames (Definition 2.34) in particular, are equal norm.

Later on, we will study unitary equivalences with emphasis on the permutations involved (often non trivial) and attempt to classify unitary equivalences through studying the associated permutations σ.

2.4 Projections and Tight Frames

Theorem 2.16. *An $n \times n$ matrix $P = [p_{j,k}]_{j,k\in J}$ is the Gramian matrix of a normalised tight frame $(f_j)_{j\in J}$ for the space $\mathcal{H} := \mathrm{span}\{f_j\}_{j\in J}$ if and only if it is an orthogonal projection matrix, i.e., $P = P^* = P^2$. Moreover,*

$$d = \dim(\mathcal{H}) = \mathrm{rank}(P) = \mathrm{trace}(P) = \sum_{j\in J} \|f_j\|^2. \tag{2.4.1}$$

Proof. ($\Rightarrow$) Let $\Phi = (f_j)_{j\in J}$ be a normalised tight frame, and $P = \mathrm{Gram}(\Phi)$. Take $f = f_\ell$ in the Parseval expansion (2.3.1). Then $f_\ell = \sum_{j\in J}\langle f_\ell, f_j\rangle f_j$ and

$$\langle f_k, f_\ell\rangle = \sum_{j\in J}\langle f_\ell, f_j\rangle\langle f_k, f_j\rangle \iff p_{\ell k} = \sum_{j\in J} p_{\ell j}p_{jk} \iff P = P^2.$$

P is Hermitian since $\overline{p_{jk}} = \overline{\langle \Phi_k, \Phi_j\rangle} = \langle \Phi_j, \phi_k\rangle = p_{kj}$ and hence P is a projection.
($\Leftarrow$) Suppose that P is an $n \times n$ orthogonal matrix such that $P = P^* = P^2$. The columns of P are $f_j := Pe_j$, $j \in J$ where $\{e_j\}_{j\in J}$ is the standard orthonormal basis of $\ell_2(J)$. Fix $f \in \mathcal{H} := \mathrm{span}\{f_j\}_{j=1}^n \subset \ell_2(J)$. Then $f = Pf$, so that

$$f = P\left(\sum_{j\in J}\langle Pf, e_j\rangle e_j\right) = \sum_{j\in J}\langle f, Pe_j\rangle Pe_j = \sum_{j\in J}\langle f, f_j\rangle f_j,$$

i.e., $(f_j)_{j=1}^n$ is a normalised tight frame for $\mathcal{H}$. Taking the trace of P gives (2.4.1) by (2.3.2). $\square$

Corollary 2.17. *Normalised tight frames are unitarily equivalent if and only if their Gramians are equal.*

Proof. Let $\Phi = (f_j)_{j\in J}$, $\psi = (g_j)_{j\in J}$ be normalised tight frames for $\mathcal{H}$ and $\mathcal{K}$ respectively.
($\Rightarrow$) If Φ and Ψ are unitarily equivalent, i.e., There exists a U unitary, and a $c > 0$ such that $g_j = cUf_j$, $\forall j \in J$, then

$$\langle g_j, g_k\rangle = \langle cUf_j, cUf_k\rangle = c^2\langle f_j, f_k\rangle,$$

but by (2.4.1) we have $c = 1$, hence the Gramians are equal.
($\Leftarrow$) Suppose the Gramians of Φ and Ψ are equal, i.e., $\langle g_j, g_k\rangle = \langle f_j, f_k\rangle$, for all $j, k \in J$. Since Theorem 2.16 tells us the Gramian is an orthogonal projection matrix in this case, we can suppose without loss of generality that $f_1, f_2, \dots, f_d$ and $g_1, g_2, \dots, g_d$ form a basis. Let U be the mapping $Uf_j = g_j$, for $j \in \{1, 2, \dots, d\}$. As the Gramians are equal, this implies U is unitary. Now we check that that U maps f_s to g_s for $d < s \leq k$. Let $d < s \leq k$ and $1 \leq j \leq d$. Then

$$\begin{aligned}\langle Uf_s - g_s, g_j\rangle &= \langle Uf_s, g_j\rangle - \langle g_s, g_j\rangle = \langle Uf_s, Uf_j\rangle - \langle g_s, g_j\rangle \\ &= \langle f_s, f_j\rangle - \langle g_s, g_j\rangle = 0, \quad \forall 1 \leq j \leq d,\end{aligned}$$

as required. Therefore U is our desired unitary map. □

In light of this corollary, when reordering is allowed, Definition 2.15 can be reformulated as follows: two tight frames Φ, Ψ, indexed by J are **unitarily equivalent** if and only if there exists a $c > 0$, and a permutation σ on J, such that

$$\mathrm{Gram}(\Phi) = c^2 P_\sigma^* \mathrm{Gram}(\Psi) P_\sigma, \tag{2.4.2}$$

where P_σ is the permutation matrix induced by σ.

Example 2.18. *Let $\omega = e^{2\pi i/3}$. Then*

$$\Phi := \left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}\omega\\\omega^2\end{bmatrix}, \begin{bmatrix}\omega^2\\\omega\end{bmatrix}\right), \Psi := \left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\\omega\end{bmatrix}, \begin{bmatrix}1\\\omega^2\end{bmatrix}\right).$$

are equal norm tight frames for $\mathbb{C}^2$ which are not unitarily equivalent since

$$\mathrm{Gram}(\Phi) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \qquad \mathrm{Gram}(\Psi) = \begin{pmatrix} 2 & 1+\omega^2 & 1+\omega \\ 1+\omega & 2 & 1+\omega^2 \\ 1+\omega^2 & 1+\omega & 2 \end{pmatrix}$$

are different.

Later on, we will present a complete characterisation of the unitary equivalence classes of a family of equal tight frames called cyclic harmonic frames(c.f., Definition 2.34) in $\mathbb{C}^2$.

Lemma 2.19. *The orthogonal projection of a normalised tight frame is a normalised tight frame.*

Proof. Let $(f_j)_{j\in J}$ be a normalised tight frame for $\mathcal{H}$ and P be an orthogonal projection of $\mathcal{H}$. Since

$$\|Pv\|^2 = \sum_{j\in J} |\langle Pv, f_j\rangle|^2 = \sum_{j\in J} |\langle P^2 v, f_j\rangle|^2 = \sum_{j\in J} |\langle Pv, Pf_j\rangle|^2, \qquad \forall v \in P(\mathcal{H}),$$

it follows that $(Pf_j)_{j\in J}$ is a normalised tight frame for $P(\mathcal{H}) \subset \mathcal{H}$. □

In particular, this means that the orthogonal projection of an orthonormal basis is a normalised tight frame. The converse is captured in the following theorem.

Theorem 2.20. *Every finite normalised tight frame is the orthogonal projection of an orthonormal basis.*

Proof. Let $\Phi = (f_j)_{j\in J}$ be a finite normalised tight frame for $\mathcal{H}$, $P = \text{Gram}(\Phi)$, $(e_j)_{j\in J}$ be the standard orthonormal basis for $\ell_2(J)$. By Theorem 2.16, P is an orthogonal projection matrix. By Lemma 2.19, $(Pf_j)_{j\in J}$ is a normalised tight frame. As

$$\langle Pe_j, Pe_k\rangle = \langle Pe_j, e_k\rangle = \big((k,j) \text{ entry of } P\big) = \langle f_j, f_k\rangle,$$

it follows that $(Pe_j)_{j\in J}$ is unitarily equivalent to Φ (see Corollary 2.17) □

This theorem is a special case of Nǎimark's theorem[AG63].

Example 2.21. *A projection of the orthonormal basis of $\mathbb{R}^3$ onto $\mathbb{R}^2$ is a tight frame for $\mathbb{R}^2$.*

Definition 2.22. *An **angle multiset** of a frame $\Phi = (\phi_j)_{j\in J}$ is*

$$Ang(\Phi) := \{\langle \phi_j, \phi_1\rangle : j \in J, \quad j \neq 1\}.$$

*Members of this multiset will be referred to as **angles**.*

Recall that multisets are sets where we allow for multiplicity of elements. For example, $\{0, 0\}$ is a multiset.

Remark 2.23. *In order for two tight frames to be unitarily equivalent, it is necessary (but not sufficient) that the two frames share the same angle multisets since they share the same Gramian matrix of inner products. This provides another way to differentiate the two frames in Example 2.18.*

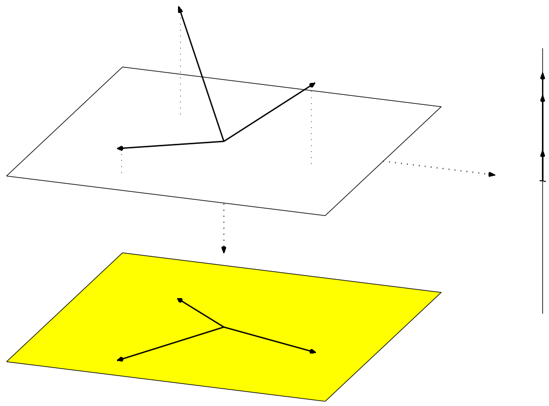

Figure 2.4.1: Graphical sketch of a tight frame in $\mathbb{R}^3$ being projected onto tight frames for $\mathbb{R}^2$ or for $\mathbb{R}$.

2.5 G–Frames

One might wonder how frames might be generated. Various linkages exist in other branches of mathematics enabling the construction of tight frames. Clear correspondences are present between graphs and tight frames with various constructions from Seidel matrices [Wal09], to Paley tournaments [Ren07]. Combinatorial constructions involving Hadamard matrices [PW02], and polygons may also be used [BF03]. See [Wal10] for an overview of various techniques.

Groups also give rise to strong relationship with frames. The idea of using groups to construct frames has been used for a long time (at least since the 1940s [BC79]) and allow us to hook in to the wonderful machinery of modern algebra. Tight frames are often highly symmetric, so it makes sense to utilise groups in some way. While most of the focus has been on abelian groups and cyclic groups in particular, some work has been done on the Heisenberg group [CV98] and other non abelian varieties [VW08].

Group frames arise by letting a group act on a non trivial vector and taking the orbit under this action to be your frame.

To begin with, we first develop the theory of G–frames in general then we will derive some nice results showing the relationship to tight frames.

Definition 2.24. *A **representation** of a finite group G is a finite dimensional Hilbert space $\mathcal{H}$, together with a group homomorphism $\rho : G \to U(\mathcal{H})$. The group action will be denoted*

$$gv := \rho(g)(v), \qquad g \in G,\ v \in \mathcal{H}.$$

Definition 2.25. *Let G be a finite group. We say that a frame $(\phi_g)_{g\in G}$ for $\mathcal{H}$ is a G–**frame** if there exists a representation $\rho = \rho_\phi : G \to U(\mathcal{H})$ such that*

$$g\phi_h := \rho(g)\phi_h = \phi_{gh}.$$

*A subspace $V \subset \mathcal{H}$ is G-**invariant** if $gv \in V$, $\forall v \in V$.*

This definition has the advantage that a G-frame is automatically an equal-norm frame.

Proposition 2.26. *The frame operator S of a G-frame commutes with G, i.e.,*

$$S(hf) = hS(f), \qquad h \in G, \forall f \in \mathcal{H}.$$

Proof. Let $\Phi = (\phi_g)_{g\in G}$ be a G-frame for $\mathcal{H}$, with frame operator $S = S_\Phi$. As $\phi(h)^* = \phi(h)^{-1} = \rho(h^{-1})$,

$$\begin{aligned} S(hf) &= \sum_{g\in G}\langle hf, \phi_g\rangle\phi_g = h\sum_{g\in G}\langle f, h^{-1}\phi_g\rangle h^{-1}\phi_g = h\sum_{g\in G}\langle f, \phi_{h^{-1}g}\rangle\phi_{h^{-1}g} \\ &= hS(f). \end{aligned}$$

□

Remark 2.27. *The Gramian of G-frames have the form*

$$\langle\phi_g, \phi_h\rangle = \langle g\phi_1, h\phi_1\rangle = \langle h^{-1}g\phi_1, \phi_1\rangle,$$

where 1 refers to the identity element of the group.

Definition 2.28. *A unitary action (representation) of a group G on $\mathcal{H}$ is **irreducible** if the only G-invariant subspaces of $\mathcal{H}$ are $\{0\}$ and $\mathcal{H}$. i.e.,*

$$\operatorname{span}\{gv\}_{g\in G} = \mathcal{H}, \qquad \forall v \neq 0,\ v \in \mathcal{H}.$$

Lemma 2.29. *S is a positive operator.*

Proof. The frame bound condition (2.1) can be rewritten as

$$\langle Af, f\rangle \leq \langle Sf, f\rangle \leq \langle Bf, f\rangle, \qquad \forall f \in \mathcal{H},$$

i.e.,

$$A\mathcal{I} \leq S \leq B\mathcal{I}.$$

Hence S is positive. □

Theorem 2.30. *If the unitary action of G on $\mathcal{H}$ is irreducible, then $(gv)_{g\in G}$ is a tight G-frame for $\mathcal{H}$ for any $v \neq 0$.*

Proof. Let $v \neq 0$ and S denote the frame operator of $(gv)_{g\in G}$. Since S is positive, we can take some eigenvalue $\lambda > 0$ with eigenvector w. S commutes with G by Proposition 2.26, so for any $g \in G$, gw is also an eigenvector for λ by the calculation

$$S(gw) = gS(w) = g(\lambda w) = \lambda(gw).$$

As $\{gw\}_{g\in G}$ spans $\mathcal{H}$ implies $S = \lambda\mathcal{I}$, by Proposition 2.7, $(gv)_{g\in G}$ is a tight frame. □

Thus, G-frames allow us to construct finite tight frames through their irreducible representations.

Example 2.31. *Let $\mathbb{Z}_n = \langle a\rangle$ be the cyclic group of order n. Then the irreducible representation*

$$\rho_{\mathbb{R}^2}(a) := \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}$$

induces the sequence $(gv)_{g\in\mathbb{Z}_n}$ of n equally spaced vectors for any $v \in \mathbb{R}^2$ of unit norm. This forms a tight frame.

