On the existence of spherical (t, t)-designs

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Abstract

Tight frames are a generalisation of orthonormal bases which may have redundancy. Special examples used in quantum information theory include MUBs (mutually unbiased bases), SIC-POVMs (symmetric, informationally complete, positive operator valued measures). They are examples of spherical (t, t)-designs. It is known that spherical (t, t)-designs exist for a sufficient large number of vectors. In this dissertation we seek to determine basic existence results for the smallest number of vectors required. We undertook a naïve numerical search which revealed some interesting features. We present our results in a set of table, give conjectures and present some notable examples.

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Chapter 1

Introduction

The concept of spherical (t,t)-designs dates from 1977 and is due to Delsarte, Goethals, and Seidel [DGS77]. Such designs have the property of allowing the integration of homogeneous polynomials of total degree $2t$ to be turned into the summation of polynomials evaluated at a finite number of points. These designs have been studied extensively in the real case and have applications to approximation theory, combinatorics, and statistics for experimental design. Their study in the complex case should however not be neglected. Justification of such study is to be found in quantum mechanics and has applications in quantum information theory, quantum computing, quantum coding and much more. It is perhaps farfetched, but not completely inconceivable, that spherical (t,t)-designs could play a role in quantum teleportation, thus bringing Star Trek to life [BPM+98]! The existence of spherical (t,t)-designs was proved by Seymour and Zaslavsky [SZ84] in 1984. They showed that such designs exist provided they are "big" enough, but gave no idea of what is meant by "big". The main goal of this project is to investigate this notion of "big".

We will start by looking at frames and tight frames. Spherical (t,t)-design are frames, and tight frames are spherical $(1,1)$-designs, so this small detour will give the reader a better appreciation of spherical (t,t)-designs.

This chapter provides a taste of what is to come. First, it provides a motivation for looking at frames. Then it proceeds to a brief discussion of frames, stating the limitations of this concept. Next, it demonstrates the idea of tight frames with examples, introduce the properties that we would like them to have, and provide examples of "nice" tight frames. After this we introduce spherical (t, t)-designs, linking them to examples of "nice" tight frames. Finally, we give a summary of what can be expected from this dissertation.

1.1 Motivation

Points in Euclidean space are usually specified by their coordinates with respect to some basis. Such coordinates have *no redundancy*, i.e., if one coordinate is lost, then the position of that point is lost. This can be problematic in some applications. One simple approach to the problem is to keep duplicate copies of each coordinate. With **frames** we have redundant vectors which give us additional coordinates, so that we can lose some coordinates and still retain the point. When the probability of losing a coordinate is small, adding a small number of extra vectors will usually result in a higher probability of retaining the point, even in high dimensional spaces. This is demonstrated in Appendix A.

1.2 Frames

The ideas outlined in Section 1.1 have been extensively studied in the infinite dimensional setting (infinite dimension Hilbert spaces). Such objects, which span the whole Hilbert space (where the Hilbert space might be real or complex), but might be linearly dependent, are called frames when they contain finitely many vectors. This is because for finite frames, it is always possible to write a vector f as a linear combination of sequences of these vectors $(f_j)_{j \in J}$, $|J| < \infty$ in the frame by having some constants $\alpha \in \mathbb{F}$ (where $\mathbb{F}$ is a Hilbert space) such that

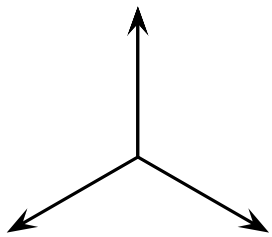
$$f = \alpha \sum_{j \in J} \langle f, f_j \rangle f_j.$$

However, when there are countably infinte many vectors in the frame, this sum is not guaranteed to converge. We shall consider some examples in Chapter 2 and give formal a definition for frames. Having a frame is desirable, but the question of how to determine the coefficients is not so clear: namely, the constant α might change depending on our f.

1.3 Tight Frames

Given the constraint on frames as mentioned above, the idea of *tight frames*, which allows us to determine the coefficients for any f given $(f_j)_{j\in J}$ follows naturally.

Example 1.1. A prototypical example of a tight frame is the Mercedes-Benz frame, shown below.



The Mercedes-Benz frame consists of three equally spaced unit vectors f_1, f_2, f_3 in $\mathbb{R}^2$, and yields the following decomposition

$$f = \frac{2}{3}\sum_{j=1}^{3} \langle f, f_j \rangle f_j \tag{1.1}$$

of any $f \in \mathbb{R}^2$. We shall revisit this example in Chapter 3.

Finite tight frames have many applications, and it is of interest to find "nice" tight frames for such applications. Some of the "nice" properties we might hope for are:

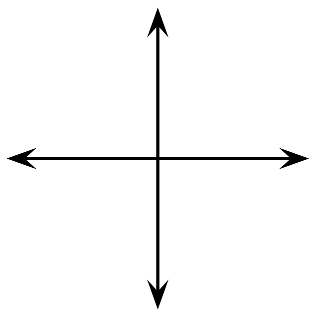
- *Equal norms.* The vectors in the tight frame have equal norm, so we can think of the vectors as lying on a sphere.
- *Symmetries.* The frame is invariant under some group G of symmetries, e.g., it could be the case that it is the rotation copy of some vectors.
- *Equiangularity/equispacing.* In $\mathbb{R}^d$, this is equivalent to having the same angles between distinct vectors. In the complex case, we need to look at the complex analogous of angle, namely the absolute value of the inner product. Sometimes having all "angles" the same can not be achieved, so we can only hope to make the number of distinct "angles" as small as possible. If there are k distinct "angle" values, then this tight frame is called *k-angular*.

- *Robustness to erasures.* Consider the example in Section 1.1. We want to be able to restore the position of our point when we can only retain small amount of information. In Example 1.1, we can lose any one coefficient $\frac{2}{3}\langle f, f_j\rangle$ for $j = 1, 2, 3$ and still be able to find our f. This will be illustrated in Appendix B.

- *Stability.* Suppose that when coefficients are "transported", they are perturbed by random error. For a tight frame expansion, the computed errors are bounded, and could even be zero even with nonzero pertubation. Such bounds cannot be achieved with orthogonal expansion. Moreover, tight frame expansions usually give us smaller error than do orthonormal expansions. We will illustrate this phenomenon with an example in Appendix C.

We now introduce a few examples which have one or more desirable properties.

Example 1.2. Here is a tight frame in $\mathbb{R}^2$, which consists of four equally spaced unit vectors f_1, f_2, f_3, f_4 in $\mathbb{R}^2$.



