Generalised Bernstein Operators on Polytopes

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Chapter 1

Introduction

The univariate Bernstein operator of degree k from $C[0,1]$ to $\mathbb{R}$ is defined by

$$B_k f(x) := \sum_{i=0}^{k} \binom{k}{i} x^{k-i}(1-x)^i f(\frac{i}{k}), \qquad f \in C[0,1]$$

Some well-known properties of the operator are linearity, positivity, degree reducing, shape-preserving on convex functions and uniform approximating continuous functions on $[0,1]$. The last property is particularly important as it gives a constructive proof of Weierstrass' theorem which states that the set of polynomials is dense in C[0,1]. Due to these interesting facts, the univariate Bernstein operator is used extensively in CAGD, and so is desired to be generalised to higher dimensions. Its most natural generalisation is by defining Bernstein operators on d-simplices ($d \geq 1$) which are the convex hull of $d+1$ affinely independent points in $\mathbb{R}^d$. It turns out that these generalised Bernstein operators on simplices also posess the aforementioned properties.

This dissertation will investigate the properties of the generalised version of Bernstein operators on simplices, that is, Bernstein operators on polytopes via generalised barycentric coordinates. Among those, spectral property is of main investigation. For Bernstein operators on simplices, results have been established by Cooper and Waldron (see [5]). It will be seen later that these results remains true for the generalised Bernstein operators on polytopes.

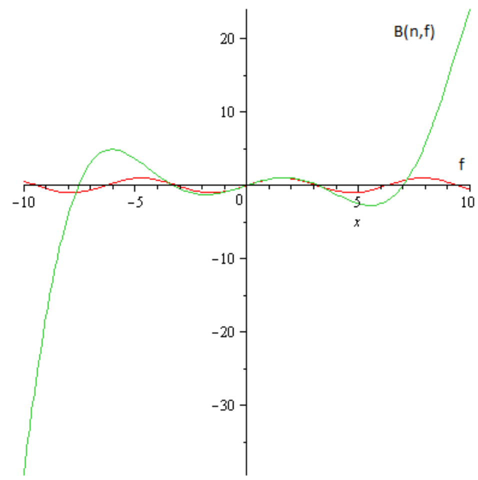

Figure 1.1: Function $f(x) = sin(x)$ and $B_{10}f(x)$

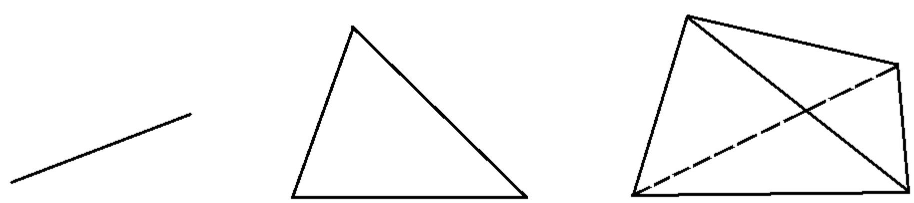

Figure 1.2: Simplices in $\mathbb{R}^d$ when $d = 1, 2, 3$

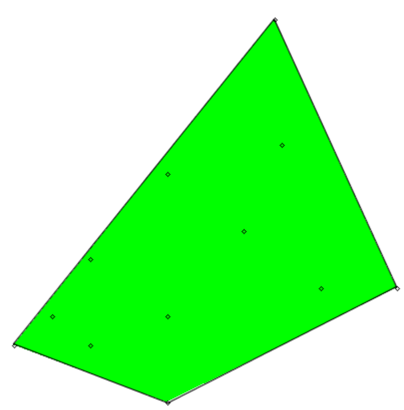

Figure 1.3: Convex hull of a sequence of points

Chapter 2

Basic facts and notation

In this chapter, we introduce some basic facts which we will need later.

2.1 Multinomial expansion

For any positive integer n and any nonnegative integer k, we have the multinomial expansion

$$(\xi_1 + \ldots + \xi_n)^k = \sum_{k_1+\ldots+k_n=k} \binom{k}{k_1, \ldots, k_n} \xi_1^{k_1} \ldots \xi_n^{k_n} \tag{2.1}$$

where

$$\binom{k}{k_1, \ldots, k_n} = \frac{k!}{k_1! \ldots k_n!}$$

For $X = \{\xi_1, \ldots, \xi_n\}$ and $\alpha \in \mathbb{Z}^X_{\geq 0}$, we can write formula (2.1) in a more compact way using standard multi-index notation as follows

$$(\xi_1 + \ldots + \xi_n)^k = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} \xi^\alpha \tag{2.2}$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

We will use (2.2) many times later.

2.2 Stirling numbers

The univariate Stirling number of the first kind $s(b, a)$ is defined via the relation

$$x(x-1)\cdots(x-a+1) = \sum_{a=0}^{b} s(b,a)x^a \tag{2.3}$$

and the univariate Stirling number of the second kind $S(b, a)$ via

$$x^b = \sum_{a=0}^{b} S(b,a)x(x-1)\cdots(x-a+1) \tag{2.4}$$

Univariate Stirling numbers of the first and second kinds can be considered as inverses of each other due to the following theorem, which can be obtained by substituting 2.3 into 2.4 (for details, see [3]):

Theorem 2.2.1 (Inversion relation of Stirling numbers of the 1^{st} and 2^{nd} kinds)**.** *For a,b,c* $\in \mathbb{Z}_{\geq 0}$

$$\sum_{0 \leq c \leq max\{b,a\}} (-1)^{|b-c|} s(b,c)S(c,a) = \delta_{ba}$$

where δ_{ba} *denotes Kronecker's delta.*

We will now derive the multivariate version of the above result and prove some lemmas related to Stirling numbers for later use.
For α,$\beta \in \mathbb{Z}_{\geq 0}^X$ and $h > 0$, define

- Multivariate shifted factorial with stepsize h by $[\xi]_h^\beta := \prod_i [\xi_i]_h^{\beta_i}$ where $[\xi_i]_h^{\beta_i} := \xi_i(\xi_i - h)\cdots(\xi_i - (\beta_i - 1)h)$
- Multivariate Stirling number of the 1^{st} kind by $s(\beta, \alpha) := \prod_i s(\beta_i, \alpha_i)$
- Multivariate Stirling number of the 2^{nd} kind by $S(\beta, \alpha) := \prod_i S(\beta_i, \alpha_i)$

Then we have the following:

Lemma 2.2.2.

$$[\xi]_h^\beta = \sum_{\alpha \leq \beta} (-h)^{|\beta - \alpha|} s(\beta, \alpha)\xi^\alpha$$

$$\xi^\beta = \sum_{\alpha \leq \beta} (h)^{|\beta - \alpha|} S(\beta, \alpha)[\xi]_h^\alpha$$

Proof. For the first equality,

$$\begin{aligned}
[\xi]_h^\beta &= \prod_i \sum_{\alpha_i \le \beta_i} (-h)^{|\beta_i - \alpha_i|} s(\beta_i, \alpha_i) \xi^{\alpha_i} \\
&= \sum_{\alpha \le \beta} \prod_i (-h)^{|\beta_i - \alpha_i|} s(\beta_i, \alpha_i) \xi^{\alpha_i} \\
&= \sum_{\alpha \le \beta} (-h)^{|\beta - \alpha|} s(\beta, \alpha) \xi^\alpha
\end{aligned}$$

The second equality is obtained in a similar manner. □

Lemma 2.2.3. *For* α,β,$\gamma \in \mathbb{Z}^P_{\ge 0}$,

$$\sum_{0 \le \gamma_i \le max\{\beta_i, \alpha_i\}} (-1)^{|\beta - \gamma|} s(\beta, \gamma) S(\gamma, \alpha) = \delta_{\beta\alpha}$$

where $\delta_{\beta\alpha} = \prod_i \delta_{\beta_i \alpha_i}$.