2.6 Harmonic Frames

Definition 2.32. *The* ***characters*** *of a finite abelian group are maps*

$$\chi : G \to \mathbb{C} \setminus \{0\}, \qquad \chi(g+h) = \chi(g)\chi(h), \quad \chi(1) = 1, \quad \forall g, h \in G.$$

It is well known that characters have a one to one correspondence with the irreducible representations of the group (up to representation equivalence). For abelian groups the value characters evaluated at each group element is just an n–th root of unity, where n is the order of the group. Under pointwise multiplication, characters form a group $\hat{G}$. The group of characters $\hat{G}$ is known to be isomorphic to the original group G.

Each character is also orthogonal to each other, i.e.,

$$\langle \xi, \eta\rangle = \sum_{g\in G} \xi(g)\overline{\eta(g)} = \begin{cases} 0, & \xi \neq \eta \\ n, & \xi = \eta \end{cases}$$

Theorem 2.33 (Pontryagin Duality)**.** *The dual of $\hat{G}$ is isomorphic to G under a canonical map, i.e., $\hat{\hat{G}} \cong G$.*

The **character table** of an abelian group G of order n is an $n \times n$ matrix with rows corresponding to characters and the columns are indexed by group elements. Since the characters are orthogonal, the character table is a scalar multiple of an orthogonal matrix. For cyclic groups of order n, the character table looks like

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}.$$

If we scale that matrix by $1/\sqrt{n}$ then it is the matrix corresponding to the discrete Fourier transform (**Fourier Matrix**).

In light of this, we can make the following definition:

Definition 2.34. *Let G be a finite abelian group of order n, with characters $(\xi_j)_{j=1}^n$ and $\hat{J} \subset \hat{G}$. Then any tight frame which is unitarily equivalent to the equal-norm tight frame given by*

$$\Phi = (\phi_g)_{g \in G}, \qquad \phi_g := (\xi_j(g))_{j \in \hat{J}} \in \mathbb{C}^{\hat{J}}$$

*is called a **harmonic frame**. If G is taken to be a cyclic group, then we call the corresponding frame a **cyclic harmonic frame**.*

This definition essentially says that if we take the character table of a finite abelian group, take subsets of the rows of the character table (i.e., subsets of characters), then the n column vectors of dimension d formed by looking at the columns of the subsequent submatrix form a tight frame called a harmonic frame.

There are two convenient ways to see why this definition is well defined. Firstly we can show by direct computation that the frame operator of the column vectors is equal to $n\mathcal{I}$ using the fact that the characters are orthogonal. Secondly, since the character table matrix is essentially an orthogonal matrix (up to a scaling by $1/\sqrt{n}$), by the theorem of Naimark (Theorem 2.20), we can use the columns of the character table (which form an orthonormal basis), and project down onto the d dimensional subspace of $\mathbb{C}^n$ formed by considering only the selected rows (zeroing out the components in other rows). We can then identify this subspace with $\mathbb{C}^d$.

Harmonic frames are also G-frames since

$$\phi_g = \rho(g)v_1, \qquad \rho(g) = \mathrm{diag}(\xi(g))_{\xi \in \hat{J}}, \qquad v_1 = (\xi(1))_{\xi \in \hat{J}},$$

implies

$$g\phi_h = \rho(g)\phi_h = \rho(g)\rho(h)v_1 = \rho(gh)v_1 = \phi_{gh}.$$

Theorem 2.35. *Let Φ be a finite equal-norm tight frame for $\mathcal{H}$. Then the following are equivalent:*

(a) Φ is a G-frame, where G is an abelian group.

(b) Φ is a harmonic frame (obtained from the character table of G).

Proof. See [VW05] for a proof. □

The groups G, H can be taken to be the same group, but this need not be the case.

Example 2.36. *The only harmonic frame arising from $\mathbb{Z}_2 \times \mathbb{Z}_2$ is unitarily equivalent to a cyclic harmonic frame of $\mathbb{Z}_4$. We will revisit this example in more detail later on in the context of unitary equivalences.*

Example 2.37.

$$\Phi = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \omega \\ \overline{\omega} \end{bmatrix}, \begin{bmatrix} \omega^2 \\ \overline{\omega}^2 \end{bmatrix}, \begin{bmatrix} \omega^3 \\ \overline{\omega}^3 \end{bmatrix}, \dots, \begin{bmatrix} \omega^{n-1} \\ \overline{\omega}^{n-1} \end{bmatrix} \right)$$

formed by taking the second and last rows of the character table of a cyclic group of order n, is a cyclic harmonic frame in $\mathbb{C}^2$. It is unitarily equivalent to n equally spaced vectors of unit length in $\mathbb{C}^2$ via the map

$$U := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix},$$

i.e.,

$$\frac{1}{\sqrt{2}} U \begin{bmatrix} \omega^j \\ \overline{\omega}^j \end{bmatrix} = \begin{bmatrix} \cos(2\pi j/n) \\ \sin(2\pi j/n) \end{bmatrix}, \qquad \forall j \in \{0, 1, 2, \dots, n-1\}.$$

Example 2.38. *The C_8 cyclic harmonic frames*

$$\Phi = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \omega \\ \omega^3 \end{bmatrix}, \begin{bmatrix} \omega^2 \\ \omega^6 \end{bmatrix}, \begin{bmatrix} \omega^3 \\ \omega^9 \end{bmatrix}, \begin{bmatrix} \omega^4 \\ \omega^4 \end{bmatrix}, \begin{bmatrix} \omega^5 \\ \omega^7 \end{bmatrix}, \begin{bmatrix} \omega^6 \\ \omega^2 \end{bmatrix}, \begin{bmatrix} \omega^7 \\ \omega^5 \end{bmatrix} \right)$$

and

$$\Phi = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \omega \\ \omega^5 \end{bmatrix}, \begin{bmatrix} \omega^2 \\ \omega^2 \end{bmatrix}, \begin{bmatrix} \omega^3 \\ \omega^7 \end{bmatrix}, \begin{bmatrix} \omega^4 \\ \omega^4 \end{bmatrix}, \begin{bmatrix} \omega^5 \\ \omega \end{bmatrix}, \begin{bmatrix} \omega^6 \\ \omega^6 \end{bmatrix}, \begin{bmatrix} \omega^7 \\ \omega^3 \end{bmatrix} \right)$$

formed by taking the first and third, first and fifth rows of the character table are not unitarily equivalent as their angle multisets differ.

Since the rows of the character matrix are orthogonal, so are its columns. We could use the construction with columns instead. Therefore we would be considering $J \subset G$ instead of $\hat{J} \subset \hat{G}$. This would still be a G-frame since $\hat{G}$ is isomorphic to G and if we define the frame to be $(w_\xi)_{\xi \in \hat{G}}$, with $w_\xi := \xi|_J$ and

$$w_\xi = \rho(\xi) w_1, \qquad \rho(\xi) := \mathrm{diag}(\xi|_J), \qquad w_1 := 1|_J,$$

we have $(w_\xi)_{\xi \in \hat{G}}$ satisfying the conditions for being a G-frame.

By *Pontryagin duality map (canonical group isomorphism)* (see Theorem 2.33),

$$G \to \hat{\hat{G}} : g \to \hat{\hat{g}}, \qquad \hat{\hat{g}}(\chi) := \chi(g), \qquad \forall \chi \in \hat{G},\, g \in G.$$

We may refer to

$$v_g = (\xi(g))_{\xi \in \hat{J}} = (\hat{\hat{g}}(\xi))_{\xi \in \hat{J}} = \hat{\hat{g}}|_{\hat{J}},$$

or $\Phi = (\xi_J)_{\xi \in \hat{G}}$ for convenience. This allows us to index our finite tight frames using subsets of the original group rather than characters when convenient.

Proposition 2.39. *Given a cyclic harmonic frame (v_j),*

$$\langle v_{j+a}, v_{k+a} \rangle = \langle v_j, v_k \rangle, \qquad \forall a \in \mathbb{Z}_n,$$

i.e., The Gramian is preserved by a constant translation of all frame vectors in the sequence.

Proof. Let $v_j = \omega^{a_1 j} + \ldots + \omega^{a_d j}$. Then,

$$\begin{aligned}
\langle v_{j+a}, v_{k+a} \rangle &= \omega^{a_1(j+a)} \overline{\omega^{a_1(k+a)}} + \ldots + \omega^{a_d(j+a)} \overline{\omega^{a_n(k+a)}} \\
&= \omega^{a_1(j+a) - a_1(k+a)} + \ldots + \omega^{a_n(j+a) - a_n(k+a)} \\
&= \omega^{a_1(j-k)} + \ldots + \omega^{a_n(j-k)} \\
&= \langle v_j, v_k \rangle.
\end{aligned}$$

□

Corollary 2.40. *If two cyclic harmonic frames $(v_j)_{j \in \mathbb{Z}_n}$ and $(w_j)_{j \in \mathbb{Z}_n}$ are unitarily equivalent, then for any $s \in \mathbb{Z}_n$, there exists a unitary map U that maps v_s to w_k for any $k \in \mathbb{Z}_n$.*

Proof. Let $(v_j)_{j \in \mathbb{Z}_n}$ and $(w_j)_{j \in \mathbb{Z}_n}$ be unitarily equivalent frames under the unitary map W. Suppose $W v_s = w_\ell = (\omega^{b_1}, \ldots, \omega^{b_d})^{\mathrm{T}}$ for some $\ell \in \mathbb{Z}_n$, and $w_k = (\omega^{a_1}, \ldots, \omega^{a_d})^{\mathrm{T}}$. Define $a = k - \ell$ and

$$P = \begin{pmatrix} \omega^{(-b_1+a_1)} & 0 & \ldots & 0 \\ 0 & \omega^{(-b_2+a_2)} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \omega^{(-b_d+a_d)} \end{pmatrix}$$

be the unitary matrix corresponding to the translation by a (as in Proposition 2.39). Since the composition of unitary maps is a unitary map, $U = PW$ is a unitary map such that

$$Uv_s = PWv_s = Pw_\ell = w_{\ell+a} = w_{\ell+k-\ell} = w_k.$$

$\square$

Remark 2.41. *Corollary 2.40 also follows by observing that the vectors in the two frames all have the same norm.*

Corollary 2.42. *Given two cyclic harmonic frames $\Phi := (\phi_g)_{g\in G}$, $\Psi := (\psi_g)_{g\in G}$ generated by the same finite group G of order n. If Φ and Ψ are unitarily equivalent, then there exists at least n unitary maps taking Φ to Ψ.*

Proof. By the previous proposition, we can assume there exists a unitary map U which maps ϕ_1 to ψ_i, for all $i \in G$. $\square$

Chapter 3

Sums of Roots of Unity

3.1 Introduction

In chapter two, we were introduced to characters of an abelian group, and saw that characters take on values which are just roots of unity. As unitary equivalence implies frames share the same angle multiset, one approach to studying unitary equivalences is to study the members of this multiset, the inner products between the frame vectors. Because the coefficients for each harmonic frame vector is just a root of unity, studying the inner products between vectors of a harmonic frame becomes a study of the sums of roots of unity. These sums give rise to some highly symmetric patterns when the resulting sums are plotted on the complex plane.

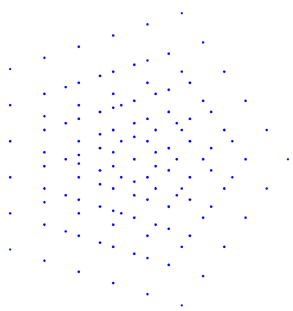

Figure 3.1.1: All possible sums of five 5–th roots of unity.

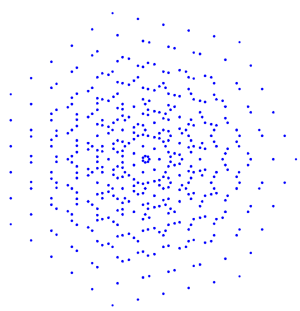

Figure 3.1.2: All possible sums of five 7–th roots of unity.

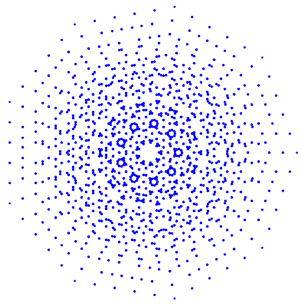

Figure 3.1.3: All possible sums of five 9–th roots of unity.

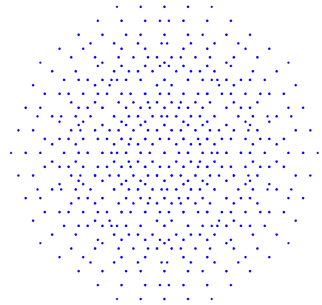

Figure 3.1.4: All possible sums of four 10–th roots of unity.

Let $\omega = e^{2\pi i/n}$ be an n–th root of unity. It is a well known fact that

$$\sum_{i=1}^{n} \omega^i = 0. \tag{3.1.1}$$

Definition 3.1. *A cyclotomic field is the smallest extension field of $\mathbb{Q}$ formed by adding an n–th root of unity. We denote this by $\mathbb{Q}[\omega]$ where $\omega := e^{2\pi i/n}$ is a root of unity.*

Ideally we want to find a nice unique way to represent the roots of unity in order to study when sums are the same. The natural approach involves finding a basis for the cyclotomic field associated with the n–th root of unity and to represent the roots in terms of this basis. One would hope that all the n–th roots of unity would form a basis. Unfortunately this is not the case.

Theorem 3.2. *The cyclotomic field $\mathbb{Q}[\omega]$, where $\omega = e^{2\pi i/n}$, has dimension $\varphi(n)$, where φ is the Euler totient function.*

The above theorem makes us hopeful that perhaps the primitive roots of unity, of which there are $\varphi(n)$, form a basis. This would be a convenient basis to use as the primitive roots of unity form a group under multiplication. Unfortunately this is also not the case.