It yields decompositions

$$f = \frac{1}{2}\sum_{j=1}^{4}\langle f, f_j\rangle f_j, \quad \forall f \in \mathbb{R}^2. \tag{1.2}$$

Note that the absolute value of the inner product (which is an analogous of the angle, hence we shall refer to it as "angle" from now on) between any two elements of the frame can only be either 0 or 1. Thus we have an example of a 2-angular tight frame. If the angle set is $\left\{0, \frac{1}{\sqrt{d}}\right\}$ where d is the dimension of the space, then we shall call the frame **mutually unbiased bases** (**MUBs**). We will demonstrate later that any MUBs is a tight frame.

Example 1.3. Using Matlab, we came up with the following example of mutually unbiased bases in $\mathbb{C}^2$:

$$\{f_1, f_2, \dots f_6\} = \left\{ \begin{bmatrix} 0.6002 + 0.7731i \\ 0.0953 + 0.1815i \end{bmatrix}, \begin{bmatrix} 0.6367 - 0.5434i \\ -0.4695 + 0.2808i \end{bmatrix}, \begin{bmatrix} 0.4769 + 0.5205i \\ -0.5952 + 0.3839i \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} -0.5197 + 0.4813i \\ 0.3857 + 0.5913i \end{bmatrix}, \begin{bmatrix} -0.1206 - 0.1658i \\ 0.4277 + 0.8803i \end{bmatrix}, \begin{bmatrix} 0.3612 + 0.4109i \\ 0.4401 + 0.7120i \end{bmatrix} \right\}.$$

Mutually unbiased bases in $\mathbb{R}^d$ have been studied extensively so we shall focus on the $\mathbb{C}^d$ context.

Example 1.4. Another example of a tight frame in $\mathbb{C}^2$ is:

$$\{f_1, f_2, f_3, f_4\} = \left\{ \begin{bmatrix} 0.2981 + 0.5298i \\ 0.5780 + 0.5444i \end{bmatrix}, \begin{bmatrix} -0.6664 - 0.0888i \\ 0.3887 + 0.6301i \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 0.1888 + 0.9693i \\ -0.1537 + 0.0323i \end{bmatrix}, \begin{bmatrix} -0.4023 + 0.2035i \\ -0.0817 - 0.8889i \end{bmatrix} \right\}.$$

This is a tight frame with the property that the absolute value of inner product between all vectors in the frame is the same. This is called a **symmetric, informationally complete, positive operator valued measure** (**SIC-POVM**).

1.4 spherical (t, t)-designs

The tight frames in the examples above all satisfy the *the cubature rule*. That is, if we think of the vectors in those tight frames as points, then for some t the integral of any homogenous polynomial of total degree $2t$ over the unit sphere is equal to the average value of the polynomial evaluated at those points. Example 1.1 and Example 1.2 satisfy this condition for $t = 1$, Example 1.3 for $t \leq 3$ and Example 1.4 for $t \leq 2$. This idea of being able to turn integration into summation can be useful in some applications. We call a sequence of vectors (f_j) satisfying the cubature rule for polynomials of degree $2t$ a **spherical** (t, t)**-design**.

Spherical designs are of value in approximation theory, quantum mechanics, statistics for experimental designs, and many other disciplines. It is of interest, for a given t and a given Hilbert space, to construct designs with *small* numbers n of vectors to make computation easy. The study of such designs in $\mathbb{R}^d$ has been extensive. However they rarely exist. In this dissertation, we shall focus on the complex setting.

1.5 Outline

The main points of what can be expected from the rest of this dissertation will now be outlined.

In chapter 2, we give a very brief overview of frames. We start with the formal definition, then a small collection of well-known facts, available in most frame theory books. In discussing these facts we make reference to the examples encountered in Chapter 1.

In chapter 3, we look at tight frames carefully so as to provide the tools needed for investigating spherical (t, t)-design. We encounter synthesis, analysis, and frame operators as well as the Gramian and angle matrix, and some examples to illustrate these concepts. We see what it means for tight frames to be equivalent. Finally, we examine the notion of frame potential and variational inequality, which can be translated directly into the spherical (t, t)-design setting.

Chapter 4 is mostly example based, with our goal being to investigate interesting cases of spherical (t, t)-designs, rather than to provide a thorough study of spherical (t, t)-designs. We only provide as much theory as needed. The majority of this chapter involves discussing examples and conjectures. Towards the end of the chapter we study in detail two families of spherical (t, t)-design that possess the equal norm and equiangular property.

In chapter 5 our main goal is to find spherical (t, t)-designs numerically. We first discuss the methodology to be used to find spherical (t, t)-designs using Matlab. Then we discuss what we observed while investigating the conjectures we came up with on the basis of the Matlab computations. Finally, we discuss some interesting examples that might be worthy of further study.

Chapter 2

Frames

In 1952, Duffin and Schaeffer [DS52] gave a "frame condition" on a sequence of vectors to make it possible to compute the coefficients of the frame expansion regardless of dimension. The notion of frame bound was introduced, and this idea contributed vastly to the development of modern frame theory. Frame theory is a substanstial topic in its own right, with a history dating back at least as far as 1937 [Sch37]. As this is not the focus of this dissertation, we shall only give a brief overview. We shall start with the formal definition from [DS52], then provide a few facts, relating them back to the examples in Chapter 1.

Definition 2.1. *A countable sequence $(f_j)_{j\in J}$ in a Hilbert space $\mathcal{H}$ is said to be a* ***frame*** *(for $\mathcal{H}$) if there exist constants A and B, $0 < A \leq B < \infty$, such that*

$$A\left\|f\right\|^2 \leq \sum_{j\in J} |\langle f, f_j\rangle|^2 \leq B\left\|f\right\|^2, \qquad \forall f \in \mathcal{H}. \tag{2.1}$$

In this case, A is called a ***lower frame bound*** *and B an* ***upper frame bound****.*

We shall establish a few facts.

Fact 2.1. To be a frame, $(f_j)_{j\in J}$ needs to span $\mathcal{H}$.

Proof. If this is not the case, then we can find $f \neq 0$ that is orthogonal to all f_j. If $(f_j)_{j\in J}$ is indeed a frame then there is a lower frame bound $0 < A$ such that

$$A\left\|f\right\|^2 \leq \sum_{j\in J} |\langle f, f_j\rangle|^2 = 0,$$

which is a contradiction. □

Fact 2.2. $(f_j)_{j\in J}$ spanning $\mathcal{H}$ is not a sufficient condition to be a frame.

Proof. The following set of vectors **is not** a frame in $\mathbb{R}^2$.

$$\{f_1, f_2, f_3, f_4, \ldots\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ldots \right\}.$$

This sequence spans $\mathbb{R}^2$, but $\sum_{j\in J} |\langle (1,1), f_j\rangle|^2 = \sum_{j\in J} 1$ does not converge. We cannot choose our upper frame bound B such that $B < \infty$. Thus, $(f_j)_{j\in J}$ is not a frame in this case. □

Fact 2.3. If J is finite, then the spanning condition is both necessary and sufficient for $(f_j)_{j\in J}$ to be a frame.