Proof. Using inversion relation of univariate Stirling numbers to yield

$$\begin{aligned}
&\sum_{0 \le \gamma_i \le max\{\beta_i, \alpha_i\}} (-1)^{|\beta - \gamma|} s(\beta, \gamma) S(\gamma, \alpha) \\
=& \sum_{0 \le \gamma_i \le max\{\beta_i, \alpha_i\}} \prod_i (-1)^{|\beta_i - \gamma_i|} s(\beta_i, \gamma_i) S(\gamma_i, \alpha_i) \\
=& \prod_i \sum_{0 \le \gamma_i \le max\{\beta_i, \alpha_i\}} (-1)^{|\beta_i - \gamma_i|} s(\beta_i, \gamma_i) S(\gamma_i, \alpha_i) \\
=& \prod_i \delta_{\beta_i \alpha_i} \\
=& \delta_{\beta\alpha}
\end{aligned}$$

□

Lemma 2.2.4. *Let* α, $\tau \in \mathbb{Z}^P_{\ge 0}$. *Suppose* $|\alpha| \ge |\tau| > 0$, *then*

$$\alpha^\tau \binom{|\alpha|}{\alpha} = \sum_{\beta \le \tau} [|\alpha|]_1^{|\beta|} S(\tau, \beta) \binom{|\alpha| - |\beta|}{\alpha - \beta}$$

Proof.

$$
\begin{aligned}
&\sum_{\beta\le\tau}[|\alpha|]_1^{|\beta|}S(\tau,\beta)\binom{|\alpha|-|\beta|}{\alpha-\beta}\\
&=\sum_{\beta\le\tau}[|\alpha|]_1^{|\beta|}S(\tau,\beta)\times\frac{(|\alpha|-|\beta|)!}{(\alpha-\beta)!}\\
&=\sum_{\beta\le\tau}[\alpha]_1^{|\beta|}S(\tau,\beta)\times\frac{(|\alpha|)!}{(\alpha)!}\\
&=\alpha^\tau\binom{|\alpha|}{\alpha}
\end{aligned}
$$

□

The next lemma plays an important role in my proof of the main theorem in Chapter 4. It basically says that a certain sum of terms of a specific order can be presented as a linear combination of lower ordered terms.

Lemma 2.2.5. *Let α, $\tau\in Z^P_{\ge 0}$ with $|\alpha|=k\in\mathbb{N}$. Suppose $|\alpha|\ge|\tau|>0$, then*

$$\sum_{|\alpha|=k}\alpha^\tau\binom{|\alpha|}{\alpha}\xi^\alpha=\sum_{\beta\le\tau}[|\alpha|]_1^{|\beta|}S(\tau,\beta)\xi^\beta$$

Proof. Following from lemma 2.2.4, we have

$$
\begin{aligned}
&\sum_{|\alpha|=k}\alpha^\tau\binom{|\alpha|}{\alpha}\xi^\alpha\\
&=\sum_{|\alpha|=k}\Big(\sum_{\beta\le\tau}[|\alpha|]_1^{|\beta|}S(\tau,\beta)\binom{|\alpha|-|\beta|}{\alpha-\beta}\Big)\xi^\alpha\\
&=\sum_{\beta\le\tau}[|\alpha|]_1^{|\beta|}S(\tau,\beta)\xi^\beta\sum_{\substack{|\alpha|=k\\ \alpha\ge\beta}}\binom{|\alpha|-|\beta|}{\alpha-\beta}\xi^{\alpha-\beta}\\
&=\sum_{\beta\le\tau}[|\alpha|]_1^{|\beta|}S(\tau,\beta)\xi^\beta
\end{aligned}
$$

□

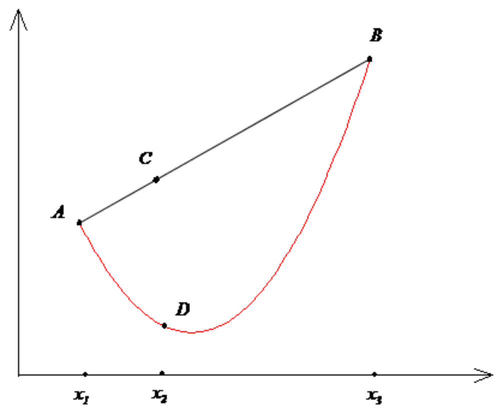

Figure 2.1: Convex function

2.3 Convex functions

Definition 2.3.1. *A function f is said to be convex on a region T if for any x_1,$x_2 \in T$ and for any $t \in [0,1]$,*

$$tf(x_1) + (1-t)f(x_2) \geq f(tx_1 + (1-t)x_2) \tag{2.5}$$

Lemma 2.3.2. *A function f is convex on T if and only if*

$$\sum_{i=1}^{n} t_i f(x_i) \geq f\Big(\sum_{i=1}^{n} t_i x_i\Big) \tag{2.6}$$

for all $n \geq 1$, $x_i \in T$ and $t_i \geq 0$ such that $\sum_i t_i = 1$.

Proof. We prove this by induction.

If $k = 1$ or $k = 2$, the statement is obviously true.

Now suppose it is also true for $k = n-1$. Consider $\sum_{i=1}^{n} t_i f(x_i)$:

If $t_1 = \cdots = t_{n-1} = 0$, the statement is true.

If $\exists t_i \neq 0$ for some $i \neq n$ then

$$\begin{aligned}
\sum_{i=1}^{n} t_i f(x_i) \geq f\Big(\sum_{i=1}^{n} t_i x_i\Big) &= (1-t_n)\sum_{i=1}^{n-1} \frac{t_i}{1-t_n} f(x_i) + t_n f(x_n) \\
&\geq (1-t_n) f\Big(\sum_{i=1}^{n-1} \frac{t_i}{1-t_n} x_i\Big) + t_n f(x_n) \\
&\geq f\Big((1-t_n)\sum_{i=1}^{n-1} \frac{t_i}{1-t_n} x_i + t_n x_n\Big) \\
&= f\Big(\sum_{i=1}^{n} t_i x_i\Big)
\end{aligned}$$

Hence the statement is true for all n by induction principle. $\square$

Chapter 3

Generalised barycentric coordinates

3.1 Barycentric coordinates

Let $P = (p_1, \ldots, p_n)$ be a sequence of points in $\mathbb{R}^d$. We say that P is *affinely independent* if each point $x \in \mathbb{R}^d$ can be written uniquely as an affine combination of these points, i.e,

$$x = \sum_j \xi_j(x) p_j, \qquad \sum_j \xi_j(x) = 1.$$

The functions ξ_j so defined are called *barycentric coordinates.*
Barycentric coordinates are nonnegative on the simplex given by the convex hull of P (See section 3.4).