Definition 3.3. *An integer n is called square free if it is a product of distinct primes, i.e., if $p \mid n$ then $p^2 \nmid n$.*

Definition 3.4. *The Möbius Function is $\mu : \mathbb{Z}^+ \mapsto \mathbb{Z}$*

$$\mu(n) := \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ square free} \\ 0 & \text{if } n \text{ is not square free} \end{cases}$$

where k is the number of prime factors of n counting multiplicities.

Theorem 3.5. *The sum of all primitive n–th roots of unity is given by the Möbius function at n. i.e.,*

$$\sum_{m \in \mathbb{Z}_n^*} \omega^m = \mu(n).$$

One immediate consequence of this theorem is that for n not square free, the primitive roots of unity for the cyclotomic field $\mathbb{Q}[\omega]$ with $\omega := e^{2\pi i/n}$ are linearly dependent and do not form a basis for $\mathbb{Q}[\omega]$ over $\mathbb{Q}$. The good news is it leaves open the possibility that for n square free, the primitive roots are a basis. Luckily this is indeed the case and we will prove this result later in Theorem 3.18.

Lemma 3.6. *Let $R > 0$, $z \in \mathbb{C}$, and $0 < |z| \leq 2R$. Then z can be uniquely written as a sum of two elements x, y such that $|x| = |y| = R$ (up to reordering). If $z = 0$ then there are an infinite number of distinct ways to represent z.*

Proof. Suppose $R > 0$ and $z \in \mathbb{C}$ such that $0 < |z| \leq 2R$. Let $C_1 = \{y \in \mathbb{C} : |y| = R\}$, $C_2 = \{y \in \mathbb{C} : |y - z| = R\}$ be two circles representing possible destination points when adding a complex number of modulus R, at the given centres. Notice that unless they are equal, C_1 and C_2 intersect at least in one place, and at most two, i.e., $|C_1 \cap C_2| \leq 2$. The intersection captures the geometric interpretation of a valid combination of complex numbers of modulus R that gives z. If $x \in C_1 \cap C_2$, then $z - x \in C_1 \cap C_2$ so $z = x + (z - x)$ is the desired sum. For $z = 0$ it is then clear that since the two circles constructed above overlap everywhere there are an infinite number of ways to construct a sum as two complex numbers of length R. The above idea is captured concisely using trig identities.

$$\begin{aligned}
z &= e^{i\theta_1} + e^{i\theta_2} \\
&= \cos(\theta_1) + \cos(\theta_2) + i(\sin(\theta_1) + \sin(\theta_2)) \\
&= 2\cos(\frac{\theta_1 + \theta_2}{2})\cos(\frac{\theta_1 - \theta_2}{2}) + i(2\sin(\frac{\theta_1 + \theta_2}{2})\cos(\frac{\theta_1 - \theta_2}{2})) \\
&= 2\cos(\frac{\theta_1 - \theta_2}{2})e^{i(\theta_1 + \theta_2)}.
\end{aligned}$$

Except for when $\theta_1 + \theta_2 = \{-\pi, \pi\}$ we can see from observing the properties of cosine and the θ_i's, that a unique complex number is produced up to reordering. $\square$

Corollary 3.7. *Let $\omega = e^{2\pi i/n}$, and $i, j \in \mathbb{Z}_n$. Then $\omega^i + \omega^j$ is a unique sum unless n is even and $j = \frac{n}{2} + i$.*

Proof. Let $R = 1$. Then $\forall i \in \mathbb{Z}_n$, $\omega^i \in C := \{x : |x| = R\}$ and so by the lemma $\omega^i + \omega^j$ is unique unless $\omega^j = -\omega^i$. Note that $-\omega^i = \omega^{\frac{n}{2}+i}$ is only possible if n is even. □

Remark 3.8. *It is only possible for two roots of unity to sum to* 0 *when n is even.*

One would hope this uniqueness of sums of roots of unity generalises over larger sums. Unfortunately this need not be the case. Even when we move to just sums of three roots of unity, things start to break down.

Example 3.9. *Let $\omega = e^{2\pi i/8}$. Then $\omega + \omega^5 + \omega^2 = \omega^3 + \omega^7 + \omega^2 = \omega^2 \neq 0$.*

In the example above, the complex number produced by the first sum on the left hand side can be produced by a different sum of roots of unity on the right. What we observe in the two dimensional case no longer holds.

Example 3.10. *Let $\omega = e^{2\pi i/9}$. Then $\omega^2 + \omega^5 + \omega^8 = \omega^4 + \omega + \omega^7 = 0$.*

We will see later that the (partial) uniqueness of sums of two roots of unity provide a pivotal role in the characterisation of unitary equivalences of cyclic harmonic frames in two dimensions.

3.2 Vanishing Sums and Linear Relations of Roots of Unity

Like (3.1.1), other so called *vanishing sums* (sums of roots of unity which give 0) give rise to relationships between the various roots of unity, especially minimal vanishing sums (where no sum of a smaller subset vanishes). One example of a related area is counting the number of vanishing sums (see [LL00],[Eve99]). A sweep through the literature surrounding the vanishing sums of the roots of unity seems to suggest it is still an active area of research. Despite the fundamental nature of this area and its extensive study, interestingly enough, an explicit classification of the sums of roots of unity is not complete. A classification up to $n \leq 12$ seems to be known from work by Mann[Man65], Conway-Jones[CJ76], and Poonen-Rubinstein[PR98] in connection with their studies of regular polygons. Notwithstanding this, significant insight into the structure of vanishing sums is known.

Definition 3.11. *Let $G = \langle a \rangle$ be a cyclic group of order n and ζ be a fixed primitive n-th root of unity. Define $\Omega : \mathbb{Z}G \to \mathbb{Z}[\zeta]$ such that $\Omega(z) = \zeta$ as the "**usual map**" from the **group ring** $\mathbb{Z}G$ into $\mathbb{Z}[\zeta]$ the ring of **cyclotomic integers**.*

Theorem 3.12 (Lam-Leung)**.** [1] *Let Ω be the usual map, $G = \langle a \rangle$ of order $n = p_1^{a_1} \dots p_r^{a_r}$, $\zeta = \zeta_n$ a primitive n-th root of unity, $P_i (1 \leq i \leq r)$ be the unique subgroup of order p_i in G, and consider $\sigma(P_i) := \sum_{g \in P_i} g \in \mathbb{Z}G$. Then*

$$ker(\Omega) = \sum_{i=1}^{r} \mathbb{Z}G \cdot \sigma(P_i), \quad \text{and} \quad ker(\Omega) = \mathbb{Z} \cdot \sigma(P_1) \quad \text{when } n = p_1.$$

This theorem sets up a map from the ring of the formal sums of roots of unity into the ring of cyclotomic integers, then gives a characterisation of the kernel, which corresponds to when the sums vanish. The subgroups can be thought of like sets of the p_i roots of unity.

Theorem 3.13. *Let a_i, $i \in I$, $I \subset \{1, 2, \dots, n\}$ be n-th roots of unity. If $a_1 + a_2 + \ldots + a_n = 0$ is a minimal vanishing sum of n-th roots of unity, then after a suitable rotation, we may assume that all a_i are m roots of unity where m is a product of distinct primes.*

Definition 3.14. *A sum of n-th roots of unity S is **similar** to another sum S' if $S' = c\alpha S$ for some $c \in \mathbb{Q} \setminus \{0\}$, α any n-th root of unity.*

The above theorems can be restated in the language of similarity if required.

Theorem 3.15. [2] *Let p be prime and ω a primitive p-th root of unity. If S is a vanishing sum, then either S is similar to $1 + \omega + \omega^2 + \ldots + \omega^{p-1}$, or $S = S' + S''$, where S' and S'' are vanishing sums such that:*

(i) The number of roots involved in S' is less than or equal *to number of roots involved in S,*

(ii) The least common order of the roots of unity of S' (and similar sums) is strictly less than *that of S,*

(iii) The number of roots involved in S'' is less than or equal *to number of roots involved in S,*

[1] This is the theorem of Rédei - de Bruijn - Schoenberg recast in terms of group rings by Lam-Leung in [LL00].

[2] Rediscovered by Conway–Jones in [CJ76].

(iv) *The least common order of the roots of unity of S'' (and similar sums) is* less than or equal *to that of S,*

Corollary 3.16. *Let $n = p^a q^b$, where p, q are primes. Then up to a rotation, the only minimal vanishing sums of n-th roots of unity are $1 + \zeta_p + \zeta_p^2 + \ldots + \zeta_p^{p-1}$ and $1 + \zeta_q + \zeta_q^2 + \ldots + \zeta_q^{q-1}$.*

Theorems 3.13 and 3.15 are telling us the vanishing sums look like (3.1.1) with n prime (up to rotation). This is essentially Theorem 3.12. In light of this, Lemma 3.6 is really just a corollary of Theorem 3.15 since the only vanishing sums are necessarily rotations of the sums of 2-th roots of unity, allowing us to deduce uniqueness of other sums.

3.3 Bases for the Cyclotomic Fields

We are interested in the finding suitable bases for cyclotomic fields $\mathbb{Q}[\omega]$ to representing our roots of unity, and hope that their basis representations may yield some useful structure in order to study the frame angle multisets.

Theorem 3.17. *The cyclotomic extension of $\mathbb{Q}[\omega]$ over $\mathbb{Q}$ has degree $\varphi(n)$, i.e., there are $\varphi(n)$ elements in a basis of $\mathbb{Q}[\omega]$.*

Theorem 3.18. *The primitive n–roots of unity form a basis for the cyclotomic extension of $\mathbb{Q}[\omega]$ over $\mathbb{Q}$ if and only if n is a product of distinct primes. Hence if n is a product of distinct primes, then any sums of primitive roots of unity of n are unique.*

Proof. We have already seen that if n is not square free, the primitive roots of unity are not linearly independent. Now we will prove that if n is square free, then the primitive roots do form a basis. A common approach is to make use of machinery in field theory to deduce this as a consequence (see [Rom05]), but we will instead give a constructive proof, since by studying the construction we are able to deduce some further properties of what the basis representation looks like.

Let $n = p_1 p_2 \ldots p_k$. Observe that all the p_j-th roots of unity are embedded in the n–th roots of unity as $e^{2\pi i n j/p_j}$ for all $1 \le j \le p_j$. Let ω^j denote $e^{2\pi i j/n}$, a n–th root of unity. Our aim is to represent ω^j as a sum of primitive roots of unity. If $\gcd(j, n) = 1$ then ω^j is a primitive root of unity so there is nothing to prove. Without loss of generality we can assume $\gcd(j, n) = c = p_1 \ldots p_s > 1$, where $1 \le s \le k$, and that $p_1 < p_2 < \ldots < p_s$. Let $j = cm$, $\gcd(m, n) = 1$. Define $\zeta_t := \omega^{n/p_t}$, i.e., a p_t-th root of unity. We will now describe an iterative method which represents ω^j as a sum of roots of unity which first have $p_2 \ldots p_s$ as factors (but none of p_α, $s < \alpha \leq k$ in the exponents of ω. On each root of unity in that sum, we wish to apply the procedure again. On each new iteration, the next smallest factor in $\{p_1, \ldots, p_s\}$, where $1 \leq s \leq k$, is eliminated from the roots of unity in the new sum, and no new factor of n is introduced. After s iterations, we will necessarily obtain a sum of the original ω^j in terms of primitive roots of unity.

We make repeated use of the fact that by equation (3.1.1),

$$\omega^\gamma \sum_{b=0}^{p_\ell - 1} \zeta_\ell^b = \omega^\gamma + \omega^\gamma \zeta_\ell + \omega^\gamma \zeta_\ell^2 + \ldots + \omega^\gamma \zeta_\ell^{p_\ell - 1} = 0,$$

hence

$$\begin{aligned} \omega^\gamma &= -\omega^\gamma \zeta_\ell - \omega^\gamma \zeta_\ell^2 - \ldots - \omega^\gamma \zeta_\ell^{p_\ell - 1} \\ &= -\omega^\gamma \omega^{n/\ell} - \omega^\gamma \omega^{2n/p_\ell} - \ldots - \omega^\gamma \omega^{(p_\ell - 1)n/p_\ell}. \end{aligned}$$

The first step in this procedure is as follows: Let $\omega^\gamma = \omega^j$ and $p_\ell = p_1$. Then

$$\begin{aligned} \omega^j &= -\omega^j \zeta_1 - \omega^j \zeta_1^2 - \ldots - \omega^j \zeta_1^{p_j - 1} \\ &= -\omega^j \omega^{n/j} - \omega^j \omega^{2n/p_1} - \ldots - \omega^j \omega^{(p_1 - 1)n/p_1}. \end{aligned}$$

As p_1 is the smallest prime divisor of j in common with n and $\frac{n}{p_1}$ does not contain p_1 as a factor, $p \nmid j + \frac{bn}{p_1}$, $p_\alpha \nmid j + \frac{b'n}{p_\alpha}$, for all $s < \alpha \leq k$, for all $1 \leq b < p_1$, and for all $1 \leq b' < p_\alpha$. Now apply the same procedure for p_2 on each of the summands generated by the procedure with p_1 and set $\gamma = j + \frac{bn}{p_1}$, for all $1 \leq b < p_1$, and $p_\ell = p_2$. At the r–th step, the procedure produces exponents of ω carrying the form

$$j + \sum_{\ell=1}^{r} \frac{b_\ell n}{p_\ell},$$

with the property that they share no common divisors with $p_{s+1} \cdots p_k$ and share only $p_{r+1} \cdots p_s$ as common divisors with n.

It is now clear that after s iterations, ω^j is expressed as a sum of primitive n–th roots of unity.