Proof. We already seen why it is necessary, so now we shall show it is sufficient. Let dimension of $\mathcal{H}$ be d. Using the Cauchy-Schwarz inequality, we have that

$$\sum_{j\in J} |\langle f, f_j\rangle|^2 \leq \sum_{j\in J} \|f\|^2 \cdot \|f_j\|^2 = \left(\sum_{j\in J} \|f_j\|^2 \right) \cdot \|f\|^2 . \tag{2.2}$$

Hence if we pick $B = \sum_{j\in J} \|f_j\|^2$, this is our upper frame bound and $B < \infty$ since $(f_j)_{j\in J}$ is a finite sequence of vectors. Now observe that

$$\sum_{j\in J} |\langle f, f_j\rangle|^2 = \sum_{j\in J} \left| \|f\| \left\langle \frac{f}{\|f\|}, f_j \right\rangle \right|^2 = \|f\|^2 \sum_{j\in J} \left| \left\langle \frac{f}{\|f\|}, f_j \right\rangle \right|^2 \tag{2.3}$$

Let $A = \inf_{h\in\mathbb{S}^d} \sum_{j\in J} |\langle h, f_j\rangle|^2$. This is well-defined due to compactness of $\mathbb{S}^d$. We then have $A \geq 0$ and $A = 0$ only when h is orthogonal to all f_j, this is impossible as $(f_j)_{j\in J}$ is a spanning set, thus $A > 0$. The A thus defined is our lower frame bound. □

From this we see that Example 1.1, 1.2, 1.3 and 1.4 are all frames.

Example 2.2. Consider the set $\left\{ \frac{1}{\sqrt{2\pi}} e^{inx} : n \in \mathbb{Z} \right\}$. It is an orthonormal basis in Hilbert space $L^2[-\pi, \pi]$. If we take its projection from $L^2[-\pi, \pi]$ onto subspace $L^2[-a, a]$, where $0 < a < \pi$ using the standard projection, then it forms a frame with an infinite number of elements.

We will now move onto tight frames, which allow us to determine the coefficients of frame expansion for any f given a tight frame with an easy formula.

Chapter 3

Tight frames

In this chapter, we discuss tight frames which overcome some of the limitations that frames possess. Although tight frames are not central to this dissertation, the understanding of tight frame provides valuable insight into the study of spherical (t, t)-designs. First we give the definition of tight frames. Then we introduce the synthesis, analysis and frame operators which will provide us with useful results that are easy to apply. Next, we discuss the unitary operator, the Gramian and the idea of equivalence of tight frames. Finally, we discuss the frame potential and variational inequality, which seems to have been underappreciated and which provides great insight into appreciation of the physical structure of tight frames.

3.1 Basic definition of tight frame

We shall now define tight frame formally.

Definition 3.1. *A countable sequence $(f_j)_{j\in J}$ in a Hilbert space $\mathcal{H}$ is said to be a **tight frame** (for $\mathcal{H}$) if there exists a (frame bound) $A > 0$, such that*

$$A \|f\|^2 = \sum_{j\in J} |\langle f, f_j\rangle|^2, \quad \forall f \in \mathcal{H}. \tag{3.1}$$

*Further, $(f_j)_{j\in J}$ is **normalised** if $A = 1$, and **finite** if J is finite.*

Example 3.2. For the case of the Mercedes-Benz frame in Example 1.1, we have $A = \frac{3}{2}$. Let $f_j = \begin{bmatrix} \cos\left(\frac{2\pi j}{3}\right) \\ \sin\left(\frac{2\pi j}{3}\right) \end{bmatrix}$, for $j = 1, 2, 3$. We will show later why this is sufficient to show the result holds for any three equally spaced unit vectors. Let $f = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right], a, b \in \mathbb{R}$, then we have

$$\begin{aligned}\sum_{j=1}^{3} |\langle f, f_j\rangle|^2 &= \sum_{j=1}^{3}\left[\cos^2\left(\tfrac{2\pi j}{3}\right)a^2 + 2\cos\left(\tfrac{2\pi j}{3}\right)\sin\left(\tfrac{2\pi j}{3}\right)ab + \sin^2\left(\tfrac{2\pi j}{3}\right)b^2\right] \\ &= a^2\left(\tfrac{1}{4}+\tfrac{1}{4}+1\right) + 2ab\left(\tfrac{-1}{2}\tfrac{\sqrt{3}}{2} + \tfrac{-1}{2}\tfrac{-\sqrt{3}}{2}\right) + b^2\left(\tfrac{3}{4}+\tfrac{3}{4}\right) \\ &= \tfrac{3}{2}\left(a^2+b^2\right) = \frac{3}{2}\|f\|^2 .\end{aligned}$$

We call (3.1) the *Bessel identity*. It is equivalent to the following identities which can be easily verified by making use of the polarisation identity[1],

$$\textit{Parseval:} \qquad f = \frac{1}{A}\sum_{j\in J}\langle f, f_j\rangle f_j, \quad \forall f \in \mathcal{H}, \tag{3.2}$$

$$\textit{Plancherel:} \qquad \langle f, g\rangle = \frac{1}{A}\sum_{j\in J}\langle f, f_j\rangle \langle f_j, g\rangle, \quad \forall f, g \in \mathcal{H}. \tag{3.3}$$

Observe that the frame bound A is simply a normalising constant wheres it is sufficient to consider only normalised tight frames. Given that $(f_j)_{j\in J}$ is a tight frame in $\mathcal{H}$ with frame bound A, let $g_j = f_j/\sqrt{A}$ for all $j \in J$. Then $(g_j)_{j\in J}$ is a normalised tight frame which satisfies

$$g = \sum_{j\in J}\langle g, g_j\rangle g_j, \qquad \forall g \in \mathcal{H}.$$

We call $(g_j)_{j\in J}$ a **Parseval frame** for $\mathcal{H}$.

The Parseval identity (3.2) can be used to calculate coordinates for f under a given frame. Identities (3.1), (3.2) and (3.3) are good for some applications, but are not very efficient for verifying and hence finding tight frames. We shall introduce a few more definitions to arrive at results which allow us to tell whether a set of vectors is a tight frame more efficiently.