3.2 Generalised barycentric coordinates

The goal of this section is to introduce functions which are similar to those discussed above, which we will call *generalised barycentric coordinates.*

For a sequence of vectors $(v_1, \dots, v_n)$ in $\mathbb{R}^d$, define the *synthesis map* by

$$V : \mathbb{R}^n \to \mathbb{R}^d : c \mapsto \sum_j c_j v_j$$

Its adjoint is

$$V^* : \mathbb{R}^d \to \mathbb{R}^n : u \mapsto (\langle u, v_j \rangle)_{j=1}^n$$

Recall from [7]:

Definition 3.2.1. *The generalised barycentric coordinates $\xi = (\xi_j)$ of a point x in $\mathbb{R}^d$ with respect to a sequence of points $P = (p_1, \dots, p_n)$ having affine hull $\mathbb{R}^d$ are given by*

$$\xi_j(x) := \langle x - c, \tilde{p_j} - c \rangle + \frac{1}{n} \tag{3.1}$$

where

$$\tilde{p_j} := (VV^*)^{-1} v_j + c, \qquad v_j := p_j - c, \qquad c := \frac{1}{n} \sum_j p_j$$

It was also shown in [7] that such generalised barycentric coordinates (ξ_j) are unique coefficients of minimal ℓ_2-norm for which

$$x = \sum_j \xi_j(x) p_j, \qquad \sum_j \xi_j(x) = 1$$

Example 3.2.2. If $P = (p_1, \dots, p_n)$ is a sequence of points in $\mathbb{R}$ then

$$\xi_j(x) = \frac{p_j - c}{\sum_k |p_k - c|^2}(x - c) + \frac{1}{n}$$

3.3 Properties

From formula (3.1), it can be seen that

- The coordinates of the barycentre c are $\xi_j(c) = \frac{1}{n} \quad \forall j$.
- The functions ξ_j are constant (and equal to $\frac{1}{n}$) $\Leftrightarrow p_j \equiv c$.
- $\xi_i = \xi_j \Leftrightarrow p_i \equiv p_j$.
- The ξ_j are continuous functions of the points $\{p_1, \dots, p_n\}$.

These imply that the (closed) set of points where the coordinates are nonnegative

$$N_P := \{x \in \mathbb{R}^d : \xi_j(x) \geq 0 \, \forall j\}$$

is the convex polytope having the barycentre as an interior point (see section 3.4).

Denote $P \backslash \{p_j\} := (p_1, \ldots, p_{j-1}, p_j, \ldots, p_n)$ and $\mathrm{Aff}(P) := \{\sum_j \xi_j p_j : \sum_j \xi_j = 1\}$.

We will now give some properties of the generalised barycentric coordinates.

Theorem 3.3.1. *The generalised barycentric coordinates* (ξ_j) *satisfy the following*

(a) *If* $\sum_{k=1}^n a_k = 1$, *then* $\xi_j(\sum_k a_k p_k) = \sum_k a_k \xi_j(p_k) \quad \forall j \in \{1, \ldots, n\}$.

(b) *If* $x = \sum_{k=1}^n \xi_k(x) p_k$, *then* $\sum_j \xi_k(p_j)\xi_j(x) = \xi_k(x)$.

(c) $\frac{1}{n} < \xi_j(p_j) \leq 1$.

(d) $\xi_j(p_k) = \xi_k(p_j)$.

(e) $\xi_j(p_j) = 1 \Leftrightarrow p_j \notin \mathit{Aff}(P \setminus \{p_j\})$, *which implies* $\xi_j = 0$ *on* $\mathit{Aff}(P \setminus \{p_j\})$.

(f) $\sum_j S(p_j) = d + 1$ *where* $S(x) := \sum_k \lambda_k^2(x)$.

Proof.

(a) This follows from formula (3.1)

$$\begin{aligned}
\xi_j\big(\sum_k a_k p_k\big) &= \big\langle \sum_k a_k p_k - c, \tilde{p_j} - c \big\rangle + \frac{1}{n} = \big\langle \sum_k a_k(p_k - c), \tilde{p_j} - c\big\rangle + \frac{1}{n} \\
&= \sum_k a_k \langle p_k - c, \tilde{p_j} - c\rangle + \frac{1}{n} \; = \sum_k a_k(\langle p_k - c, \tilde{p_j} - c + \frac{1}{n}\rangle) \\
&= \sum_k a_k \xi_j(p_k)
\end{aligned}$$

(b) Apply ξ_k to both side of the equation $x = \sum_{k=1}^n \xi_k(x) p_k$ to obtain

$$\xi_k(x) = \xi_k(\sum_{k=1}^n \xi_k(x) p_k) = \sum_j \xi_k(p_j)\xi_j(x)$$

Note that the last equation follow from (a).

(c),(d),(e),(f) See [7] □

Theorem 3.3.2. *Let* $P = (p_1, \ldots, p_n)$ *be a sequence of points in* $\mathbb{R}^d$ *with affine hull* $\mathbb{R}^d$. *Then we can rearrange* P *into the sequence* $Q = (q_1, \ldots, q_n)$ *such that* *the set of points $S = \{q_1, \ldots, q_{d+1}\}$ is affinely independent. Suppose $(\mu_i)_{i=1}^n$ is the generalised barycentric coordinates of a point $x \in \mathbb{R}^d$ with respect to Q, then $\{\mu_i\}_{i=1}^{d+1}$ is a linearly independent set.*

Proof. Since $\mathrm{Aff}(P) = \mathbb{R}^d$, there must exist a set of $d+1$ affinely independent points. So such a rearrangment is possible.
Suppose $\{\mu_1, \ldots, \mu_{d+1}\}$ is not linearly independent. Then

$$\mu_i(x) = \sum_{\substack{j=1 \\ j\neq i}}^{d+1} a_j \mu_j(x) \text{ for some } i \in \{1, \ldots, d+1\}$$

Let $x = c$ then

$$\mu_i(c) = \sum_{\substack{j=1 \\ j\neq i}}^{d+1} a_j \mu_j(c) \Rightarrow \frac{1}{n} = \sum_{\substack{j=1 \\ j\neq i}}^{d+1} a_j \frac{1}{n} \Rightarrow \sum_{\substack{j=1 \\ j\neq i}}^{d+1} a_j = 1$$