Finally, notice that the number of primitive roots of unity equals the dimension, and hence they form a basis. $\square$

Corollary 3.19. *If ω^j is an n^{th} root of unity and*

$$\omega^j = \sum_{b \in \mathbb{Z}_n^*} c_b \omega^b, \qquad c_b \in \mathbb{R}$$

is the basis representation of ω^j in terms of the primitive roots of unity, then all the c_b are integers and have the same sign. Furthermore, if $\gcd(j, n) = p_1 \dots p_s$ then

$$c_b = (-1)^s |c_b|. \tag{3.3.1}$$

Proof. Clear from the proof of Theorem 3.18. By construction, the signs of the sums are changing at each level in unison. Hence we obtain (3.3.1). □

Corollary 3.20. *If n is prime, then the set of roots of unity(excluding 1) form a basis. Hence any sums of roots of unity of n are unique.*

Example 3.21. *Let $n = 3 \times 5 = 15$. Take $\omega = e^{2\pi i/15}$. We will use the algorithm in the proof of Theorem 3.18 to construct ω^6's basis representation. Since $\gcd(6, 15) = 3$, we only need to apply the procedure once with the sums of the third roots of unity (ω^5).*

$$\begin{aligned} \omega^6 &= -\omega^6\omega^5 - \omega^6\omega^{10} = -\omega^{6+5} - \omega^{6+10} = -\omega^{11} - \omega^{16} \\ &= -\omega^{11} - \omega. \end{aligned}$$

Example 3.22. *Let $n = 6$, ω a primitive 3-th root of unity. If we pick ω^2, ω^3 to be our basis, then $\omega = \omega^2 - \omega^3$. It demonstrates that the nice property of Corollary 3.19 is not universal to all basis representations of the roots of unity.*

This way of representing the roots of unity when n is square free is extremely convenient as the primitive roots can be thought of as $\mathbb{Z}_n^*$, the multiplicative group of units of $\mathbb{Z}_n$. It is closely related to the notion of multiplicative equivalence, which will be defined in the next chapter. Of particular interest is Corollary 3.19, a useful observation from the way we construct the basis representation. It is a key cog in a later proof which characterises unitary equivalence of a particular family of cyclic harmonic frames, where the generating group has order n (square free).

In general, the first $\varphi(n)$ roots of unity form a basis for the cyclotomic integers $\mathbb{Z}[\omega] \subset \mathbb{Q}[\omega]$ and various other bases are also possible, with a basis representation being recovered by using all the minimal vanishing sums to provide linear relations. See [Bos90] for some theorems where $n \not\equiv 2 \bmod 4$, for which a subset of the roots of unity are an integral basis, and for a way to construct an integral basis that has an integral basis for every cyclotomic subfield of $\mathbb{Q}[\omega]$).

We do not explore the various other ways used to represent cyclotomic fields in practice (often using some representation of elements with polynomials in the field, see [Fie06]).

However, in our context of studying unitary equivalences, we wish to exploit the group structure of the primitive roots because of their relationship to ($\mathbb{Z}_n^*$ as the automorphism group of $\mathbb{Z}_n$), as we shall see in the next chapter. There is scope here for future research in developing more machinery in attempt to exploit these non favourable basis representations using the linear relations between the roots of unity.

Chapter 4

Unitary Equivalences

4.1 Introduction

The unitary equivalences between harmonic frames will be studied here. With the exception of the next section, and notably Theorem 4.13, the rest of the sections will deal with cyclic harmonic frames.

We already met what it means for two finite tight frames to be equivalent in the introductory chapter (Definition 2.15), but we will restate it here for convenience.

Definition 4.1. *Two tight frames $(\Phi_j)_{j\in J}$, $(\psi_k)_{k\in K}$ are* ***unitarily equivalent*** *if there exists a bijection $\sigma : J \to K$, a unitary map U and a $c > 0$ such that*

$$\Phi_j := cU\psi_{\sigma j}, \qquad \forall j \in J.$$

Recall this is equivalent to (4.1.1) in chapter two,

$$\mathrm{Gram}(\Phi) = c^2 P_\sigma^* \mathrm{Gram}(\Psi) P_\sigma. \tag{4.1.1}$$

The significance in establishing a unitary equivalence of any two given tight frames is the power to inherit a lot of properties of the other frame, given that the Gramian matrix plays such an important role in many results concerning finite tight frames. This helps in the classification of finite tight frames as it gives a way to group classes of similarly behaving frames together.

The problem of computing whether any two given finite harmonic frames are unitarily equivalent is in general a difficult problem. Past approaches have involved a brute force approach in an attempt to construct a unitary operator that maps one frame to the other. Usually this involves a polynomial (of degree d, the dimension of the frames) time algorithm that maps a basis to a basis, but this becomes computationally infeasible as n and d become large (see [HW06]).

Optimisation can be made to this method by first comparing the angle multisets of the two frames and only proceeding if they are the same. Alternatively, by comparing angle multisets and then by constructing the permutations that are possible between the two frames by first fixing where one element is mapped to, e.g., from the first frame vectors to each other, a legitimate operation due to Corollary 2.40). Once fixed, looking to the Gramian, we can deduce the possible options left for the other vectors, and proceed to enumerate all possible permutations to try.

In practice this second approach seems to reduce the computational time drastically as the angle multisets first rule out most inequivalences, and also in many frames the number of repeating angles in the angle multiset is small. Hence the number of permutations to check is also small. However, as the number of repeated angles increases, this approach has a worse case time cost exceeding the first approach, as complexity is now in terms of multiplying factorials of the number of repeating angles, e.g., if there are three repeated angles, first repeats 3 times, second repeats 5 times and third repeats 2 times, then there will be 3!5!2! permutations to check.

We can reduce this complexity by computing the expected cost of either approach beforehand and then calling the routine which will be less complex in each given case.

This computational approach becomes unwieldy with large n and d but also fails to give insight into why two harmonic frames may be unitarily equivalent. In the search for a nice characterisation, we are led to making the following definitions.

4.2 Unitary Equivalence and the Preservation of Group Structure

Let $\mathrm{Aut}(G)$ denote the group of automorphisms of G, i.e., isomorphisms $\sigma : G \to G$.

Definition 4.2. *We say G-frames $\Phi = (\phi_g)_{g\in G}$, $\Psi = (\psi_g)_{g\in G}$ are **unitarily equivalent via an automorphism** if the map $\sigma : G \to G$ in Definition 4.1 can be taken to be in* $\mathrm{Aut}(G)$.

Example 4.3. *If G-frames Φ and Ψ are equal, then the set of permutations σ in the unitary equivalences* (4.1.1) *between them form a group called the symmetry group of Φ (see [VW09]). This group, denoted by* $\mathrm{Sym}(\Phi)$*, contains* *a subgroup of order $|G|$ consisting of the permutations*

$$\sigma : g \mapsto hg, \qquad h \in G,$$

with only the identity being an automorphism of G.

The above example shows that if two frames are unitarily equivalent via an automorphism, then there still exist permutations out there inducing a unitary equivalence which do not preserve the group structure. Another way to see this is by taking a group automorphism which induces a permutation, then apply Corollary 2.40 to obtain another permutation which does not map the identity element to the identity element, and hence cannot be an automorphism, but still preserves unitary equivalence.

Definition 4.4. *We say subsets J and K of a finite abelian group G are **multiplicatively equivalent** if there is an automorphism $\sigma : G \to G$ for which $K = \sigma J$.*

Example 4.5. *For $G = \mathbb{Z}_n$, each $\sigma \in \mathrm{Aut}(G)$ has the form $g \mapsto ag$, with $a \in \mathbb{Z}_n^*$, and hence J, $K \subset \mathbb{Z}_n$ are multiplicatively equivalent if and only if $K = aJ$ for some $a \in \mathbb{Z}_n^*$. This is the given condition for multiplicative equivalence when dealing with cyclic harmonic frames.*

Definition 4.6. *A harmonic frame $\Phi = (f_j)$ is said to be **unlifted** if $\sum_j f_j = 0$, otherwise it is **lifted**.*

Definition 4.7. *A tight frame is called **real** if its Gramian only has real entries. Otherwise it is a **complex** tight frame.*

Definition 4.8. *A finite tight frame is called **geometrically uniform** (see [BE03]) if its vectors are the orbits of a single non trivial vector $v \in \mathcal{H}$ under the action of a finite abelian group G of unitary matrices, i.e., $\Phi = (gv)_{g\in G}$. These frames have distinct vectors.*

The conditions on J for such a harmonic frame to have distinct vectors, to be real, and to be lifted are as follows.

Theorem 4.9. *Let G be an abelian group of order n, and $\Phi = \Phi_J = (\xi|_J)_{\xi\in\hat{G}}$ be the harmonic frame of n vectors for $\mathbb{C}^d$ given by a choice $J \subset G$, where $|J| = d$. Then*

(a) Φ has distinct vectors if and only if J generates G.

(b) Φ is a real frame if and only J is closed under taking inverses.

(c) Φ is a lifted frame if and only if the identity is an element of J.

Proof. We will now utilise some results below which are found in any introductory book on character theory. See [JL93] for example.

(a) Let H be the subgroup generated by J. Then Φ has distinct vectors if and only if the composition of maps $\hat{G} \to \hat{H} \to \mathbb{C}^J$, $\xi \mapsto \xi|_H \mapsto \xi|_J$ is 1–1. Since each $h \in H$ can be written as a sum of elements in J, and ξ is a character, $\xi(h)$ is determined by $\xi|_J$, and so $\xi|_H \mapsto \xi|_J$ is 1–1. Hence $\xi \mapsto \xi|_J$ is 1–1 if and only if the group homomorphism given by $\hat{G} \mapsto \hat{H} : \xi \mapsto \xi|_H$ is 1–1, i.e., $\hat{G} = \hat{H}$ and so $G = H = \langle J \rangle$.

(b) The frame Φ is real if and only if its multiset of angles is real, i.e.,

$$\sum_{j \in J} \xi(j) = \sum_{j \in J} \hat{\hat{j}}(\xi) \in \mathbb{R}, \quad \forall \xi \in \hat{G} \iff \psi := \sum_{j \in J} \hat{\hat{j}} \in \mathbb{R}^{\hat{G}}.$$

Suppose that J is closed under taking inverses, and $j \in J$. Then either j is its own inverse, so $\xi(j) = \xi(-j) = \overline{\xi(j)} \in \mathbb{R}$, or $j, -j \in J$, so they contribute $\xi(j) + \xi(-j) = \xi(j) + \overline{\xi(j)} \in \mathbb{R}$ to the sum for the angle. Thus we conclude each angle is zero. Conversely, suppose the multiset of angles is real, so that $\overline{\psi} = \psi$. Let $\langle \zeta, \chi \rangle$ be the inner product on $\mathbb{C}^{\hat{G}}$ for which the characters of $\hat{G}$ are orthogonal, i.e., $\langle \zeta, \chi \rangle := \frac{1}{|\hat{G}|} \sum_{\xi \in \hat{G}} \zeta(\xi)\overline{\chi(\xi)}$. Recall the Pontryagin duality map

$$G \to \hat{\hat{G}} : g \to \hat{\hat{g}}, \qquad \hat{\hat{g}}(\chi) := \chi(g), \qquad \forall \chi \in \hat{G},\, g \in G.$$

Since $\overline{\chi(g)} = \chi(g^{-1})$, then

$$\overline{\hat{\hat{j}}(\chi)} = \overline{\chi(j)} = \chi(-j) = (\hat{\hat{-j}})(\chi).$$

Hence,

$$j \in J \iff \langle \psi, \hat{\hat{j}} \rangle = 1$$

if and only if,

$$\langle \overline{\psi}, \overline{\hat{\hat{j}}} \rangle = \langle \psi, \overline{\hat{\hat{j}}} \rangle = \langle \psi, (\hat{\hat{-j}}) \rangle = 1$$

if and only if,

$$-j \in J.$$

(c) By the orthogonality relations for characters, Φ is unlifted if and only if

$$\sum_{\xi \in \hat{G}} \xi|_J = 0$$

if and only if,

$$\sum_{\xi \in \hat{G}} \xi(j)\overline{\xi(1)} = \sum_{\xi \in \hat{G}} \xi(j) = 0, \forall j \in J,$$

if and only if,

$$j \neq 1, \quad \forall j \in J.$$

$\square$

Corollary 4.10. *Let G be a finite abelian group, and d^* be the minimum number of generators for G. Then there is a G-frame of distinct vectors for $\mathbb{C}^d$ if and only if $d \geq d^*$.*

In the cyclic harmonic frame context, it is only necessary to study geometrically uniform tight frames.

The theorem implies that we do *not* get distinct vectors if and only if we take a subset of $\mathbb{Z}_n$ such that there is a k, where $k \mid n$ and $k \mid x$ for all x in the subset, e.g. the elements of $\{0, 2, 4\}$ have a common factor of 2, and would not generate $\mathbb{Z}_6$. The set $\{0, 2, 3\}$ would. Even though 2 and 3 have common factors with 6, they are different ones. Observe that in this scenario, the there exists some $m < n$ so that for each character χ, we have $\chi^m = 1$. This tells us that we can obtain the particular harmonic frame as a geometrically uniform frame (i.e., has distinct vectors) by using a smaller group instead. Thus, we can restrict our study to frames of distinct vectors.

Example 4.11. *Let $G = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ (k times), where p is prime. Then G gives harmonic frames of distinct vectors for $\mathbb{C}^d$ only for $d \geq k$ ($d^* = k$ since $0 \neq g \in G$ has order p).*

Example 4.12. *In $G = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, the non zero elements have order 2, and so are their own inverses. So the harmonic frames generated by this group are real. Note that the order of the group elements here implies that the characters are real too.*

The next theorem captures the connection between multiplicative equivalence and unitary equivalence via automorphisms. In some senses these are the *nice* kinds of unitary equivalences since the permutations respect the group structure of the harmonic frames involved.

Theorem 4.13. *Suppose J and K are subsets of a finite abelian group G. Then the following are equivalent*

(a) The subsets J and K are multiplicatively equivalent (see Definition 4.4).

(b) The harmonic frames given by J and K are unitarily equivalent via an automorphism.

Proof. (a) $\Longrightarrow$ (b): Suppose that $K = \sigma J$, where $\sigma \in \mathrm{Aut}(G)$.