[1] The *polarisation identity* for an Hilbert space $\mathcal{H}$ is: for all $f, g \in \mathcal{H}$ we have that

$$\begin{aligned}\Re\langle f, g\rangle &= \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2\right), \\ \Im\langle f, g\rangle &= \frac{1}{4}\left(\|f+ig\|^2 - \|f-ig\|^2\right), \quad \text{for } \mathcal{H} \text{ complex.}\end{aligned}$$

3.2 The synthesis, analysis and frame operators

Definition 3.3. *For a finite sequence $(f_j)_{j\in J}$ in $\mathcal{H}$ the* ***synthesis operator*** *(****reconstruction operator*** *or* ***pre-frame operator****) is the linear map $V\colon \ell_2(J) \to \mathcal{H}$ defined by*

$$V\left(\sum_{j\in J} a_j e_j\right) = \sum_{j\in J} a_j f_j. \tag{3.4}$$

In the foregoing $(e_j)_{j\in J}$ is the canonical basis for $\ell_2(J)$. If V is the synthesis operator determined by a finite sequence $(f_j)_{j\in J}$ we write $V := [f_j]_{j\in J}$. The adjoint V^* of the synethesis operator is the ***analysis operator*** *(or* ***frame transform operator****). Note that*

$$V^*\colon \mathcal{H} \to \ell_2(J) \quad \text{and} \quad V^* f \mapsto \sum_{j\in J} \langle f, f_j\rangle\, e_j. \tag{3.5}$$

The product, $VV^* = S\colon \mathcal{H} \to \mathcal{H}$ is known as the ***frame operator****. It has the property that*

$$Sf = VV^* f = \sum_{j\in J} \langle f, f_j\rangle\, f_j, \quad \forall f \in \mathcal{H}. \tag{3.6}$$

Note that S is self adjoint as we have $S^* = (VV^*)^* = (V^*)^*V^* = VV^* = S$. Using this we obtain

$$\operatorname{trace}(S) = \operatorname{trace}(VV^*) = \operatorname{trace}(V^*V) = \sum_{j\in J} \|f_j\|^2, \tag{3.7}$$

$$\operatorname{trace}(S^2) = \operatorname{trace}(S^*S) = \operatorname{trace}((VV^*)^*(VV^*)) = \sum_{j\in J}\sum_{k\in J} |\langle f_j, f_k\rangle|^2. \tag{3.8}$$

Hence we have the following results:

Proposition 3.4. *A finite sequence $(f_j)_{j\in J}$ in $\mathcal{H}$ is a tight frame for $\mathcal{H}$ (with frame bound A) if and only if*

$$S = VV^* = AI_{\mathcal{H}}, \quad V := [f_j]_{j\in J}. \tag{3.9}$$

In the foregoing $I_{\mathcal{H}}$ is the identity operator for $\mathcal{H}$. In particular, a tight frame satisfies

$$\sum_{j\in J} \|f_j\|^2 = dA, \quad \text{where} \quad d = \dim(\mathcal{H}), \tag{3.10}$$

and

$$\sum_{j\in J}\sum_{k\in J} |\langle f_j, f_k\rangle|^2 = \frac{1}{d}\left(\sum_{j\in J} \|f_j\|^2\right)^2. \tag{3.11}$$

Proof. Equation (3.9) follows directly from the Parseval identity (3.2).

$$\sum_{j\in J}\|f_j\|^2 = \operatorname{trace}(S) = \operatorname{trace}(AI_{\mathcal{H}}) = dA,$$

$$\sum_{j\in J}\sum_{k\in J}|\langle f_j, f_k\rangle|^2 = \operatorname{trace}(S^2) = \operatorname{trace}(A^2 I_{\mathcal{H}}) = A^2 d = \frac{1}{d}(Ad)^2 = \frac{1}{d}\left(\sum_{j\in J}\|f_j\|^2\right)^2. \qquad \square$$

Example 3.5. Recall the vectors for Example 1.2. The synthesis operator is

$$V = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix},$$

and the frame operator is

$$S = VV^* = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I_{\mathbb{R}^2}.$$

Thus we see that those four equally spaced vectors in $\mathbb{R}^2$ form a tight frame. This demonstrates that using the synthesis operator to show that something is a tight frame is easier than proceeding directly from the definition.

3.3 Unitary operators and tight frames in $\mathbb{R}^2$

We shall now introduce the idea of a unitary operator. We will use this concept to show that in Example 3.2 and Example 3.5, it is justifiable to demonstrate only that a particular representation of a class of related frames is tight. We will conclude with a brief discussion of tight frames in $\mathbb{R}^2$ which possess the equal norm and equiangular property.

Definition 3.6. *Let $U\colon \mathcal{H} \to \mathcal{K}$ be a linear operator. We say that U is* ***unitary*** *if U is surjective, and $\langle Uf, Ug\rangle = \langle f, g\rangle$ for all $f, g \in \mathcal{H}$.*

Theorem 3.7. *A tight frame will remain a tight frame after unitary transformation.*

Proof. Let $V\colon \ell_2(J) \to \mathcal{H}$ be the synthesis operator for a given tight frame $(f_j)_{j\in J}$, $A > 0$ the frame bound and $U\colon \mathcal{H} \to \mathcal{K}$ a unitary operator. Observe that $U^*U = I_{\mathcal{H}}$, $UU^* = I_{\mathcal{K}}$. Note that UV is the synthesis operator for the sequence $(Uf_j)_{j\in J}$. Now we have

$$(UV)(UV)^* = UVV^*U^* = UAI_{\mathcal{H}}U^* = AUI_{\mathcal{H}}U^* = AI_{\mathcal{K}}. \qquad \square$$

From this we see that in Example 3.2 and Example 3.5, verifying that specific instance of those frames are tight is justified, as rotation is a unitary transformation. In the first example the reciprocal of the frame bound is $\frac{2}{3}$, whereas in the second example it is $\frac{2}{4}$. We can think of this as distributing two dimensions worth of information onto three or four coordinates, so in the first example, each coordinate gets $\frac{2}{3}$ dimensions worth of information, and in the second example, $\frac{2}{4}$ dimensions worth.

Example 3.8. In 1937, Schönhardt [Sch37] generalised the idea of Example 3.2 and Example 3.5 to n equally spaced unit vectors $f_1, \dots, f_n \in \mathbb{R}^2$. The decomposition

$$f = \frac{2}{n} \sum_{j=1}^{n} \langle f, f_j \rangle f_j, \quad \forall f \in \mathbb{R}^2, \tag{3.12}$$

follows naturally.

In 1940, Brauer and Coxeter [BC40] gave an example of the continuous analogy

$$f = \frac{1}{\pi} \int_0^{2\pi} \langle f, f_\theta \rangle f_\theta \, \mathrm{d}\theta, \quad \forall f \in \mathbb{R}^2, \quad f_\theta := \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \tag{3.13}$$

The collection $(f_\theta)_{\theta \in [0, 2\pi)}$ is called a **continuous tight frame**.
The use of unitary operators in the study of tight frames will be further demonstrated in the next section.