Let $x = p_k$ then

$$\mu_k(p_i) = \mu_i(p_k) = \sum_{\substack{j=1 \\ j\neq i}}^{d+1} a_j \mu_j(p_k) = \mu_k\Big(\sum_{\substack{j=1 \\ j\neq i}}^{d+1} a_j p_j\Big)$$

where the last equation follows from theorem 3.3.1(a).
Therefore,

$$\Big(p_i - \sum_{\substack{j=1 \\ j\neq i}}^{d+1} a_j p_j\Big) \perp (\tilde{p_k} - c) \qquad \forall k \in \{1, \ldots, n\}$$

which implies

$$p_i - \sum_{\substack{j=1 \\ j\neq i}}^{d+1} a_j p_j = 0 \qquad \text{or} \qquad p_i = \sum_{\substack{j=1 \\ j\neq i}}^{d+1} a_j p_j$$

which is absurd since $S = \{q_1, \ldots, q_{d+1}\}$ is affinely independent. □

3.4 Region of nonnegativity

The *region of nonnegativity* for a sequence of points $P = (p_1, \ldots, p_n)$ is defined as

$$N_P := \{x \in \mathbb{R}^d : \xi_j \geq 0 \ \forall j\}$$

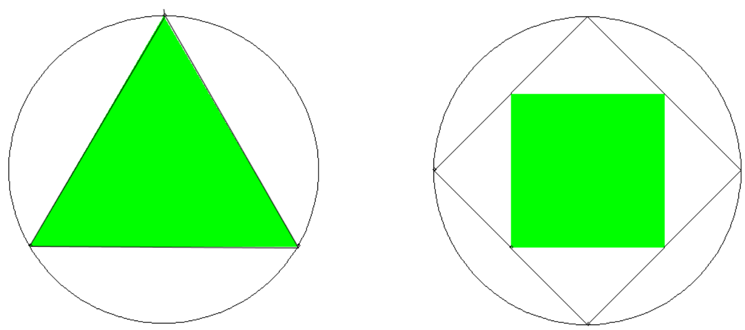

Figure 3.1: Unit triangle and unit square with their region of nonnegativity shaded

Since $\xi_j(c) = \frac{1}{n}\ \forall j$, the barycentre c is contained in N_P.
Define the half space

$$H_{j,P} := \{x \in \mathbb{R}^d : \xi_j(x) \geq 0\}$$

Then,

$$N_P = \bigcap_{\{j : p_j \neq c\}} H_{j,P}$$

Hence, the region of nonnegativity N_P is the bounded convex polytope formed by the half spaces (called the convex hull of P) containing the barycentre c.
If P consists of exactly $d+1$ points which are vertices of a simplex, then the region of nonnegativity will be the entire simplex. One way to see this is by noting that $\xi_j(p_j) = 1\ \forall j$ and $\xi_k(p_j) = 0\ \forall k \neq j$, which implies $\{p_1, \ldots, p_n\} \subseteq N_P$.

3.5 Generalised Bernstein operators

By the mutinomial expansion (2.2),

$$1 = (\xi_1 + \ldots + \xi_n)^k = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} \xi^\alpha$$

Let $B_\alpha := \binom{|\alpha|}{\alpha}\xi^\alpha$, then the set $\{B_\alpha : |\alpha| = k\}$ form a partition of unity. Also, theorem 3.3.2 implies that $\{B_\alpha : |\alpha| = k\}$ are linearly independent if and only

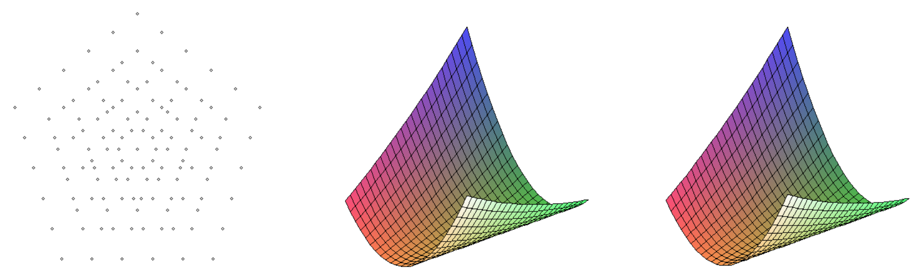

Figure 3.2: Points v_α when $k = 5$, $n = 5$ with function $f = \xi_1^2$ and $B_{k,P}f$

if $n = d + 1$.

Denote T to be the convex hull of the sequence of points $P = (p_1, \dots, p_n)$. We will now give the definition of the *generalised Bernstein operators on polytopes.*

Definition 3.5.1. *The generalised multivariate Bernstein operator of degree $k \geq 1$ is defined as*

$$B_{k,P}f := \sum_{|\alpha|=k} B_\alpha f(v_\alpha)$$

where $f \in C(T)$ is a continuous function on T and $v_\alpha := \sum_j \frac{\alpha_j}{|\alpha|} p_j$.

For any generalised barycentric coordinate ξ_l, it is observed that

$$B_{k,P}(\xi_l) = \sum_{|\alpha|=k} B_\alpha \xi_l(v_\alpha) = \sum_{|\alpha|=k} B_\alpha \xi_l(\sum_j \frac{\alpha_j}{|\alpha|} p_j)$$

Since ξ_l is affine, $\xi_l(\sum_j \frac{\alpha_j}{|\alpha|} p_j) = \sum_j \frac{\alpha_j}{|\alpha|} \xi_l(p_j)$. Substitute this in the above equation and exchange the summations to obtain

$$\begin{aligned}
B_{k,P}(\xi_l) &= \sum_j \xi_l(p_j)\xi_j \sum_{\substack{|\alpha|=k \\ \alpha_j>0}} \frac{\alpha_j}{|\alpha|} \frac{|\alpha|!}{\alpha!} \xi^{\xi - e_j} \\
&= \sum_j \xi_l(p_j)\xi_j \sum_{|\beta|=k-1} \frac{(|\alpha|-1)!}{\beta!} \xi^\beta = \sum_j \xi_l(p_j)\xi_j \\
&= \xi_l
\end{aligned}$$

where $e_j(p_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

The last equation follows from lemma 3.3.1.

Hence, $B_{k,P}$ reproduces the linear polynomials $\prod_1 = \mathrm{span}\{\xi_l\}$. This is a typical property of Bernstein operators and is a special case of theorem 4.1.1 in the next chapter.

Chapter 4

Diagonalisation

This chapter presents my main work on the generalised Bernstein operators discussed earlier. In particular, I will show that these operators are diagonalisable.

4.1 Generalised Bernstein operators are diagonalisable

Let $\Pi_k := \{\xi^\alpha : |\alpha| \leq k\}$ be the sets of all monomials in the generalised barycentric coordinates. We will now show that the generalised Bernstein polynomials of monomials of order k can be written as linear combinations of mononials in Π_k, which implies the operator is degree reducing. If we present this in a matrix form with respect to Π_k, then the operator corresponds to a triangular matrix, hence the term *diagonalisation*.