The natural action of $\mathrm{Aut}(G)$ on $\hat{G}$, which is given by

$$T\chi = \hat{T}\chi := \chi \circ T^{-1}, \qquad T \in \mathrm{Aut}(G), \quad \chi \in \hat{G},$$

induces automorphisms of $\hat{G}$, since

$$\hat{T}(\xi\eta) = (\xi\eta) \circ T^{-1} = (\xi \circ T^{-1})(\eta \circ T^{-1}) = (\hat{T}\xi)(\hat{T}\eta), \qquad \xi, \eta \in \hat{G}.$$

Using $\chi(j) = (\chi \circ \sigma^{-1})(\sigma j) = \hat{\sigma}\chi(\sigma j)$, we calculate

$$\begin{aligned}\langle \xi|_J, \eta|_J \rangle &= \sum_{j\in J} \xi(j)\overline{\eta(j)} = \sum_{j \in J} \hat{\sigma}\xi(\sigma j)\overline{\hat{\sigma}\eta(\sigma j)} = \sum_{k\in K} \hat{\sigma}\xi(k)\overline{\hat{\sigma}\eta(k)} \\ &= \langle \hat{\sigma}\xi|_K, \hat{\sigma}\eta|_K \rangle.\end{aligned}$$

Hence, by the condition (4.1.1), the $\hat{G}$–frames $(\xi|_J)_{\xi\in\hat{G}}$ and $(\xi|_K)_{\xi\in\hat{G}}$ are unitarily equivalent via the automorphism $\hat{\sigma} : \hat{G} \to \hat{G} : \chi \mapsto \chi \circ \sigma^{-1}$.

(b) $\Longrightarrow$ (a): Suppose the harmonic frames given by $J, K \subset G$ are unitarily equivalent via an isomorphism $\hat{\sigma} : \hat{G} \to \hat{G}$, i.e.,

$$\langle \xi|_J, \eta|_J \rangle = \langle \hat{\sigma}\xi|_K, \hat{\sigma}\eta|_K \rangle, \qquad \forall \xi, \eta \in \hat{G}.$$

Taking $\eta = 1$, the trivial character, above, gives

$$\sum_{j\in J} \xi(j) = \sum_{k\in K} (\hat{\sigma}\xi)(k), \qquad \forall \xi \in \hat{G}. \tag{4.2.1}$$

We now seek to define an automorphism $\sigma = \tau^{-1} : G \to G$ satisfying

$$(\hat{\sigma}\chi)(g) = (\chi \circ \sigma^{-1})(g), \qquad \forall \chi \in \hat{G}, \quad \forall g \in G.$$

Since $\hat{\sigma} : \hat{G} \to \hat{G}$ is an automorphism, $\chi \mapsto \hat{\sigma}\chi(g)$ belongs to $\hat{\hat{G}}$, and so we can use Pontryagin duality to define τg by

$$\widehat{\widehat{\tau g}}(\chi) := \hat{\sigma}\chi(g), \qquad \forall \chi \in \hat{G}.$$

This map $\tau : G \to G$ is a bijection, since

$$\begin{aligned}\tau g = \tau h \iff \hat{\sigma}\chi(g) = \hat{\sigma}\chi(h), \quad \forall \chi \in \hat{G} &\iff \hat{\hat{g}}(\hat{\sigma}\chi) = \hat{\hat{h}}(\hat{\sigma}\chi), \quad \forall \chi \in \hat{G} \\ \iff \hat{\hat{g}}(\xi) = \hat{\hat{h}}(\xi), \quad \forall \xi \in \hat{G} &\iff \hat{\hat{g}} = \hat{\hat{h}} \iff g = h,\end{aligned}$$

and it is a homomorphism since

$$\begin{aligned}\hat{\sigma}\xi \in \hat{G} &\implies (\hat{\sigma}\xi)(g+h) = (\hat{\sigma}\xi)(g)(\hat{\sigma}\xi)(h), \quad \forall \xi \in \hat{G} \\ &\iff (\widehat{\widehat{\tau(g+h)}})(\xi) = \widehat{\widehat{\tau g}}(\xi)\widehat{\widehat{\tau h}}(\xi), \quad \forall \xi \in \hat{G} \\ &\iff (\widehat{\widehat{\tau(g+h)}}) = (\widehat{\widehat{\tau g}})(\widehat{\widehat{\tau h}}) \iff \tau(g+h) = \tau g + \tau h\end{aligned}$$

(where we write the group operation in $\hat{\hat{G}}$ as $\cdot$). Thus $\sigma := \tau^{-1} \in \mathrm{Aut}(G)$, which satisfies

$$(\hat{\sigma}\xi)(k) = \widehat{\widehat{\sigma^{-1}k}}(\xi),$$

Hence, by Pontryagin duality, (4.2.1) gives

$$\sum_{j \in J} \hat{\hat{j}}(\xi) = \sum_{k \in K} \widehat{\widehat{\sigma^{-1}k}}(\xi), \quad \forall \xi \in \hat{G} \implies \sum_{j \in J} \hat{\hat{j}} = \sum_{k \in K} \widehat{\widehat{\sigma^{-1}k}}.$$

Since characters of a finite abelian group are linearly independent, we conclude

$$\{\hat{\hat{j}} : j \in J\} = \{\widehat{\widehat{\sigma^{-1}k}} : k \in K\} \implies \{j : j \in J\} = \{\sigma^{-1}k : k \in K\}.$$

Hence $K = \sigma J$, i.e., J and K are multiplicatively equivalent subsets of G. $\square$

The number of generating sets of G is essentially given by the Eulerian function multiplied by a normalising amount to stop the over count introduced by ordering (see [Hal36].) One approach to calculating this function could be to employ the machinery of measure theory on special zeta functions(see [dS00]).[1]

We note here that computational evidence in magma seems to suggest that multiplicative equivalence captures the vast majority of unitarily equivalent cyclic harmonic frames and that a very small number fall outside this category.

Example 4.14 (Four vectors in $\mathbb{C}^2$)**.** *Consider $G = \mathbb{Z}_4$. The automorphism group has order* 2*, generated by $\sigma : g \mapsto 3g$ ($\mathbb{Z}_4^* = \{1,3\}$). Thus the multiplicative equivalence classes of* 2*–element subsets of G, which are the orbits under the action of* $\mathrm{Aut}(G)$*, are*

$$\big\{\{0,1\},\{0,3\}\big\}, \qquad \big\{\{1,2\},\{2,3\}\big\}, \qquad \big\{\{1,3\}\big\}, \qquad \big\{\{0,2\}\big\}.$$

[1] For cyclic harmonic frames of prime order, M. Hirn came with a recursive formula for doing a count [Hir09]. See also [Wal10] for an alternative way to count cyclic harmonic frames of prime order.

n	8	16	24	32	40
Mul. eq. classes.	16	66	238	278	571
Mul. ineq, uni. eq.	2	6	8	14	12

Table 4.1: Cyclic harmonic frames of order n, in $\mathbb{C}^3$. Number of multiplicative equivalence classes compared with number of multiplicatively inequivalent classes that are unitarily equivalent to another.

The first three give cyclic harmonic frames with distinct vectors(since 1 *generates* G*), while the last does not. None are unitarily equivalent, since the (respective) angle multisets are*

$$\{-i+1, 0, i+1\}, \qquad \{0, -i-1, i-1\}, \qquad \{0, 0, -2\}, \qquad \{0, 0, 2\}.$$

Example 4.15. *Consider* $G = \mathbb{Z}_2 \times \mathbb{Z}_2$*, which is generated by any two of its three elements* $\{a, b, a+b\}$ *of order* 2*. The automorphism group has order* 6*, with an automorphism corresponding to each permutation of* $\{a, b, a+b\}$*. Thus the multiplicative equivalence classes are*

$$\big\{\{a, b\}, \{a, a+b\}, \{b, a+b\}\big\}, \qquad \big\{\{0, a\}, \{0, b\}, \{0, a+b\}\big\}.$$

Only the first gives a harmonic frame with distinct vectors. This real frame has angles $\{0, 0, -2\}$*, and is unitarily equivalent to the cyclic harmonic frame with these angles.*

Example 4.16 (Seven vectors in $\mathbb{C}^3$)**.** *For* $G = \mathbb{Z}_7$*, the seven multiplicative equivalence classes have representatives*

$$\{1,2,6\},\ \{1,2,3\},\ \{0,1,2\},\ \{0,1,3\},\ \{1,2,5\},\ \{0,1,6\} \quad \text{(size 6)}$$

$$\{0,1,6\} \quad \text{(size 3)} \qquad \{1,2,4\} \quad \text{(size 2)}.$$

Each gives a cyclic harmonic frame of distinct vectors (as nonzero elements generate G*). None of these are unitarily equivalent since their angle multisets are different.*

A finite abelian group G can be written as a direct sum of p–groups

$$G_p = \mathbb{Z}_{p^{e_1}} \oplus \mathbb{Z}_{p^{e_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{e_m}}$$

where p are the prime divisors of $|G|$. The automorphism group of G_p has order

$$|\mathrm{Aut}(G_p)| = \prod_{k=1}^{m}(p^{d_k} - p^{k-1}) \prod_{j=1}^{m}(p^{e_j})^{m-d_j} \prod_{i=1}^{m}(p^{e_i-1})^{m-c_i+1}, \tag{4.2.2}$$

d=2			d=3			d=4		
n	non	cyc	n	non	cyc	n	non	cyc
4	0	3	4	0	3	4	0	1
8	1	7	8	5	16	8	8	21
9	1	6	9	3	15	9	5	23
12	2	13	12	11	57	12	30	141
16	4	13	16	28	74	16	139	228
18	2	18	18	19	121	18	80	494
20	3	19	20	29	137	20	154	622
24	6	27	24	89	241	24	604	1349
25	1	15	25	8	115	25	37	636
27	3	18	27	33	159	27	202	973
28	4	25	28	57	255	28	443	1697
32	9	25	32	158	278	32	1379	2152

Table 4.2: the number of inequivalent *non*-cyclic and *cyc*lic harmonic frames of $n \leq 32$ distinct vectors in $\mathbb{C}^d$.

where

$$c_k := \min\{r : e_r = e_k\} \leq k, \qquad d_k := \max\{r : e_r = e_k\} \geq k,$$

and so the order of $\mathrm{Aut}(G)$ is the product of these orders (see [HR07]).

Computational data suggests most harmonic frames are cyclic and that as the group G became less cyclic, the number of cyclic harmonic frames from G decreased.[HW06] Theorems 4.13, 4.9, and the automorphism order (4.2.2) seem to suggest the following mechanisms are at play:

- As G becomes less cyclic, $|\mathrm{Aut}(G)|$ becomes larger, and so the number of multiplicative equivalence classes becomes smalller.

- As G becomes less cyclic, the orders of its elements become smaller, so $J \subset G$ is less likely to generate G, and hence give a harmonic frame with distinct vectors.

4.3 The Cases of $\mathbb{C}^1$ and $\mathbb{C}^2$

In this section, we will show that in one dimension, there is only one unique harmonic frame of n distinct vectors, and that in two dimensions, multiplicative equivalence completely characterises the unitary equivalence classes of cyclic harmonic frames, i.e., that two cyclic harmonic frames in $\mathbb{C}^2$ are unitarily equivalence if and only if they are multiplicatively equivalent. This seems like a surprising result at first given that exceptional families seem to exist in families outside of one and two dimensions. But on further reflection, the theorems of 3.12, 3.15 and Lemma 3.6 conjures up an explanation for why no exceptions exist in $\mathbb{C}^2$.

Theorem 4.17 (One Dimension)**.** *There is a unique harmonic frame of n distinct vectors for $\mathbb{C}^1$, namely the cyclic harmonic frame given by the n–th roots of unity.*

Proof. Use Theorems 4.9 and 4.13. If g generates an abelian group G of order n, then G must be $\mathbb{Z}_n$. If g_1,g_2 generate $\mathbb{Z}_n$, then $\{g_1\}$,$\{g_2\}$ are multiplicatively equivalent (as $g_1 \mapsto g_2$ gives an automorphism of G), and so give unitarily equivalent frames. □

Example 4.18. *There is a unique lifted harmonic frame of n vectors for $\mathbb{C}^2$, i.e., the cyclic harmonic frame given by the subset $\{0, a\}$ where $\mathbb{Z}_n = \langle a \rangle$. The angle multiset of the cyclic harmonic frame for $\mathbb{C}^2$ given by $\{j_1, j_2\} \subset \mathbb{Z}_n$ is*

$$\{\omega^{aj_1} + \omega^{aj_2} : a \in \mathbb{Z}_n, a \neq 0\}, \qquad \omega := e^{2\pi i/n}.$$

Now we turn our attention to the two dimensional setting. Because of those strong results concerning vanishing sums, we can show that if two cyclic harmonic frames in $\mathbb{C}^2$ are not multiplicatively equivalent, then their angle multisets differ. This is done by constructing an angle in one of the frame's angle multiset and not in the other. We then combine this with Theorems 4.13 and 4.9 to show that cyclic harmonic frames in $\mathbb{C}^2$ are multiplicatively equivalent if and only if they are unitarily equivalent.

Lemma 4.19. *Let $\omega = e^{2\pi i/n}$ and $j_1, j_2 \in \mathbb{Z}_n$. If $\omega^{j_1} + \omega^{j_2} = 0$, then n is even, and*

$$\omega^{aj_1} + \omega^{aj_2} = \begin{cases} 0, & a \text{ odd;} \\ 2\omega^{aj_1}, & a \text{ even.} \end{cases}$$

Proof. If $\omega^{j_1} + \omega^{j_2} = 0$, then $\omega^{j_2 - j_1} = -1$, so n is even, and $j_2 - j_1 = \frac{n}{2}$. This gives

$$\omega^{aj_1} + \omega^{aj_2} = \omega^{aj_1} + \omega^{a(j_1 + \frac{n}{2})} = \omega^{aj_1} + (-1)^a \omega^{aj_1},$$

as supposed. □

Recall the cyclic group $\mathbb{Z}_n$ has a unique cyclic subgroup of each order dividing n, and no other subgroups. Thus, if $j_1, j_2 \in \mathbb{Z}_n$ have the same order, then they generate the same subgroup, i.e.,

$$\mathrm{ord}(j_1) = \mathrm{ord}(j_2) \iff \langle j_1 \rangle = \langle j_2 \rangle.$$

We will also repeatedly use the facts

$$\mathrm{ord}(aj) \le \mathrm{ord}(j), \quad \forall a \in \mathbb{Z},\ j \in \mathbb{Z}_n, \qquad \mathrm{ord}(b) = n \iff b \in \mathbb{Z}_n^*. \quad (4.3.1)$$

Theorem 4.20. *Cyclic frames of n distinct vectors for $\mathbb{C}^2$ are unitarily equivalent if and only if the subsets of $\mathbb{Z}_n$ that give them are multiplicatively equivalent.*

Proof. Suppose the subsets $\{j_1, j_2\}$ and $\{k_1, k_2\}$ of $\mathbb{Z}_n$ are not multiplicatively equivalent, and give harmonic frames of distinct vectors, i.e., $\langle j_1, j_2\rangle = \langle k_1, k_2\rangle = \mathbb{Z}_n$. We will show that the cyclic harmonic frames they give have different angle multisets, and so are not unitarily equivalent. Since multiplicatively equivalent subsets give the same angle multisets, it suffices to consider the following cases.