3.4 The Gramian and unitarily equivalence

We shall now define the Gram matrix. We have seen the importance of the frame operator $S = VV^*$. Composing V and its adjoint V^* the other way gives us the **Gramian**, $V^*V \colon \ell_2(J) \to \ell_2(J)$.

Definition 3.9. *For a finite sequence of n vectors $(f_j)_{j \in J}$ in $\mathcal{H}$, the **Gramian** or **Gram matrix** is the $n \times n$ Hermitian matrix*

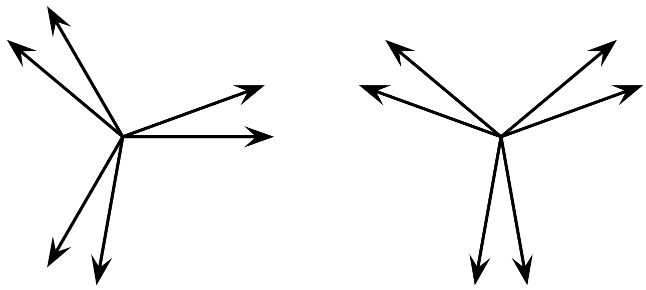
$$\mathrm{Gram}((f_j)_{j \in J}) := [\langle f_k, f_j \rangle]_{j,k \in J} \,. \tag{3.14}$$

To facilitate the full appreciation of the Gramian we shall now introduce the idea of equivalence for finite tight frames.

Definition 3.10. *Two normalised tight frames $(f_j)_{j\in J}$ for $\mathcal{H}$ and $(g_j)_{j\in J}$ for $\mathcal{K}$, with the same index set J are (unitarily) equivalent if there is a unitary transformation $U:\mathcal{H}\to\mathcal{K}$, such that $g_j = Uf_j$, for all $j\in J$.*

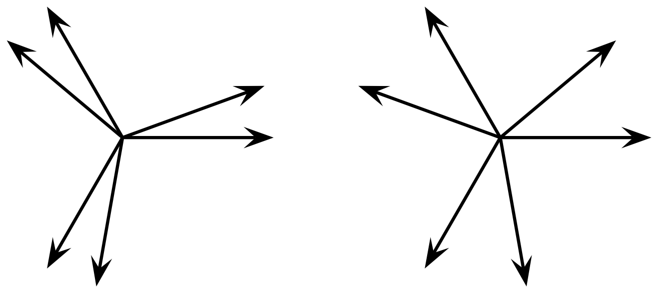
Example 3.11. The following two tight frames Φ and Ψ **are** unitarily equivalent. See below for their graphical representation in $\mathbb{R}^2$ by letting $a+ib=(a,b)$. Let $\omega := e^{\frac{2\pi i}{3}}, \nu := e^{\frac{\pi i}{9}}$.

$$\Phi := \left(1,\nu,\omega,\nu\omega,\omega^2,\nu\omega^2\right), \quad \Psi := (\nu,\nu^2,\nu\omega,\nu^2\omega,\nu\omega^2,\nu^2\omega^2).$$



The following two tight frames Φ and Θ **are not** unitarily equivalent.

$$\Phi := \left(1,\nu,\omega,\nu\omega,\omega^2,\nu\omega^2\right), \quad \Theta := (1,\nu^2,\omega,\nu^2\omega,\omega^2,\nu^2\omega^2).$$



This is because the unitary map must preserves the inner product between vectors (analogy of angle), which is the case for Φ and Ψ but clearly not the case for Φ and Θ. We will demonstrate this example numerically in Example 3.13.

Theorem 3.12. *Normalised tight frames are unitarily equivalent if and only if their Gramians are equal.*

Proof. Let $(f_j)_{j\in J}$, $(g_j)_{j\in J}$ be normalised tight frames for $\mathcal{H}$ and $\mathcal{K}$ respectively.
If $(f_j)_{j\in J}$ and $(g_j)_{j\in J}$ are unitarily equivalent, then there exists a unitary map $U\colon\mathcal{H}\to\mathcal{K}$ such that $g_j = Uf_j$ for all $j\in J$. We then have

$$\langle g_j, g_k\rangle = \langle Uf_j, Uf_k\rangle = \langle f_j, f_k\rangle, \quad \forall j,k\in J$$

so their Gramians are equal.
If the Gramians of $(f_j)_{j\in J}$ and $(g_j)_{j\in J}$ are equal, that is, $\langle g_j, g_k\rangle = \langle f_j, f_k\rangle$ for all $j, k \in J$. Then, there exists a unitary $U\colon \mathcal{H} \to \mathcal{K}$ with $g_j = Uf_j$ for all $j \in J$. Thus $(f_j)_{j\in J}$ and $(g_j)_{j\in J}$ are unitarily equivalent. ☐

Theorem 3.12 implies that: *Any properties of a tight frame that is invariant under unitary transformation can be determined from its Gramian.*

Example 3.13. Recall Example 3.11. We will show that Φ and Ψ are unitarily equivalent by calculating the difference between their Gramians in Matlab:

```
>>w=exp(2*pi*1i/3);v=exp(pi*1i/9);
>>Phi=[1 v w v*w w^2 v*w^2]; Psi=[v v^2 v*w v^2*w v*w^2 v^2*w^2];
>>Phi'*Phi -Psi'*Psi
ans =
  0  0  0  0  0  0
  0  0  0  0  0  0
  0  0  0  0  0  0
  0  0  0  0  0  0
  0  0  0  0  0  0
  0  0  0  0  0  0
```

Therefore the Gramians for Φ and Ψ are the same, hence Φ and Ψ are unitarily equivalent. In contrast the difference of the Gramians for Φ and Θ is

```
>>Theta=[1 v^2 w v^2*w w^2 v^2*w^2];
>> Phi'*Phi - Theta'*Theta
ans =
     0       0.2 - 0.3i      0       0.2 + 0.3i      0      -0.4 + 0.00i
 0.2 + 0.3i      0      -0.4 + 0.0i      0       0.2 - 0.3i      0
     0      -0.4 - 0.0i      0       0.2 - 0.3i      0       0.2 + 0.30i
 0.2 - 0.3i      0       0.2 + 0.3i      0      -0.3 - 0.0i      0
     0       0.2 + 0.3i      0      -0.3 + 0.0i      0       0.2 - 0.30i
-0.4 - 0.0i      0       0.2 - 0.3i      0       0.2 + 0.3i      0
```

Hence Φ and Θ are not unitarily equivalent.

We shall now define the angle matrix. This definition is not standard, but it will be useful further on in determining whether a set of vectors is k-angular.