For simplicity, we only prove the diagonalisation of Bernstein operator in $\mathbb{R}^d$ when $d = 1$. The proof can then be extended verbatim to higher dimensions.

Theorem 4.1.1 (Degree reducing)**.** *For $k \geq m$ and any generalised barycentric coordinate ξ_l*

$$B_{k,P}\xi_l^m = \frac{[k]_1^m}{k^m}\xi_l^m + \sum_{\substack{|\tau|=m \\ \beta<\tau}} \binom{m}{\tau} \frac{[k]_1^{|\beta|}}{k^m} S(\tau,\beta)\xi^\tau(p_l)\xi^\beta$$

Proof. By definition of Bernstein operator,

$$B_{k,P}\xi_l^m = \sum_{|\alpha|\leq k} \binom{|\alpha|}{\alpha} \xi^\alpha \xi_l^m \Big(\sum_j \frac{\alpha_j}{|\alpha|} p_j \Big) \tag{4.1}$$

Since generalised barycentric coordinates are affine, we have

$$\begin{aligned} \xi_l^m \Big(\sum_j \frac{\alpha_j}{|\alpha|} p_j \Big) = \Big(\sum_j \frac{\alpha_j}{|\alpha|} \xi_l(p_j) \Big)^m &= \sum_{|\tau|=m} \binom{m}{\tau} \frac{\alpha^\tau}{|\alpha|^m} \xi_l^{\tau_1}(p_1) \dots \xi_l^{\tau_n}(p_n) \\ &= \sum_{|\tau|=m} \binom{m}{\tau} \frac{\alpha^\tau}{|\alpha|^m} \xi^\tau(p_l) \end{aligned}$$

where the last equation follows from the fact that $\xi_i(p_j) = \xi_j(p_i)$ (see lemma 3.3.1) Substituting this into (4.1) and exchanging the summations yield

$$B_{k,P}\xi_l^m = \sum_{|\tau|=m} \binom{m}{\tau} \frac{\alpha^\tau}{|\alpha|^m} \xi^\tau(p_l) \Big(\sum_{|\alpha|\leq k} \binom{|\alpha|}{\alpha} \xi^\alpha \Big)$$

By lemma 2.2.5, $\sum_{|\alpha|\leq k} \binom{|\alpha|}{\alpha} \xi^\alpha = \sum_{\beta\leq\tau} \frac{[k]_1^{|\beta|}}{k^m} S(\tau,\beta)\xi^\beta\xi^\alpha$
Therefore,

$$B_{k,P}\xi_l^m = \frac{[k]_1^m}{k^m} \sum_{|\tau|=m} \binom{m}{\tau} \xi^\tau(p_l)\xi^\tau + \sum_{\substack{|\tau|=m \\ \beta<\tau}} \binom{m}{\tau} \frac{[k]_1^{|\beta|}}{k^m} S(\tau,\beta)\xi^\tau(p_l)\xi^\tau \tag{4.2}$$

Note that $\frac{[k]_1^m}{k^m} \sum_{|\tau|=m} \binom{m}{\tau} \xi^\tau(p_l)\xi^\tau = \frac{[k]_1^m}{k^m} \Big(\sum_i \xi_l(p_i)\xi_i \Big)^m = \frac{[k]_1^m}{k^m} \xi_l^m$.
Substitute this into (4.2) to yield the result. □

We now state the multivariate version of the above theorem. For a proof, see our paper [8].

Theorem 4.1.2. *For* $\mu,\tau_i \in \mathbb{Z}_{\geq 0}^P$, $i \in \{1,\dots,n\}$ *and* $k \geq |\mu|$,

$$B_{k,P}\xi^\mu = \frac{[k]_1^{|\mu|}}{k^{|\mu|}} \xi^\mu + \sum_{|\beta|<|\mu|} \frac{[k]_1^{|\beta|}}{k^{|\mu|}} a(\beta,\mu)\xi^\beta \tag{4.3}$$

where

$$a(\beta,\mu) := \sum_{\substack{|\tau_i|=\mu_i \\ \tau=\tau_1+\cdots+\tau_n \\ \beta<\tau}} S(\tau,\beta) \binom{\mu_1}{\tau_1} \xi^{\tau_1}(p_1) \cdots \binom{\mu_n}{\tau_n} \xi^{\tau_n}(p_n) \tag{4.4}$$

Corollary 4.1.3. *Let f be any polynomial. Then $B_{k,P}f \to f$ uniformly on the convex hull of P.*

Proof. For monomials f, this follows directly from 4.3 since $lim_{k\to\infty}\lambda_{|\mu|}^{(k)} = 1$ and $lim_{k\to\infty}\frac{[k]_1^{|\beta|}}{k^{|\mu|}} = 0$. In addition, note that $B_{k,P}$ is linear. $\square$

We are now ready to establish the diagonalisability of the generalised Bernstein operators. For a proof, see our paper [8].

Theorem 4.1.4 (Diagonalisation). *The generalised Berntein operator $B_{k,P}$ is diagonalisable, with eigenvalues*

$$\lambda_m^{(k)} := \frac{[k]_1^m}{k^m} = \frac{k!}{(k-m)!k^m},\ k = 1,\dots,m, \qquad 1 = \lambda_1^{(k)} > \lambda_2^{(k)} > \dots > \lambda_k^{(k)} > 0$$

Let $P_{m,V}^{(k)}$ denote the $\lambda_m^{(k)}$-eigenspace. Then

$$P_{1,V}^{(k)} = \Pi_1(\mathbb{R}^s), \qquad \forall m$$

For $m > 1$, $P_{m,V}^{(k)}$ consists of polynomials of exact degree m, and is spanned by

$$p_{\xi^\mu}^{(k)} = \xi^\mu + \sum_{|\beta|<|\mu|} c(\beta,\mu,k)\xi^\beta, \qquad |\mu| = m$$

where the coefficients can be calculated using (4.4) and the recurrence formula

$$c(\beta,\mu,k) := \frac{a(\beta,\mu)}{1-|\mu|}, \qquad |\beta| = |\mu|-1$$

$$c(\beta,\mu,k) := \frac{k^{|\beta|}}{\lambda_{|\mu|}^{(k)} - \lambda_{|\beta|}^{(k)}}\Big(\frac{a(\beta,\mu)}{k^{|\mu|}} + \sum_{|\beta|<|\gamma|<|\mu|} c(\gamma,\mu,k)\frac{a(\beta,\gamma)}{k^{|\gamma|}}\Big),\ |\beta| < |\mu|-1.$$

4.2 Some applications

Applying the generalised Bernstein operators to shifted factorials, we obtain the following interesting result:

Proposition 4.2.1. *Let $P = (p_1,\dots,p_n)$ be a sequence of points in $\mathbb{R}^d$ with affine hull $\mathbb{R}^d$ and $\{\xi_1,\dots,\xi_n\}$ are generalised barycentric coordinates. If $V = \{p_1,\dots,p_s\} \subset P$ and $p_j \notin Aff(P\setminus\{p_j\})$, then*

$$B_{k,P}([\xi]_{1/k}^\mu) = \lambda_{|\mu|}^{(k)}\xi^\mu$$

where $\mu \in \mathbb{Z}_{\geq 0}^V$.