Case (a). $\omega^{j_1} + \omega^{j_2} \ne 0$. By Lemma 3.6, if this angle appears in the second frame as $\omega^{bk_1} + \omega^{bk_2}$, $b \in \mathbb{Z}_n$, then $\{j_1, j_2\} = \{bk_1, bk_2\}$. Since the frames are not multiplicatively equivalent, we must have $b \notin \mathbb{Z}_n^*$, and hence $\langle b\rangle \ne \mathbb{Z}_n$. But this implies $\langle j_1, j_2\rangle = \langle bk_1, bk_2\rangle \subset \langle b\rangle \ne \mathbb{Z}_n$, and so $\omega^{j_1} + \omega^{j_2}$ cannot be an angle in the second frame.

Case (b). $\omega^{aj_1} + \omega^{aj_2} = \omega^{bk_1} + \omega^{bk_2} = 0$, $\forall a, b \in \mathbb{Z}_n^*$. Suppose first that there is a unit in each of the subsets. Then by going to multiplicatively equivalent subsets, we may assume that $j_1 = k_1 = 1$, and thus obtain $\omega + \omega^{j_2} = 0 = \omega + \omega^{k_2}$, which gives $j_2 = k_2$, and so the two subsets are equal. Thus we may assume that $j_1, j_2 \notin \mathbb{Z}_n^*$. By Lemma 4.19, n is even, and the nonzero angles of the first frame are $\{2\omega^{2kj_1} : 1 \le k \le \frac{n}{2}\} = \{2\omega^{2kj_2} : 1 \le k \le \frac{n}{2}\}$, and we conclude $\langle 2j_1\rangle = \langle 2j_2\rangle$. Since j_1, j_2 are not units, they cannot have the same order (and generate $\mathbb{Z}_n$), and so we can assume that $\mathrm{ord}(j_1) < \mathrm{ord}(j_2)$. The group $\langle 2j_1\rangle$ is either equal to $\langle j_1\rangle$, or has half its order, and similarly for $\langle j_2\rangle$. Thus the only way to have $\langle 2j_1\rangle = \langle 2j_2\rangle$ is for $\langle j_1\rangle = \langle 2j_1\rangle$, in which case $j_1 \in \langle 2j_2\rangle \subset \langle j_2\rangle$, and $\langle j_1, j_2\rangle = \langle j_2\rangle \ne \mathbb{Z}_n$. We conclude that case (b) can never occur.

□

Remark 4.21. *A careful reading of the proof shows that if $\omega^{j_1} + \omega^{j_2} \ne 0$, then*

$$\{\omega^{aj_1} + \omega^{aj_2} : \omega^{aj_1} + \omega^{aj_2} \ne 0, a \in \mathbb{Z}_n^*\} \qquad (4.3.2)$$

is a set of nonzero angles, which is unique to frame given by $\{j_1, j_2\}$ (or any multiplicatively equivalent subset), and that for n even, there is a unique(up to unitary equivalence) frame in which the angles given by (4.3.2) are all zero, namely that given by $\{1, 1+\frac{n}{2}\}$. The frame generated by this subset will have the zero angle repeating $\frac{n}{2}$ times in the angle multiset.

Example 4.22 (A noncyclic harmonic frame in $\mathbb{C}^2$)**.** *There a seven unitarily inequivalent cyclic harmonic frames of $n = 8$ distinct vectors for $\mathbb{C}^2$. We now list them, giving a representative of the multiplicative equivalence class they correspond to, followed by the 4 angles given by (4.3.2) – note these are unique, and then the remaining 3 angles.*

$\{0,1\}$	$1+\omega, 1+\omega^3, 1+\omega^5, 1+\omega^7$	$1+\omega^2, 1+\omega^4=0, 1+\omega^6$
$\{1,2\}$	$\omega+\omega^2, \omega^3+\omega^6, \omega^5+\omega^2, \omega^7+\omega^6$	$\omega^2+\omega^4, \omega^4+1=0, \omega^6+\omega^4$
$\{1,3\}$	$\omega+\omega^3, \omega^5+\omega^7(\text{twice})$	$\omega^2+\omega^6=0, \omega^4+\omega^4, \omega^6+\omega^2=0$
$\{1,4\}$	$\omega+\omega^4, \omega^3+\omega^4, \omega^5+\omega^4, \omega^7+\omega^4$	$\omega^2+1, \omega^4+1=0, \omega^6+1$
$\{1,5\}$	$\omega+\omega^5=\omega^3+\omega^7=0(\text{twice})$	$\omega^2+\omega^2, \omega^4+\omega^4, \omega^6+\omega^6$
$\{1,6\}$	$\omega+\omega^6, \omega^3+\omega^2, \omega^5+\omega^6, \omega^7+\omega^2$	$\omega^2+\omega^4, \omega^4+1=0, \omega^6+\omega^4$
$\{1,7\}$	$\omega+\omega^7, \omega^3+\omega^5(\text{twice})$	$\omega^2+\omega^6=0, \omega^4+\omega^4, \omega^6+\omega^2$

There are two harmonic frames of distinct vectors given by the group $G = \mathbb{Z}_4 \times \mathbb{Z}_2$. Here is a representative subset giving them, followed by the angle multiset.

$\{(0,1),(1,0)\}$	$0,0,1+\omega^2,1+\omega^6,\omega^2+\omega^4,\omega^4+\omega^4,\omega^6+\omega^4$
$\{(1,0),(1,1)\}$	$0,0,0,0,\omega^2+\omega^2,\omega^4+\omega^4,\omega^6+\omega^6$

The last of these has the same angles as the cyclic harmonic frame given by $\{1,5\}$, and it is easy to check that it is unitarily equivalent to it. The angle multiset of the first is not shared by any cyclic harmonic frame, and so is an example of a noncyclic harmonic frame. This noncyclic harmonic frame $(\xi|_J)_{\xi\in\hat{G}}$ for $J = \{(0,1),(1,0)\}$ is

$$\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}i\\1\end{bmatrix}, \begin{bmatrix}i\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\end{bmatrix}, \begin{bmatrix}-i\\1\end{bmatrix}, \begin{bmatrix}-i\\-1\end{bmatrix}\right\}.$$

4.4 Equivalences Breaking the Group Structure

Example 4.23. *($n = 8$, $d = 3$). For $\mathbb{Z}_8$ there are 17 multiplicative equivalence classes of 3–element subsets which generate it. Only two of these give frames with the same angles, namely*

$$\big\{\{1,2,5\},\{3,6,7\}\big\}, \qquad \big\{\{1,5,6\},\{2,3,7\}\big\}.$$

The common angle multiset is

$$\{-1,i,i,-i,-i,-2i-1,2i-1\} \qquad (\omega^2 = i,\ \omega^4 = -1,\ \omega^6 = -i,\ \omega := e^{\frac{2\pi i}{8}}).$$

Notice here that in many of the angles $\omega^{aj_1}+\omega^{aj_2}+\omega^{aj_2}$, $a\neq 0$ there is cancellation, as outlined in Lemma 4.19. This explains why the angles multisets for multiplicatively inequivalent subsets can be the same. These frames are unitarily equivalent (to be proved),but not via an automorphism.

Definition 4.24. *Let p be prime, $p^2 \mid n$. Define*

$$B_{p,n} := \{mb : m \in \mathbb{Z}_n^*,\ 1 \leq b < n,\ p^2 b \text{ divides } n \text{ in } \mathbb{Z}\} \subset \mathbb{Z}_n.$$

Lemma 4.25. *Let p be prime, $d = p+1$, $n \geq d$ with $p^2 \mid n$, and*

$$A := \frac{n}{p}\mathbb{Z}_n + a = \{a, \frac{n}{p}+a, \frac{2n}{p}+a, \ldots, (p-1)\frac{n}{p}+a\}, \quad a \in \mathbb{Z}_n.$$

Then the cyclic harmonic frames for $\mathbb{C}^d$ given by the subsets,

$$A \cup \{b\}, \quad A \cup \{b + r\frac{n}{p}\}, \qquad b, b + r\frac{n}{p} \in B_{p,n}, \quad b \notin A,$$

of $\mathbb{Z}_n$ are unitarily equivalent.

Proof. Since multiplicative equivalence implies unitary equivalence (Theorem 4.13), we can multiply our subsets by some $m \in \mathbb{Z}_n^*$. This gives subsets of the same form since $mA = \frac{n}{p}\mathbb{Z}_n + ma$ and $mB_{p,n} = B_{p,n}$. Hence, in view of the definition of $B_{p,n}$, we can suppose without loss of generality that $p^2 b \mid n$.

Let $\zeta = \omega^{n/p} = e^{2\pi i/p}$ and $(f_j)_{j\in\mathbb{Z}_n}$, $(g_j)_{j\in\mathbb{Z}_n}$ be the cyclic frames given by $A \cup \{b\}$ and $A \cup \{b + r\frac{n}{p}\}$, where $p^2 b \mid n$. Then

$$\begin{aligned}\langle f_j, f_k\rangle &= \omega^{aj}\overline{\omega^{ak}} + \ldots + \omega^{(a+(p-1)\frac{n}{p})j}\overline{\omega^{(a+(p-1)\frac{n}{p})k}} + \omega^{bj}\overline{\omega^{bk}} \\ &= \omega^{a(j-k)}\{1 + \zeta^{j-k} + \ldots + \zeta^{(p-1)(j-k)}\} + \omega^{b(j-k)},\end{aligned}$$

and similarly,

$$\langle g_j, g_k\rangle = \omega^{a(j-k)}\{1 + \zeta^{j-k} + \ldots + \zeta^{(p-1)(j-k)}\} + \omega^{(b+r\frac{n}{p})(j-k)}.$$

Since $p^2 b \mid n$ we can define a permutation σ of $\mathbb{Z}_n$ given by

$$\sigma j := j - t_j^*, \quad j \equiv j^* \bmod p, \qquad t_j^* := r\frac{n}{pb}j^*.$$

This is well defined and 1–1 since

$$\begin{aligned}\sigma j = \sigma k \quad &\Longrightarrow \quad j - r\frac{n}{pb}j^* = k - r\frac{n}{pb}k^* \\ &\Longrightarrow \quad j \equiv k \bmod p \quad (\text{since } p \text{ divides } r\tfrac{n}{pb}) \\ &\Longrightarrow \quad j^* = k^* \quad \Longrightarrow \quad j = k.\end{aligned}$$

We now show that σ gives a unitary equivalence, i.e., $\langle g_{\sigma j}, g_{\sigma k}\rangle = \langle f_j, f_k\rangle$, $\forall j, k$. If $j - k \equiv 0 \bmod p$, then $\sigma j - \sigma k = j - k$, so that

$$\langle g_{\sigma j}, g_{\sigma k}\rangle = p\omega^{a(j-k)} + \omega^{(b+r\frac{n}{p})(j-k)}, \qquad \langle f_j, f_k\rangle = p\omega^{a(j-k)} + \omega^{b(j-k)},$$

which are equal since $\omega^{r\frac{n}{p}(j-k)} = \zeta^{r(j-k)} = \zeta^0 = 1$.

Now consider $j - k \not\equiv 0 \bmod p$. Since $\zeta^{j-k} \neq 1$ is a p–th root of unity we have by equation (3.1.1) that

$$\langle f_j, f_k\rangle = \omega^{b(j-k)}, \qquad \langle g_{\sigma j}, g_{\sigma k}\rangle = \omega^{(b+r\frac{n}{p})(\sigma j-\sigma k)} = \omega^{(b+r\frac{n}{p})(j-k+t_{j^*}-t_{k^*})} =: \omega^c.$$

Since $bp^2 \mid n$ and $j^* - k^* = j - k + px$, $x \in \mathbb{Z}$ we have

$$\begin{aligned} c &= (b + r\tfrac{n}{p})\big(j - k - r\tfrac{n}{pb}(j^* - k*)\big) \equiv b\big(j - k - r\tfrac{n}{pb}(j^* - k^*)\big) + r\tfrac{n}{p}(j - k) \\ &\equiv b\big(j - k - r\tfrac{n}{pb}(j - k + px)\big) + r\tfrac{n}{p}(j - k) \equiv (j - k)\{b - r\tfrac{n}{p} + r\tfrac{n}{p}\} \\ &\equiv b(j - k) \bmod n. \end{aligned}$$

Hence $\langle g_{\sigma j}, g_{\sigma k}\rangle = \omega^c = \omega^{b(j-k)} = \langle f_j, f_k\rangle$. □

Lemma 4.26. *Let p be prime, $d = p+1$, $n = p^2 z$, $z \in \mathbb{N}$, and $A := \frac{n}{p}\mathbb{Z}_n + a$, where $a \in \{1, 2, \ldots, p-1\}$, $b \notin A$ and $p \mid b$. Then the cyclic harmonic frames given by*

$$J_r = A \cup \{b + r\tfrac{n}{p}\}, \quad r \in \{0, 1, \ldots, p - 1\},$$

are not multiplicatively equivalent.