Definition 3.14. *For a finite sequence of n vectors $(f_j)_{j\in J}$ in $\mathcal{H}$, the **angle matrix** is the $n \times n$ matrix*

$$\operatorname{angle}\left((f_j)_{j\in J}\right) := [|\langle f_k, f_j\rangle|]_{j,k\in\mathcal{K}}. \tag{3.15}$$

The angle matrix is the Gramian after taking entry-wise absolute value.

We shall refer to $|\langle f, g\rangle|$ as the *angle* between f and g. This terminology is not standard. Some writers refer to $|\langle f, g\rangle|^2$ as the angle.

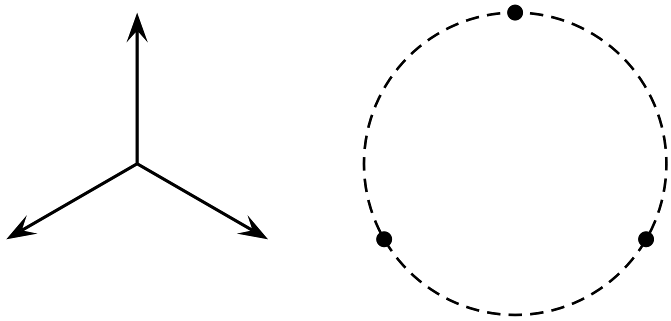
3.5 The frame potential and the variational inequality

The idea of the frame potential provides a nice framework for conceptualizing frames from a physical point of view. Recall (3.11) from Proposition 3.4. The function $\mathrm{FP}\colon \mathbb{S}^n \to [0,\infty)$ defined by

$$\mathrm{FP}(f_1,\ldots,f_n) = \sum_{j=1}^{n}\sum_{k=1}^{n} |\langle f_j, f_k\rangle|^2 \tag{3.16}$$

was called the **frame potential** by Fickus [Fic01] in 2001 because it was derived from a *frame force*, where $\mathbb{S}^n$ is the n dimensional unit sphere.

In physics, potential energy is the energy stored in a body or in a system due to its position in a force field or due to its configuration. When the bodies are free to move, they will attempt to change position to arrive at the configuration that minimises the potential energy. For example, a spring will return to its unextended state when there is no outside force, electrons will move away from each other, and objects in the air will fall to the ground. The frame potential is analogous to the potential energy of objects in the physical world. It is minimised when vectors form a tight frame. Example 3.2 motivates us to think of electrons placed on a wired circle. This system will reach equilibrium when the electrons are equally spaced. See below for a comparison between the Mercedes-Benz frame and three electrons on a wire frame.



However, this analogy is not perfect. If we had only two electrons, they would go to opposite ends to maximise the distance between them and thus manimises the potential energy. In contrast in the tight frame situation, unit vectors form a normalised tight frame if and only if they are an orthonormal basis. Tight frames seem to occur when the vectors are "as orthogonal as possible". Thus we must adjust the analogy, by considering a system where vectors are subject to a repellent force when the angle between them is less than $\frac{\pi}{2}$, and subject to an attraction force when the angle between them is greater than $\frac{\pi}{2}$. In this system the configuration that minimises the potential energy is exactly a tight frame.

If we have $f_1, \ldots, f_n$ being $n \geq d$ unit vectors in $\mathbb{C}^d$, then

$$\mathrm{FP}(f_1, \ldots, f_n) = \sum_{j=1}^{n} \sum_{k=1}^{n} |\langle f_j, f_k \rangle|^2 \geq \frac{n^2}{d}.$$

This is known as the **Welch bound**, after [Wel74]. Unit vectors $f_1, \ldots, f_n$ which give equality are called **Welch bound equality sequences** (**WBE sequences**), a notion made popular by Massey and Mittelholzer [MM93]. A WBE sequence is the same thing as a tight frame. The following theorem extends the Welch bound to the case where the vectors may have arbitrary lengths.

Theorem 3.15. *Let $f_1, \ldots, f_n$ be vectors in $\mathcal{H}$, not all zero, and $d = \dim(\mathcal{H})$. Then*

$$\sum_{j=1}^{n} \sum_{k=1}^{n} |\langle f_j, f_k \rangle|^2 \geq \frac{1}{d} \left(\sum_{j=1}^{n} \|f_j\|^2 \right)^2, \tag{3.17}$$

with equality if and only if $(f_j)_{j=1}^n$ is a tight frame for $\mathcal{H}$.

Proof. Given VV^* is positive definite, it is unitarily diagonalisable with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d \geq 0$. Then using the Cauchy-Schwarz inequality we have

$$(\mathrm{trace}(S))^2 = \left(\sum_{j=1}^{d} \lambda_j \right)^2 = \langle (1), (\lambda_j) \rangle^2 \leq \|(1)\|^2 \|(\lambda_j)\|^2 = d \sum_{j=1}^{d} \lambda_j^2 = d \ \mathrm{trace}(S^2),$$

The result follows from (3.7) and (3.8). ☐

We call (3.17) the *variational inequality*. When equality is attained, it is known as the *variational formula*, and is equivalent to

$$\int_{\mathbb{S}} p(x) \ \mathrm{d}\sigma(x) = \frac{1}{\sum_{\ell=1}^{n} \|f_\ell\|^2} \sum_{j=1}^{n} p(f_j), \tag{3.18}$$

which is the *cubature rule*, where p is a homogeneous polynomial of total degree 2. A proof can be found in [Wal11]. From this, we see that all tight frames are **spherical**

$(1,1)$**-designs**. We have also seen previously that some of our examples of tight frames are spherical (t,t)-designs, for $t > 1$. In the next Chapter we will further investigate spherical (t,t)-designs.

3.6 Summary

In this chapter we have developed results that will help us with the study of tight frames. Many of these results can be generalised to the spherical (t,t)-design setting, which we will see in Chapter 4. We have seen that *Any properties of a tight frame that is invariant under unitary transformation can be determined from its Gramian.*. This tells us that in studying tight frames we need only look at their Gramians to understand them. In fact we will demonstrate in Chapter 4 through examples that the same applies to spherical (t,t)-designs. What is more, most of the time we need in fact only look at the angle matrix.

Chapter 4

spherical (t,t)-designs

We have seen that certain tight frames are spherical (1,1)-designs. It should not be surprising that there may be some relationship between tight frames and spherical (t,t)-designs. We have seen that tight frames are helpful in integrating homogenous polynomials of total degree 2. It is of interest to generalise this to homogenous polynomials of higher total degree. In this chapter, we first look at the basic mathematics of spherical (t,t)-designs. Then, motivated by examples, we consider some special kinds of designs, instances of such designs include MUBs and SIC-POVMs. Along the way, we will point out patterns that seems to be worth-while investigating, which we will do numerically in Chapter 5.