Proof. Since $p_j \notin \mathrm{Aff}(P \setminus \{p_j\})$, we have $\xi_i(p_j) = \delta_{ij}$. Therefore,

$$\begin{aligned}
B_{k,P}([\xi]_{1/k}^{\mu}) &= B_{k,P}\left(\sum_{\gamma\leq\mu}\frac{(-1)^{|\mu-\gamma|}}{k^{|\mu-\gamma|}}s(\mu,\gamma)\xi^{\gamma}\right) = \sum_{\gamma\leq\mu}\frac{(-1)^{|\mu-\gamma|}}{k^{|\mu-\gamma|}}s(\mu,\gamma)B_{k,P}(\xi^{\gamma}) \\
&= \sum_{\gamma\leq\mu}\frac{(-1)^{|\mu-\gamma|}}{k^{|\mu-\gamma|}}s(\mu,\gamma)\left(\sum_{\substack{|\tau_i|=\gamma_i \\ \tau=\tau_1+\cdots+\tau_n \\ \beta<\tau}}\binom{\gamma}{\tau}\frac{[k]_1^{|\beta|}}{k^{|\gamma|}}S(\tau,\beta)\xi^{\tau}(p)\xi^{\beta}\right) \\
&= \sum_{\gamma\leq\mu}\frac{(-1)^{|\mu-\gamma|}}{k^{|\mu-\gamma|}}s(\mu,\gamma)\left(\sum_{\beta<\gamma}\frac{[k]_1^{|\beta|}}{k^{|\gamma|}}S(\gamma,\beta)\xi^{\beta}\right) \\
&= \frac{1}{k^{|\mu|}}\sum_{\beta\leq\mu}[k]_1^{|\beta|}\xi^{\beta}\left(\sum_{\gamma\leq\mu}(-1)^{|\mu-\gamma|}s(\mu,\gamma)S(\gamma,\beta)\right) \\
&= \frac{1}{k^{|\mu|}}\sum_{\beta\leq\mu}[k]_1^{|\beta|}\xi^{\beta}\delta_{\mu\beta} \\
&= \lambda_{|\mu|}^{(k)}\xi^{\mu}
\end{aligned}$$

where the second to last equation follows from the inversion relation between Stirling numbers of the first and sencond kinds (see lemma 2.2.3). □

From our construction of the generalised Bernstein operators, it is not expected that they preserve nonnegativity of functions, that is, mapping nonnegative funtions to nonnegative functions. The next result says that this property does not vanish completely but remains true for certain functions in some special cases.

Proposition 4.2.2. *Let* $P = (p_1, \ldots, p_n)$ *be a sequence in* $\mathbb{R}^d$ *with barycenter* c *be such that* $p_i + p_{n+1-i} \ = 2c$.
If $d \geq 2$*, then there exist a* $j \in \{1, \ldots, n\}$ *such that* $B_{k,P}\xi_j^2 \geq 0$ *and* $B_{k,P}\xi_{n+1-j}^2 \geq 0 \ \forall k > 1 \ \forall x \in \mathbb{R}^d$.

Proof. If n is odd then some points will be the barycentre c. So without loss of generality, assume n is even. By degree reducing formula,

$$B_{k,P}\xi_l^2 = (1-\frac{1}{k})\xi_l^2 + \frac{1}{k}\sum_{i=1}^{n}\xi_i^2(p_l)\xi_i$$

Since $p_i + p_{n+1-i} = 2c$, we have $\xi_i + \xi_{n+1-i} = \frac{2}{n}$. Therefore,

$$\xi^2_{n+1-i}(p_l) = (\frac{2}{n} - \xi_i(p_l))^2 = \frac{4}{n^2} - \frac{4}{n}\xi_i(p_l) + \xi_i^2(p_l)$$

So we have

$$\begin{aligned}
\sum_{i=1}^{n} \xi_i^2(p_l)\xi_i &= \sum_{i=1}^{n/2} \xi_i^2(p_l)\xi_i + \sum_{i=1}^{n/2} \Big[\frac{4}{n^2} - \frac{4}{n}\xi_i(p_l) + \xi_i^2(p_l)\Big]\xi_{n+1-i} \\
&= \sum_{i=1}^{n/2} \xi_i^2(p_l)(\xi_i + \xi_{n+1-i}) + \sum_{i=1}^{n/2} \Big[-\frac{4}{n^2} + \frac{4}{n}\xi_{n+1-i}(p_l)\Big]\xi_{n+1-i} \\
&= \frac{2}{n}\sum_{i=1}^{n/2} \xi_i^2(p_l) - \frac{4}{n^2}\sum_{i=1}^{n/2} \xi_{n+1-i} + \frac{4}{n}\sum_{i=1}^{n/2} \xi_{n+1-i}(p_l)\xi_{n+1-i}
\end{aligned}$$

By symmetry,

$$\sum_{i=1}^{n} \xi_i^2(p_l)\xi_i = \frac{2}{n}\sum_{i=1}^{n/2} \xi^2_{n+1-i}(p_l) - \frac{4}{n^2}\sum_{i=1}^{n/2} \xi_i + \frac{4}{n}\sum_{i=1}^{n/2} \xi_i(p_l)\xi_i$$

Sum the two equations and yield

$$\sum_{i=1}^{n} \xi_i^2(p_l)\xi_i = \frac{1}{n}\sum_{i=1}^{n} \xi_i^2(p_l) - \frac{2}{n^2}\sum_{i=1}^{n} \xi_i + \frac{2}{n}\sum_{i=1}^{n} \xi_i(p_l)\xi_i \quad = \frac{1}{n}S(p_l) - \frac{2}{n^2} + \frac{2}{n}\xi_l$$

Therefore,

$$B_{k,P}\xi_l^2 = (1 - \frac{1}{k})\xi_l^2 + \frac{2}{nk}\xi_l + \frac{1}{nk}S(p_l) - \frac{2}{n^2k}$$

and $B_{k,P}\xi_l^2$ is the quadratic polynomial with respect to ξ_l.

$$\begin{aligned}
B_{k,P}\xi_l^2 \geq 0 \ \forall k > 1 \ \forall x \in \mathbb{R}^d &\Leftrightarrow \Delta = \frac{1}{n^2k^2} - (1 - \frac{1}{k})(\frac{1}{nk}S(p_l) - \frac{2}{n^2k}) \leq 0 \\
&\Leftrightarrow S(p_l) \geq \frac{2k-1}{n(k-1)} \ \ \forall k > 1
\end{aligned}$$

Note that the function $f(k) = \frac{2k-1}{n(k-1)}$ is decreasing so that $\max\{f(k), k > 1\} = f(2) = \frac{3}{n}$.