Proof. Suppose by way of contradiction that J_{r_1} and J_{r_2}, $r_1 \neq r_2$ are multiplicatively equivalent, i.e., $mJ_{r_1} = J_{r_2}$, $m \in \mathbb{Z}_n^*$. As $mA = \frac{n}{p}\mathbb{Z}_n + ma$, this implies

$$ma = a + s\frac{n}{p}, \qquad m(b + r_1\tfrac{n}{p}) = b + r_2\frac{n}{p}.$$

$$(a + s\tfrac{n}{p})(b + r_1\tfrac{n}{p}) = ma(b + r_1\tfrac{n}{p}) = a\big(m(b + r_1\tfrac{n}{p})\big) = a(b + r_2\tfrac{n}{p})$$

which gives

$$s\frac{b}{p}n = a(r_2 - r_1)\frac{n}{p} - s\frac{n}{p^2}r_1 n$$

Hence

$$0 \equiv a(r_2 - r_1)\frac{n}{p} \not\equiv 0 \bmod n,$$

a contradiction. Therefore J_{r_1} and J_{r_2} are not multiplicatively equivalent. □

Theorem 4.27. *Let p be prime, $d = p + 1$, $n \geq d$ with $p^2 \mid n$, and*

$$A := \frac{n}{p}\mathbb{Z}_n + a = \{a, \tfrac{n}{p} + a, \tfrac{2n}{p} + a, \ldots, (p-1)\tfrac{n}{p} + a\}, \quad a \in \mathbb{Z}_n.$$

Then the cyclic harmonic frames for $\mathbb{C}^d$ given by the subsets,

$$A \cup \{b\}, \quad A \cup \{b + r\tfrac{n}{p}\}, \qquad b, b + r\tfrac{n}{p} \in B_{p,n}, \quad b \notin A$$

of $\mathbb{Z}_n$ are unitarily equivalent if and only if they are multiplicatively equivalent.

Proof. By combining lemmata 4.25 and 4.26, we have the desired result. □

Note that Example 4.23 is one type of frame caught by Theorem 4.27, as the theorem suggests, there are an infinite number of examples in this class living in various dimensions higher than two.

Lemma 4.28. *Let p be prime, $d = p + 1$, $n \geq d$, and*

$$A := \frac{n}{p}\mathbb{Z}_n + a = \{a, \tfrac{n}{p} + a, \tfrac{2n}{p} + a, \ldots, (p-1)\tfrac{n}{p} + a\}, \quad a \in \mathbb{Z}_n$$

Then the cyclic harmonic frames for $\mathbb{C}^d$ given by the subsets,

$$A \cup \{b\}, \quad A \cup \{bq\}, \quad b \in \mathbb{Z}_n, \quad b \notin A, \quad q \in \mathbb{Z}_n \setminus \{0\}$$

of $\mathbb{Z}_n$ are unitarily equivalent if for some $m \in \mathbb{Z}_n$,

$$qb \equiv b \ mod\ \frac{n}{p}, \qquad mqb \equiv qb - b \ mod\ n, \qquad m \not\equiv 1 \ mod\ p.$$

Proof. Observe $qb \equiv b \bmod \frac{n}{p}$ is equivalent to $qb = b + r\frac{n}{p}$, for some $r \in \mathbb{Z}_n$. As before,

$$\begin{aligned}\langle f_j, f_k \rangle &= \omega^{aj}\overline{\omega^{ak}} + \ldots + \omega^{(a+(p-1)\frac{n}{p})j}\overline{\omega^{(a+(p-1)\frac{n}{p})k}} + \omega^{bj}\overline{\omega^{bk}} \\ &= \omega^{a(j-k)}\{1 + \zeta^{j-k} + \ldots + \zeta^{(p-1)(j-k)}\} + \omega^{b(j-k)},\end{aligned}$$

and similarly

$$\langle g_{\sigma j}, g_{\sigma k} \rangle = \omega^{a(\sigma j - \sigma k)}\{1 + \zeta^{\sigma j - \sigma k} + \ldots + \zeta^{(p-1)(\sigma j - \sigma k)}\} + \omega^{bq(\sigma j - \sigma k)}.$$

For these to be equal, we seek a permutation σ of the form

$$\sigma j := j - mj^*, \qquad j^* := j \bmod p.$$

We now check that this defines a permutation. Observe

$$\sigma j = \sigma k \iff \exists x_j, x_k \in \mathbb{Z}_n : j^* + px_j - mj^* = k^* + px_k - mk^*.$$

If $j \equiv k \bmod p$ i.e., $j^* = k^*$, this implies $px_j = px_k$, hence $j = k$.
If $j \not\equiv k \bmod p$ i.e., $j^* \neq k^*$, this implies

$$\sigma j = \sigma k \iff 0 \not\equiv (j^* - k^*)(1 - m) = p(x_k - x_j) \equiv 0 \bmod p,$$

a contradiction. Therefore σ defines a permutation.

We now show that σ gives a unitary equivalence, i.e., $\langle g_{\sigma j}, g_{\sigma k}\rangle = \langle f_j, f_k\rangle$, for all j, k. If $j - k \equiv 0 \bmod p$, then $\sigma j - \sigma k = j - k$, so that

$$\langle f_j, f_k\rangle = p\omega^{a(j-k)} + \omega^{b(j-k)}, \qquad \langle g_{\sigma j}, g_{\sigma k}\rangle = p\omega^{a(j-k)} + \omega^{qb(j-k)},$$

which are equal since $j - k = px$, $\exists x \in \mathbb{Z}_n$, so

$$\omega^{qb(j-k)} = \omega^{qb(px)} = \omega^{bpx} = \omega^{b(j-k)}.$$

Now consider $j - k \not\equiv 0 \bmod p$. Since $\zeta^{j-k} \neq 1$ is a p^{th} root of unity we have by equation (3.1.1) that

$$\langle f_j, f_k\rangle = \omega^{b(j-k)}, \qquad \langle g_{\sigma j}, g_{\sigma k}\rangle = \omega^{qb(\sigma j - \sigma k)} = \omega^{qb\left(j-k-m(j^*-k^*)\right)}.$$

Since $j^* - k^* = j - k + px$, $x \in \mathbb{Z}$ we have

$$\begin{aligned}
&qb\big(j - k - m(j^* - k^*)\big)\\
&\quad = (b + r\tfrac{n}{p})\big(j - k - m(j^* - k^*)\big)\\
&\quad = b(j - k) - bm(j^* - k^*) + r\tfrac{n}{p}\big(j^* - k^* + px - m(j^* - k^*)\big)\\
&\quad = b(j - k) + (j^* - k^*)\{-m(b + r\tfrac{n}{p}) + r\tfrac{n}{p}\} + rnx\\
&\quad = b(j - k) + (j^* - k^*)\{-mqb + (qb - b)\} + rnx\\
&\quad \equiv b(j - k) \bmod n.
\end{aligned}$$

Hence $\langle g_{\sigma j}, g_{\sigma k}\rangle = \omega^c = \omega^{b(j-k)} = \langle f_j, f_k\rangle$. □

Example 4.29. *A special case of Lemma 4.28 is when*

$$b \not\equiv 0 \ mod\ \frac{n}{p} \ : \ p^2 b \mid n, \qquad qb := b + r\frac{n}{p} \not\equiv 0 \ mod\ \frac{n}{p}, \qquad m := r\frac{n}{pb},$$

i.e., the situation of Lemma 4.25.

Theorem 4.30. *Let p be prime, $d = p+1$, $n \geq d$, and*

$$A := \frac{n}{p}\mathbb{Z}_n + a = \{a, \tfrac{n}{p} + a, \tfrac{2n}{p} + a, \dots, (p-1)\tfrac{n}{p} + a\}, \quad a \in \mathbb{Z}_n.$$

Then the cyclic harmonic frames for $\mathbb{C}^d$ given by the subsets,

$$A \cup \{b\}, \quad A \cup \{bq\}, \quad b \in \mathbb{Z}_n, \quad b \notin A, \quad q \in \mathbb{Z}_n \setminus \{0\}$$

of $\mathbb{Z}_n$ are unitarily equivalent and not multiplicatively equivalent if for some $m \in \mathbb{Z}_n$,

$$qb \equiv b \ mod \ \frac{n}{p}, \qquad mqb \equiv qb - b \ mod \ n, \qquad m \not\equiv 1 \ mod \ p$$

and

$$zb \neq zbq \text{ for all } z \in \mathbb{Z}_n^* \text{ such that } zA = A. \tag{4.4.1}$$

In particular, if $b \in \mathbb{Z}_n^*$, then $q \notin \mathbb{Z}_n^* \cup \{0\}$ is sufficient.

Proof. Lemma 4.28 gives the result in one direction. For the converse, we note that the extra condition (4.4.1) added to Lemma 4.28, by definition implies the sets cannot be multiplicatively equivalent. □

Example 4.31. *($n = 9$, $d = 4$) For $\mathbb{Z}_9$, the following multiplicative equivalence classes of 4–element subsets give cyclic frames with the same angles*

$$\big\{\{1,4,6,7\},\{2,3,5,8\}\big\}, \qquad \big\{\{1,3,4,7\},\{2,5,6,8\}\big\}.$$

The common angle multiset is

$$\{\omega^3, \omega^3, \omega^3, \omega^6, \omega^6, \omega^6, 1+3\omega^3, 1+3\omega^6\}, \qquad \omega := e^{2\pi i/9}.$$

It can be verified by Theorem 4.30 that the frames these give are unitarily equivalent (but not via an automorphism). Here the permutation σ is

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 \\ 1 & 3 & 2 & 4 & 6 & 5 & 7 & 0 & 8 \end{pmatrix} \qquad (\text{ for } \ \{1,4,6,7\} \ \text{ and } \ \{1,3,4,7\} \ ).$$

Lemma 4.32. *Let $n = p^3 m$, p prime, $m \in \mathbb{N}$, and*

$$A := \frac{n}{p}\mathbb{Z}_n + a = \{a, \tfrac{n}{p} + a, \tfrac{2n}{p} + a, \dots, (p-1)\tfrac{n}{p} + a\}, \quad a \in \{1, 2, \dots, p-1\},$$

$$B_1 := \{p, pz_1, \dots, pz_m\}, \qquad z_j \in \mathbb{Z}_n^*,$$
$$B_2 := \{p + \tfrac{n}{p}, pz_1 + \tfrac{n}{p}, \dots, pz_m + \tfrac{n}{p}\},$$

with $A \cap B_1 = A \cap B_2 = \emptyset$ and $B_1 \neq B_2$. Then the frames generated by the sets $A \cup B_1$ and $A \cup B_2$, are unitarily equivalent if $z_j \equiv 1$ mod p for all $j \in \{1, \dots, m\}$.

Proof. We will call the two frames (f_i) and (g_i) respectively.
Define the permutation as follows

$$\sigma j := j - j'\frac{n}{p^2}, \qquad j' := j \bmod \frac{n}{p}.$$

To show this is well defined, we will prove that σ is injective. Observe

$$\sigma j = \sigma k \iff j' - j'\frac{n}{p^2} + \frac{n}{p}x_j = k' - k'\frac{n}{p^2} + \frac{n}{p}x_k$$

If $j \equiv k \bmod \frac{n}{p}$, then

$$\sigma j = \sigma k \implies x_j = x_k \implies j = k$$

If $j \not\equiv k \bmod \frac{n}{p}$, suppose for a contradiction that $\sigma j = \sigma k$. Then,

$$0 = \sigma j - \sigma k = j' - j'\frac{n}{p^2} + \frac{n}{p}x_j - k' + k'\frac{n}{p^2} - \frac{n}{p}x_k,$$

if and only if,

$$(j' - k')(1 - \tfrac{n}{p^2}) = -\frac{n}{p}(x_j - x_k).$$

Let $d = \frac{n}{p^2}$. Then $\frac{n}{p} = pd$. Note $1 - d = -(d-1)$ and d is coprime to $d-1$. Since $p^3 \mid n$, $d = \frac{n}{p^2} = pk$ so that $d - 1 = pk - 1$ is coprime to p. Hence $\gcd(1-d, pd) = 1$, i.e., $1 - \frac{n}{p^2}$ is a unit in $\mathbb{Z}_{\frac{n}{p}}$. It follows that

$$(j' - k')(1 - \tfrac{n}{p^2}) = (j' - k')(1 - d) \equiv 0 \bmod \frac{n}{p},$$

if and only if,

$$(j' - k')(1 - d)(1 - d)^{-1} \equiv 0 \bmod \frac{n}{p} \iff j' - k' \equiv 0 \bmod \frac{n}{p},$$

if and only if,

$$j' \equiv k' \bmod \frac{n}{p},$$

a contradiction. Therefore, σ is a permutation.

Now we show the permutation σ induces a unitary equivalence. We will again proceed similarly to the above proofs. The argument will proceed on the individual z_j involved, but for simplicity we will use z to denote it.