4.1 Outline of spherical (t,t)-designs

The original definition of spherical (t,t)-designs from [DGS77] is not useful for our discussion. Hence we shall start with an equivalent definition of spherical (t,t)-designs. A proof of equivalence can be found in [Wal11].

Definition 4.1. *Let $f_1, \dots, f_n$ be vectors in $\mathcal{H} = \mathbb{F}^d$ (where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$), not all zero. Then*

$$\sum_{j=1}^{n}\sum_{k=1}^{n} |\langle f_j, f_k\rangle|^{2t} \geq c_t(d,\mathbb{F}) \left(\sum_{j=1}^{n} \|f_j\|^{2t}\right)^2. \tag{4.1}$$

Where

$$c_t(d,\mathbb{C}) = \frac{1}{\binom{d+t-1}{t}}, \qquad c_t(d,\mathbb{R}) = \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2t-1)}{d(d+2)\cdots(d+2(t-1))}.$$

When equality is attained, $(f_i)_{i=1}^n$ is called a ***spherical*** (t,t)***-design*** *for $\mathbb{F}^d$. When f_i is a unit vector for all $1 \leq i \leq n$, $(f_i)_{i=1}^n$ is referred to as a* ***weighted spherical*** (t,t)***-design****.*

Similar to the tight frame case, we refer to (4.1) as the *variational inequality* and call it the *variational formula* when equality is attained. The variational formula in this case is also equivalent to the *cubature rule*,

$$\int_{\mathbb{S}} p(x) \, \mathrm{d}\sigma(x) = \frac{1}{\sum_{\ell=1}^n \|f_\ell\|^{2t}} \sum_{j=1}^n p(f_j), \tag{4.2}$$

for all p where p are homogeneous polynomials on $\mathcal{H}$ of total degree $2t$.
From now on, we shall use the phrase t-design and spherical (t,t)-design interchangeably, and sometimes omit the word weighted if it is clear from the context.
Checking whether a sequence of vectors is a t-design is easy using (4.1) and calculations involving the angle matrix.

Example 4.2. Recall the MUBs in Example 1.3, which is the union of three mutually unbiased bases in $\mathbb{C}^2$. This was obtained when we were trying to find a 3-design in $\mathbb{C}^2$ with 6 vectors. Let $\Phi = \{f_1, f_2, \ldots, f_6\}$ be the sequence of vectors from Example 1.3. It has the angle matrix

$$\operatorname{angle}(\Phi) = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \end{bmatrix}, \quad \text{and} \quad \begin{aligned} c_3(2,\mathbb{C}) &= \tfrac{1}{\binom{4}{3}} = \tfrac{1}{4}, \\ c_2(2,\mathbb{C}) &= \tfrac{1}{\binom{3}{2}} = \tfrac{1}{3}, \\ c_1(2,\mathbb{C}) &= \tfrac{1}{\binom{2}{1}} = \tfrac{1}{2}. \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{j=1}^6 \sum_{k=1}^6 |\langle f_j, f_k \rangle|^6 &= 24 \times \frac{1}{8} + 6 = \frac{1}{4} \times 6^2 = \frac{1}{4} \left(\sum_{j=1}^6 \|f_j\|^6 \right)^2, \\ \sum_{j=1}^6 \sum_{k=1}^6 |\langle f_j, f_k \rangle|^4 &= 24 \times \frac{1}{4} + 6 = \frac{1}{3} \times 6^2 = \frac{1}{3} \left(\sum_{j=1}^6 \|f_j\|^4 \right)^2, \\ \sum_{j=1}^6 \sum_{k=1}^6 |\langle f_j, f_k \rangle|^2 &= 24 \times \frac{1}{2} + 6 = \frac{1}{2} \times 6^2 = \frac{1}{2} \left(\sum_{j=1}^6 \|f_j\|^2 \right)^2. \end{aligned}$$

Ergo we see that Φ is a weighted 3-design and a weighted 2-design as well as a weighted 1-design (tight frame).

Example 4.3. Now consider Example 1.4, which is a SIC-POVM in $\mathbb{C}^2$. This was obtained when we were trying to find a 2-design in $\mathbb{C}^2$ with 4 vectors. Let $\Psi = \{f_1, f_2, f_3, f_4\}$ be the sequence of vectors from Example 1.4. It has the angle matrix

$$\operatorname{angle}(\Psi) = \begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 1 \end{bmatrix}, \quad \text{and} \quad \begin{aligned} c_2(2,\mathbb{C}) &= \tfrac{1}{\binom{3}{2}} = \tfrac{1}{3}, \\ c_1(2,\mathbb{C}) &= \tfrac{1}{\binom{2}{1}} = \tfrac{1}{2}. \end{aligned}$$

Then we have

$$\sum_{j=1}^{4}\sum_{k=1}^{4} |\langle f_j, f_k\rangle|^4 = 12 \times \frac{1}{9} + 4 = \frac{1}{3} \times 4^2 = \frac{1}{3}\left(\sum_{j=1}^{4} \|f_j\|^4\right)^2,$$
$$\sum_{j=1}^{4}\sum_{k=1}^{4} |\langle f_j, f_k\rangle|^2 = 12 \times \frac{1}{3} + 4 = \frac{1}{2} \times 4^2 = \frac{1}{2}\left(\sum_{j=1}^{4} \|f_j\|^2\right)^2.$$

Hence we see that Ψ is a weighted 2-design as well as a weighted 1-design (tight frame).

From the above examples and other computations we can make a few conjectures which we shall investigate further.

- If a set of equal norm vectors Φ is a t-design, then it is also an r-design for $1 \le r \le t$. It is to be hoped that we can effect a scaling when the norms of the vector are not equal and still have this conjecture hold.
- When trying to find a t-design, we tend to get vectors sets with equal norm properties.
- Attempt to find t-designs generate vectors sets with nice equiangular properties.
- In $\mathbb{C}^d$ we can get $d+1$ mutually unbiased bases.

We shall answer the first conjecture with the following proposition.