Also, by lemma 3.3.1(f), $\sum_l S(p_l) = d+1 \geq 3 \ \forall d \geq 2$, which implies there exists a j such that $S(p_j) \geq \frac{3}{n} \geq \frac{2k-1}{n(k-1)} \ \forall k > 1$. And therefore, $B_{k,P}\xi_j^2 \geq 0 \ \forall k > 1 \ \forall x \in \mathbb{R}^d$. In addition, $\xi_{n+1-j}(x+c) = \xi_j(-x+c) \ \forall x$, which implies also that $B_{k,P}\xi^2_{n+1-j} \geq 0$.

□

Chapter 5

Korovkin theorem

It was proved that the sequence of univariate Bernstein polynomials $\{B_k f\}_k$ on $[0, 1]$ converges uniformly to f for any continuous function f. This result is amazing, which gives a constructive proof of the Weierstrass's approximation theorem. And even more amazing, in the 1950s, Korovkin obtained a generalisation of this result which pinpoints the crucial properties of the Bernstein operator are that $B_k f \to f$ uniformly for $f = 1, x, x^2$ and that B_k is linearly monotone. These properties are enough to ensure such uniform convergence to happen.

5.1 Generalised Korovin Theorem

We first state two important results which will be needed in proving the generalised version of Korovkin theorem.

Lemma 5.1.1. *In a finite dimensional Banach space, every closed and bounded set is compact.*

Lemma 5.1.2 (Heine-Cantor theorem)**.** *Every continuous function on a compact set is uniformly continuous.*

We now prove the generalised version of this observation on polytopes. The proof is a modified version of that presented in [2].

Theorem 5.1.3 (Generalised Korovkin theorem)**.** *Let (L_m) denote a sequence of monotone linear operators that map a function $f \in C(T)$, where T is the convex hull formed by a sequence of points $P = (p_1, \dots, p_n)$ in $\mathbb{R}^d$, to a function $L_m f \in C(T)$, and let $L_m f \to f$ uniformly on T for any monic monomials of the form $g_0 = 1$, $g_{1i} = \xi_i$, $g_{2i} = \xi_i^2$, where $i \in \{1, \dots, n\}$. Then $L_m f \to f$ uniformly for all $f \in C(T)$.*

Proof. Let t, $x \in T$ and

$$t = \xi_1(t)p_1 + \cdots + \xi_n(t)p_n$$
$$x = \xi_1(x)p_1 + \cdots + \xi_n(x)p_n$$

Denote $\phi_i(x) := (\xi_i(t) - \xi_i(x))^2$ and $\phi_t(x) := \sum_{i=1}^n \phi_i(x) = \|t - x\|^2$.
Consider $(L_m\phi_i)(t)$:
Since L_m is linear, we obtain

$$(L_m\phi_i)(t) = \xi_i^2(t)(L_m g_0)(t) - 2\xi_i(t)(L_m g_{1i})(t) + L_m g_{2i}(t)$$

where $g_0 = 1$, $g_{1i} = \xi_i$, $g_{2i} = \xi_i^2$.
Therefore,

$$(L_m\phi_i)(t) = \xi_i^2(t)[(L_m g_0(t) - 1] - 2\xi_i(t)[(L_m g_{1i})(t) - \xi_i(t)] + [(L_m g_{2i})(t) - \xi_i^2(t)]$$

Since $\xi_i(t)$'s are generalised barycentric coordinates of the point t which is in the convex hull T and T is compact (due to lemma 5.1.1), $\xi_i(t)$'s are bounded above by some $M > 0$.
Denote $\|.\|$ to be the uniform norm, we deduce that

$$\|L_m\phi_i\| \leq M^2\|L_m g_0 - g_0\| + 2M\|L_m g_{1i} - g_{1i}\| + \|L_m g_{2i} - g_{2i}\|$$

By hypothesis, $\|L_m f - f\| \to 0$ as $m \to \infty$ for $f \in \{g_0, g_{1i}, g_{2i}\}$, so $L_m\phi_t \to 0$ as $m \to \infty$ uniformly in t.
Hence so does $(L_m\phi_t)(t)$ as L_m is linear.

Now let f be any function in $C(T)$. Since T is compact, f is also uniformly continuous on T by lemma 5.1.2.
Therefore,

$$\forall\epsilon > 0 \exists\delta > 0 : \forall t, x \in T, \|t - x\| < \delta \Rightarrow |f(t) - f(x)| < \epsilon \tag{5.1}$$

Now if $\|t - x\| \geq \delta$, we have

$$|f(t) - f(x)| \leq 2\|f\| \leq 2\|f\|\frac{\|t - x\|^2}{\delta^2} = \alpha\phi_t(x) \tag{5.2}$$

where $\alpha = 2\|f\|/\delta^2$. From 5.1 and 5.2, we see that for all $t, x \in T$:

$$|f(t) - f(x)| \leq \epsilon + \alpha\phi_t(x)$$

So

$$-\epsilon - \alpha\phi_t(x) \leq f(t) - f(x) \leq \epsilon + \alpha\phi_t(x)$$

Since L_m is monotone, apply L_m to both sides of the above inequality and then evaluate each obtained functions at $x = t$, we have

$$-\epsilon(L_m g_0)(t) - \alpha(L_m\phi_t)(t) \leq f(t)(L_m g_0)(t) - (L_m f)(t) \leq \epsilon(L_m g_0)(t) + \alpha(L_m\phi_t)(t)$$

Notice that $(L_m\phi_t)(t) \geq 0$ since L_m is monotone and $\phi_t(x) \geq 0\ \forall x \in T$. Thus we obtain

$$|f(t)(L_m g_0)(t) - L_m f)(t)| \leq \epsilon\|L_m g_0\| + \alpha(L_m\phi_t)(t) \tag{5.3}$$

On writing $L_m g_0 = 1 + L_m g_0 - g_0$, we have

$$\|L_m g_0\| \leq 1 + \|L_m g_0 - g_0\| \tag{5.4}$$

From 5.3 and 5.4,

$$|f(t)(L_m g_0)(t) - (L_m f)(t)| \leq \epsilon(1 + \|L_m g_0 - g_0\|) + \alpha(L_m\phi_t)(t)$$

We now write

$$f(t) - (L_m f)(t) = [f(t)(L_m g_0)(t) - (L_m f)(t)] + [f(t) - f(t)(L_m g_0)(t)]$$

and hence obtain the inequality

$$|f(t) - (L_m f)(t)| \leq |f(t)(L_m g_0)(t) - (L_m f)(t)| + |f(t) - f(t)(L_m g_0)(t)|$$

But

$$|f(t) - f(t)(L_m g_0)(t)| \leq \|f\| \cdot \|L_m g_0 - g_0\| \tag{5.5}$$

From 5.3 and 5.5,

$$|f(t) - (L_m f)(t)| \leq \epsilon + (\|f\| + \epsilon)\|L_m g_0 - g_0\| + \alpha(L_{mt})(t)$$

Therefore, with n large enough, we will have

$$|f(t) - (L_m f)(t)| \leq 3\epsilon$$

uniformly in t.