Case (a). $j - k \equiv 0 \bmod p$.
Note that $\sigma j - \sigma k = j - k - (j' - k')\frac{n}{p^2} = j - k$. It follows that

$$\omega^{a(j-k)} \sum_{t=0}^{p-1} \zeta^{j-k} = \omega^{a(\sigma j - \sigma k)} \sum_{t=0}^{p-1} (\zeta^{\sigma j - \sigma k})^t = p\omega^{a(j-k)}.$$

But also for $z \in \mathbb{Z}_n^*$,

$$\begin{aligned} \omega^{(pz+\frac{n}{p})(\sigma j - \sigma k)} &= \omega^{(pz+\frac{n}{p})\left(j-k-(j'-k')\frac{n}{p^2}\right)} = \omega^{(pz+\frac{n}{p})(j-k)} \\ &= \omega^{pz(j-k)+\frac{n}{p}(j-k)} = \omega^{pz(j-k)+\frac{n}{p}ps} = \omega^{pz(j-k)} \end{aligned}$$

Case (b). $j - k \not\equiv 0 \bmod p$.
As $p^3 \mid n$, $\sigma j - \sigma k = j - k - (j' - k')\frac{n}{p^2} \equiv j - k \bmod p$, and $(j - k, p) = 1$,

$$\sum_{t=0}^{p-1} (\zeta^{\sigma j - \sigma k})^t = \sum_{t=0}^{p-1} (\zeta^{j-k})^t = 0.$$

But also for $z \in \mathbb{Z}_n^*$,

$$\begin{aligned} \omega^{(pz+\frac{n}{p})(\sigma j - \sigma k)} &= \omega^{pz(\sigma j - \sigma k)+\frac{n}{p}(\sigma j - \sigma k)} \\ &= \omega^{pz(j-k)-pz(j'-k')(\frac{n}{p^2})+\frac{n}{p}(j-k)-\frac{n}{p}(j'-k')(\frac{n}{p^2})} \\ &\equiv \omega^{pz(j-k)-pz(j'-k')(\frac{n}{p^2})+\frac{n}{p}(j-k)} \\ &\equiv \omega^{pz(j-k)-\frac{n}{p}(j-k)+\frac{n}{p}(j-k)} \\ &\equiv \omega^{pz(j-k)} \bmod n. \end{aligned}$$

In both cases we have shown that $\langle g_{\sigma j}, g_{\sigma k} \rangle = \langle f_j, f_k \rangle$. i.e., σ induces a unitary equivalence between the two frames. $\square$

Lemma 4.33. *Let $n = p^3 m$, p prime, $m \in \mathbb{N}$, and*

$$A := \frac{n}{p}\mathbb{Z}_n + a = \{a, \tfrac{n}{p}+a, \tfrac{2n}{p}+a, \ldots, (p-1)\tfrac{n}{p}+a\}, \quad a \in \{1, 2, \ldots, p-1\} \cap \mathbb{Z}_n^*,$$

$$B_1 := \{p, pz_1, \ldots, pz_m\}, \qquad z_j \in \mathbb{Z}_n^*,$$
$$B_2 := \{p + \tfrac{n}{p}, pz_1 + \tfrac{n}{p}, \ldots, pz_m + \tfrac{n}{p}\},$$

with $A \cap B_1 = A \cap B_2 = \emptyset$ and $B_1 \neq B_2$. Then the frames generated by the sets $A \cup B_1$ and $A \cup B_2$, are not multiplicatively equivalent.

Proof. Without loss of generality, assume $a = 1$. If it such a $u \in \mathbb{Z}_n^*$ exists to make the two frames multiplicatively equivalent, observe that we must have $u = \frac{sn}{p} + 1$ for some s. Then

$$upz_j = (\tfrac{sn}{p} + 1)pz_j = pz_j$$

for each z_j, i.e., all elements of the set are fixed. Therefore the two sets are not multiplicatively equivalent. □

Theorem 4.34. *Let $n = p^3 m$, p prime, $m \in \mathbb{N}$, and*

$$A := \frac{n}{p}\mathbb{Z}_n + a = \{a, \tfrac{n}{p}+a, \tfrac{2n}{p}+a, \dots, (p-1)\tfrac{n}{p}+a\}, \quad a \in \mathbb{Z}_n^* \cap \{1, 2, \dots, p-1\}$$

$$B_1 := \{p, pz_1, \dots, pz_m\}, \qquad z_j \in \mathbb{Z}_n^*,$$
$$B_2 := \{p + \tfrac{n}{p}, pz_1 + \tfrac{n}{p}, \dots, pz_m + \tfrac{n}{p}\},$$

with $A \cap B_1 = A \cap B_2 = \emptyset$ and $B_1 \neq B_2$ as before. Then the frames generated by $A \cup B_1$ and $A \cup B_2$, with $z_j \equiv 1 \bmod p$ for all $j \in \{1, \dots, m\}$ are unitarily equivalent and not multiplicatively equivalent.

Proof. Apply Lemma 4.32 and Lemma 4.33. □

Example 4.35. *($n = 16$, $d = 4$). The harmonic frames generated by subsets $\{1, 6, 9, 10\}$ and $\{1, 2, 9, 14\}$ are unitarily equivalent but not via an automorphism by Theorem 4.34. Here, the elements $1, 9$ correspond to the set A, and $2 \times 3 = 6$, $2 \times 5 = 10$, $2 \times 1 = 2$, $2 \times 7 = 14$ correspond to the various bz_j, with $b = 2$.*

Definition 4.36. *Let J, K be two sets which generate cyclic harmonic frames that are unitarily equivalent. An element $\ell \in \mathbb{Z}_n$ is* **permutation invariant***, if adding or removing that element from* **both** *J and K will not affect the unitary equivalence of J and K.*

Proposition 4.37. *Some permutation invariants for the permutation constructions corresponding to Theorems 4.27, 4.30, 4.34 are as follows:*

(a) For $\sigma j = j - r\frac{n}{p}$, $\ell = p^\alpha t$, $\alpha \in \mathbb{N}$, $t \in \mathbb{N}_0$.

(b) For $\sigma j = j - mj'$, $j' := j \bmod p$, $\ell = \frac{n}{m}t$, $t \in \mathbb{N}_0$.

(c) For $\sigma j = j - j'\frac{n}{p^2}$, $j' := j \bmod \frac{n}{p}$, $\ell = p^2 t$, $t \in \mathbb{N}_0$.

Proof. We show that under each type of permutation, $\omega^{\ell(j-k)} = \omega^{\ell(\sigma j - \sigma k)}$.

Case (a). Let $\ell = p^\alpha t$, $\alpha \in \mathbb{N}$, $t \in \mathbb{N}_0$. Then

$$\omega^{\ell(\sigma j - \sigma k)} = \omega^{\ell(j-k-r\frac{n}{p}j'+r\frac{n}{p}k')} = \omega^{\ell(j-k)-\ell r\frac{n}{p}j'+\ell r\frac{n}{p}k'} = \omega^{\ell(j-k)}.$$

Case (b). Let $\ell = \frac{n}{m}t$, $t \in \mathbb{N}_0$. Then

$$\omega^{\ell(\sigma j - \sigma k)} = \omega^{\ell(j-k-m(j'-k'))} = \omega^{\ell(j-k)-\ell m(j'-k')} = \omega^{\ell(j-k)}.$$

Case (c). Let $\ell = p^2 t$, $t \in \mathbb{N}_0$. Then

$$\omega^{\ell(\sigma j - \sigma k)} = \omega^{\ell(j-k-(j'\frac{n}{p^2}-k'\frac{n}{p^2}))} = \omega^{\ell(j-k)-\ell j'\frac{n}{p^2}-\ell k'\frac{n}{p^2}} = \omega^{\ell(j-k)}.$$

$\square$

Example 4.38. *Lifted frames contain permutation invariant characters in the subsets which generate them. The number* $0 \in \mathbb{Z}_n$ *which corresponds to the lifting character (trivial one) is a permutation invariant for all* σ*. In this way we can construct more types of cyclic harmonic frames where unitary equivalence is possible but not via an automorphism, e.g.,*
($n = 8$*,* $d = 4$*) We can 'lift' the Example 4.23, i.e., add* 0 *to each subset to obtain (multiplicative equivalence classes)*

$$\big\{\{0,1,2,5\},\{0,3,6,7\}\big\}, \qquad \big\{\{0,1,5,6\},\{0,2,3,7\}\big\}.$$

These are still multiplicative equivalence classes, since $m0 = 0$*,* $m \in \mathbb{Z}_n$*, and by the same reasoning are not multiplicatively equivalent. They still give the same angles, since the angle* $\theta = \omega^{aj_1} + \omega^{aj_2} + \omega^{aj_3}$ *transforms to* $\omega^0 + \omega^{aj_1} + \omega^{aj_2} + \omega^{aj_3} = 1 + \theta$*, and they are unitarily equivalent.*

Example 4.39. *A less obvious example is in* $n = 8$*,* $d = 4$ *where*

$$\big\{\{1,4,5,6\},\{2,3,4,7\}\big\}, \qquad \big\{\{1,2,4,5\},\{3,4,6,7\}\big\}.$$

Here, the σ*–invariant is* 4 *which is fixed under Theorems 4.27 and 4.30.*

The above results contribute to a classification of the cyclic harmonic frames falling outside of the criteria stipulated in Theorem 4.13. Ultimately, a full classification remains a small cove of mystery for future adventurers to explore.

4.5 Equivalences Respecting Group Structure

We saw in the last section a few families of frames which were unitarily equivalent but not via an automorphism, and hence not multiplicatively equivalent. Here, we present some families of frames which are unitarily equivalent if and only if they are multiplicatively equivalent, and conjecture about some other families which may exhibit this property.

Theorem 4.40. *Cyclic harmonic frames of n (square free) distinct vectors, generated by subsets of $\mathbb{Z}_n^*$ are multiplicatively equivalent if and only if they are unitarily equivalent.*

Proof. Let $\omega = e^{2\pi i/n}$, J, K be two subsets of $\mathbb{Z}_n^*$ (with n square free) which generate harmonic frames Φ and Ψ. One direction follows directly from Theorem 4.13.

Conversely suppose Φ and Ψ are not multiplicatively equivalent. Let ω a primitive n–th root of unity. Then $\sum_{j\in J}\omega^j$ is an angle in the angle multiset of Φ but not in Ψ. This follows from the fact that ω^j, $j \in J\cup K$ are all basis elements of the basis consisting of the primitive roots of unity (Theorem 3.18). Since the sets J, K are not multiplicatively equivalent, $\sum_{j\in J}\omega^j$ could only be an angle of Ψ if and only if there exists a $z\in\mathbb{Z}_n\setminus\mathbb{Z}_n^*$ such that, $\sum_{j\in J}\omega^j = \sum_{k\in K}\omega^{zk}$. Since $\gcd(zk,n) = \gcd(z,n)$ for all $k\in K$, we have by Corollary 3.19 that the basis representation of $\sum_{k\in K}\omega^{zk}$ is different to $\sum_{j\in J}\omega^j$, and hence this is not possible. □

Corollary 4.41. *Cyclic harmonic frames of p (prime) distinct vectors are multiplicatively equivalent if and only if they are unitarily equivalent.*

Proof. This follows immediately from the theorem as all non zero elements of Z_p are in $\mathbb{Z}_p^*$. □

Let θ be the angle map on d–element subsets of $\mathbb{Z}_n$ given by

$$\theta(J) := \sum_{j\in J}\omega^j.$$

Theorem 4.42. *Let $\mathcal{C}_d$ be the collection of d–element subsets of $\mathbb{Z}_n$ given by*

$$\mathcal{C}_d := \big\{J : \theta^{-1}(\theta(J)) = \{J\}\big\}.$$

If $J\in\mathcal{C}_d$, then J and K are give unitarily equivalent cyclic harmonic frames of distinct vectors if and only if they are multiplicatively equivalent subsets.

Proof. Suppose, by way of contradiction, that J and K are not multiplicatively equivalent. Then the angle $\theta(J) = \sum_{j\in J} \omega^j$ in the frame given by J is in the frame given by K if and only if

$$\sum_{j\in J} \omega^j = \sum_{k\in K} \omega^{bk} \quad \implies \quad J = bK \quad (\text{since } \theta \text{ is } 1\text{–}1),$$

where $b \notin \mathbb{Z}_n^*$ (since the frames are not multiplicatively equivalent). Since the frame given by J has distinct vectors, $\mathbb{Z}_n = \langle J \rangle$, and we have

$$\mathbb{Z}_n = \langle J \rangle = \langle bK \rangle \subset \langle b \rangle \neq \mathbb{Z}_n,$$

a contradiction. □

The subsets in $\mathcal{C}_d$ exhibit properties similar to a basis of a cyclotomic field. For any $J \in \mathcal{C}_d$, when the sum of all the roots of unity associated with each element in J, we get a number which is only obtainable as a sum of the roots of unity associated with that subset.

Example 4.43. *Consider the cyclic harmonic frames when $d = 2$, n odd. Here $\mathcal{C}_2$ is all 2–element subsets of $\mathbb{Z}_n$ (see Theorem 4.20).*

Example 4.44. *($n = p$ a prime). Here the p–th roots of unity are linearly independent over $\mathbb{Q}$, and form a basis, and thus $\mathcal{C}_d$ is all d–element subsets of $\mathbb{Z}_p$. Moreover, unitarily inequivalent frames share no angles.*

Conjecture 4.45. *Let $n = p_1 p_2 \cdots p_k$, $p_1 > p_2 > \ldots > p_k$ (square free). The cyclic harmonic frames of n distinct vectors in dimension less than or equal to p_1 are unitarily equivalent if and only if they are multiplicatively equivalent.*

This conjecture is based on the intuition from Theorem 3.15 which tells us the minimal vanishing sums for $\mathbb{Q}[\omega]$ (with ω an n–th root of unity), are at least of length p_1.

Conjecture 4.46. *The cyclic harmonic frames of n (square free) distinct vectors are unitarily equivalent if and only if they are multiplicatively equivalent.*

This conjecture stems from the observation in computational data which strongly suggests this is the case. No counterexample has been found. In $\mathbb{C}^3$, data strongly suggests non multiplicatively equivalent frames always have different angle multisets. In $\mathbb{C}^4$ this behaviour is no longer the case. For example, when $n = 10$, the harmonic frames generated by the subsets $\{1, 2, 4, 7\}$

and $\{1, 2, 4, 7\}$ are not multiplicatively equivalent, but share the same angle multiset $\{0, -\omega^3 - \omega^2 - 1, -\omega^3 - \omega^2, -\omega^3 + \omega^2 - 2\omega, -\omega^3 + \omega^2 - 2\omega + 1, \omega^3 - \omega^2 + 2\omega - 2, \omega^3 - \omega^2 + 2\omega - 1, \omega^3 + \omega^2 - 1, \omega^3 + \omega^2\}$. Nevertheless, despite the angle multiset overlap, they are unitarily inequivalent, so we are unable to produce a counterexample to this conjecture. It does mean if it is true, then new techniques need to be applied to prove it.

Conjecture 4.47. *The cyclic harmonic frames generated by subsets of* $\mathbb{Z}_n^*$ *are unitarily equivalent if and only if they are multiplicatively equivalent.*

The data strongly suggests multiplicatively inequivalent frames produce different angle multisets under these conditions.

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