□

5.2 Application to Bernstein operators

We now check to see that the generalised Bernstein operators satisfy all the requirements in the generalised Korovkin theorem on the region of nonnegativity N_P of their domains, and hence approximate all continuous function uniformly on N_P . Despite the fact that the rate of convergence may be slow, this is one of the most remarkable property of all Bernstein operators in general.

Recall that the generalised Bernstein operator of degree k on the convex hull T of a sequence of points $P = (p_1, \ldots, p_n)$ in $\mathbb{R}^s$ is defined to be

$$B_{k,P} f = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} \xi^\alpha f\Big(\sum_j \frac{\alpha_j}{|\alpha|} p_j\Big)$$

It is not hard to see that $(B_{k,P})$ is continuous and linear on T. On the region of nonnegativity of N_P, it is ensured that $\xi^\alpha f\Big(\sum_j \frac{\alpha_j}{|\alpha|} p_j\Big) \geq \xi^\alpha g\Big(\sum_j \frac{\alpha_j}{|\alpha|} p_j\Big)$ whenever $f \geq g$, where f,$g \in C(T)$. The remaining conditions are less obvious so we will obtain them through a lemma.

Lemma 5.2.1. *The sequence of generalised Berntein operators* $(B_{k,P}f)$ *converge to* f *uniformly on the region of nonnegativity* N_P *for any monic monomials of the form* $g_0 = 1$, $g_1 = \xi_i$, $g_2 = \xi_i^2$, *where* $i \in \{1, \dots, n\}$.

Proof. As direct substitution, we see that for g_0
$B_{k,P}g_0 = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} \xi^\alpha g_0 \Big(\sum_j \frac{\alpha_j}{|\alpha|} p_j \Big) = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} \xi^\alpha = 1 = g_0$
For g_1 and g_2, we use theorem 4.1.2 and deduce that
$B_{k,P}g_1 = \xi_1^{(k)} \xi_i = \xi_1 = g_1$
$B_{k,P}g_2 = \xi_2^{(k)} \xi_i^2 + \frac{1}{|\alpha|} \sum_j \xi^2(p_i)\xi_j \to \xi_i^2 = g_2$ uniformly on T as $k \to \infty$. □

To sum up, we have the following:

Theorem 5.2.2. *For any function* $f \in C(T)$, *where* T *is the convex hull of a sequence of points* $P = (p_1, \dots, p_n)$ *in* $\mathbb{R}^d$ *with affine hull* $\mathbb{R}^d$, $B_{k,P}f$ *converges to* f *uniformly on the region of nonnegativity* N_P *of its domain.*

Chapter 6

Shape preserving properties

In this chapter, we investigate the behaviour of the generalised Bernstein operators when applied to convex functions. The following theorems say that they preserve the convexity of functions on the region of nonnegativity N_P.

Theorem 6.0.3. *Let f be a convex function defined on $C(T)$, where T is the convex hull of a sequence of points $P = (p_1, \ldots, p_n)$ in $\mathbb{R}^d$ with affine hull $\mathbb{R}^d$. Then $B_{k,P} f \geq f$ on the region of nonnegativity of T.*

Proof. Since x lies in the region of nonnegativity of T, $\xi_i(x) \geq 0 \, \forall i \in \{1, \ldots, n\}$. Also, f is convex on T and $\sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} \xi^\alpha = 1$. Therefore,

$$\begin{aligned}
B_{k,P} f = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} \xi^\alpha f(v_\alpha) &= \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} \xi^\alpha f(\sum_j \frac{\alpha_j}{|\alpha|} p_j) \\
&\geq f(\sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} \xi^\alpha \sum_j \frac{\alpha_j}{|\alpha|} p_j) \\
&= f\Big(\sum_j \xi_j p_j \sum_{\substack{|\alpha|=k \\ \alpha_j > 0}} \binom{|\alpha|}{\alpha - e_j} \lambda^{\alpha - e_j} \Big) \\
&= f(\sum_j \xi_j p_j) = f
\end{aligned}$$

where $e_j(p_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$, and the last two equation follows from the fact that

$$\sum_{\substack{|\alpha|=k\\ \alpha_j>0}} \binom{|\alpha|}{\alpha - e_j} \xi^{\alpha - e_j} = 1, \qquad \sum_j \xi_j(x) p_j = x \; \forall x \in T.$$

□

Theorem 6.0.4 (Shape preserving property)**.** *Let f be a convex function defined on $C(T)$, where T is the convex hull of a sequence of points $P = (p_1, \dots, p_n)$ in $\mathbb{R}^d$ with affine hull $\mathbb{R}^d$. Then $B_{k-1,P} f \geq B_{k,P} f$ on the region of nonnegativity of T.*

Proof. Without loss of generality, assume $\xi_n \neq 0$. Set $\tau_i = \frac{\xi_i}{\xi_n} \geq 0$, then $\tau_1 + \cdots + \tau_n = \xi_n^{-1}$ and we have

$$\begin{aligned}
\xi_n^{-k}(B_{k-1,P} f - B_{k,P} f) &= \xi_n^{-k} \Big(\sum_{|\alpha|=k-1} \binom{k-1}{\alpha} \xi^\alpha f(v_\alpha) - \sum_{|\beta|=k} \binom{k}{\beta} \xi^\beta f(v_\beta) \Big) \\
&= \Big(\sum_i \tau_i \Big) \sum_{|\alpha|=k-1} \binom{k-1}{\alpha} \tau^\alpha f(v_\alpha) - \sum_{|\beta|=k} \binom{k}{\beta} \tau^\beta f(v_\beta) \\
&= \sum_i \sum_{|\alpha|=k-1} \binom{k-1}{\alpha} \tau^{\alpha+e_i} f(v_\alpha) - \sum_{|\beta|=k} \binom{k}{\beta} \tau^\beta f(v_\beta) \\
&= \binom{k}{\beta} \sum_{|\beta|=k} c_\beta \tau^\beta
\end{aligned}$$

where $c_\beta = \sum_i \frac{\beta_i}{k} f(v_{\beta - e_i}) - f(v_\beta)$. Note that $\sum_i \frac{\beta_i}{k} v_{\beta - e_i} = v_\beta$ so that $c_\beta \geq 0$ by convexity of f. □

Chapter 7

Summary

The generalised Bernstein operators on polytopes retain most of the basic properties of the Bernstein operators on a simplices. They are degree reducing and therefore are diagonalisable. A direct consequence of this result is that the set of polynomials are uniformly approximated on the whole convex hull of the points by its Bernstein polynomials. For other properties such as shape-preserving on convex functions as well as Korovkin theorem, the region of nonnegativity plays a central role in justifying the results which have been established on simplices.

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