Proposition 4.4. *Let $(f_j)_{j=1}^n$ be a t-design for $\mathbb{F}^d$. Then $(\|f_j\|^{t/r-1} f_j)_{j=1}^n$ is an r-design for $\mathbb{F}^d$, $1 \le r \le t$.*

Proof. Let $g_j := \|f_j\|^{t/r-1} f_j$. To show that $(g_j)_{j=1}^n$ is an r-design it is sufficient to show that it satisfies the cubature rule, as the cubature rule is equivalent to the variational formula. Let q be an arbitrary homogeneous polynomial on $\mathbb{F}^d$ with total degree $2r$. Then if we let $p(x) = \|x\|^{2(t-r)} q(x)$, it is a homogeneous polynomial on $\mathbb{F}^d$ with total degree $2t$. Using this we obtain

$$\sum_{j=1}^{n} \frac{\|g_j\|^{2r}}{\sum_{\ell=1}^{n} \|g_\ell\|^{2r}} q\Big(\frac{g_j}{\|g_j\|}\Big) = \sum_{\substack{j=1 \\ f_j \neq 0}}^{n} \frac{\|f_j\|^{2t}}{\sum_{\ell=1}^{n} \|f_\ell\|^{2t}} p\Big(\frac{f_j}{\|f_j\|}\Big) = \int_{\mathbb{S}} p \, d\sigma = \int_{\mathbb{S}} q \, d\sigma. \tag{4.3}$$

Thus, g is an r-design. □

From this, we see that if a set of vectors Φ is a weighted t-design then it is also weighted r-design for $1 \leq r \leq t$. Therefore weighted t-designs can be used to integrate homogenous polynomials of total even degree up to $2t$, not just for degree $2t$. We shall now look more closely at MUBs and SIC-POVMs which not only have the equal-norm property but also have the desirable property of having a small angle set.

4.2 MUBs

We stated that in Section 1.3 that it is desirable to find tight frames with equiangularity (such as MUBs) for applications and we now briefly explain the motivation for MUBs. In physics, there is an interest in quantum mechanical observables that are complementary: that is, where precise knowledge of one of them gives no information on the probability of the outcomes of the other. This idea was first introduced by Bohr [Boh28] in 1928. One application of this idea is the quantum key exchange protocol introduced by Bennett and Brassard [BB84]. This protocal exploits complementarity to secure a key exchange against eavesdropping. The eigenbases of non-degenrate complementary observables turn out to be mutually unbiased. The converse is also true: we can associate a pair of mutually unbiased bases with a pair of non-degenerate complementary observables. There are properties of MUBs that are invaluable in quantum information processing. However, in view of the limited scope of this dissertation we shall not go into the physics. We will now formally define MUBs and look into a few theorems to further our understanding.

Definition 4.5. *Two orthonormal bases Φ and Ψ of $\mathbb{C}^d$ are called* ***mutually unbiased*** *if and only if $|\langle f_i, g_j\rangle|^2 = 1/d$ holds for all $f_i \in \Phi$ and $g_j \in \Psi$. A collection of* *orthonormal bases in $\mathbb{C}^d$ that are pairwise mutually unbiased are referred to as* ***mutually unbiased bases (MUBs)*** *in $\mathbb{C}^d$.*

We shall now examine a few results to gain further insight into MUBs.

Theorem 4.6. *The union Φ of $d+1$ mutually unbiased bases in $\mathbb{C}^d$ forms a weighted 2-design with angle set $\left\{0, 1/\sqrt{d}\right\}$ and $d(d+1)$ elements. Moreover Φ is a tight frame.*

Proof. Recall that $c_1(d,\mathbb{C}) = \binom{d}{1}^{-1} = \frac{1}{d}, c_2(d,\mathbb{C}) = \binom{d+1}{2}^{-1} = \frac{2}{d(d+1)}$. Using the variation formula (when (4.1) attain equality) we can verify that

$$\frac{\sum_{f,g\in\Phi} |\langle f,g\rangle|^2}{\left(\sum_{f\in\Phi} \|f\|^2\right)^2} = \frac{d(d+1)+d(d+1)d^2\frac{1}{d}}{(d(d+1))^2} = \frac{1}{d} = c_1(d,\mathbb{C}),$$

$$\frac{\sum_{f,g\in\Phi} |\langle f,g\rangle|^4}{\left(\sum_{f\in\Phi} \|f\|^4\right)^2} = \frac{d(d+1)+d(d+1)d^2\frac{1}{d^2}}{(d(d+1))^2} = \frac{2}{d(d+1)} = c_2(d,\mathbb{C}).$$

Whence Φ is a 1-design and 2-design. $\square$

We see that in Example 4.2 we have that $d=2$, angle set $\left\{0, 1/\sqrt{2}\right\}$ and $6 = 2(2+1)$ vectors. The converse of Theorem 4.6 is also true.

Theorem 4.7. *A weighted 2-design Φ in $\mathbb{C}^d$ with angle set $\left\{0, 1/\sqrt{d}\right\}$ and $d(d+1)$ elements is the union of $d+1$ mutually unbiased bases.*

For the purpose of applications, it is of interest to construct a maximal number of mutually unbiased bases in any dimension $d \geq 2$. For prime power dimensions the problem is solved, the maximal set of $d+1$ mutually unbiased bases in $\mathbb{C}^d$ exists. One of the earlier constructions is by Wotters and Fields [WF89] in 1989. More recent methods include one by Bandyopadhyay [BBRV02] in 2002 and one making use of Galois rings by Klappenecker [KR04] in 2004. The case of non-prime power dimensions remains an open problem. It is known that three mutually unbiased bases exist in dimension six, and it is believed that four mutually unbiased bases do not exist in dimension six. No proof has been given for this yet, but Grassl [Gra04] in 2004 has shown that particular sets of three MUBs cannot be extended to four MUBs.

4.3 SIC-POVMs

The family of SIC-POVMs is another family of frames which, similar to MUBs has recently become popular due to its use in quantum computing. It has the nice property of being equal-norm and equiangular. As is the case for MUBs, the construction of SIC-POVMs remains a challenging task. Scott and Grassl [SG10] has found solutions for up to 67 dimensions. Here, we formally define this family then show that members of this family actually forms weighted 2-designs. We then discuss what work has been done so far.

Definition 4.8. *A* ***symmetric, informationally incomplete, positive operator valued measure (SIC-POVM)*** *is a family of unit vectors $(f_j)_{j=1}^{d^2}$ in $\mathbb{C}^d$ such that*

$$|\langle f_i, f_j\rangle|^2 = \frac{1}{d+1}$$

for all $1 \le i,\ j \le d^2$ where $i \neq j$.

It turns out that the problem of finding SIC-POVMs is equivalent to the **Grassmannian packing problem**: how do we pack d^2 lines into $\mathbb{C}^d$, such that they all pass through a single point, and the angles between any two of them are as large as possible?

Theorem 4.9. *Let Ψ be a SIC-POVM in $\mathbb{C}^d$, then Ψ forms a 2-design with the angle set $\left\{1/\sqrt{d+1}\right\}$ and d^2 elements.*

Proof. Using the variational formula (when (4.1) attains equality) we can verify that Φ is a 1-design as well as a 2-design.

$$\frac{\sum_{f,g\in\Psi} |\langle f, g\rangle|^2}{\left(\sum_{f\in\Psi} \|f\|^2\right)^2} = \frac{d^2 + \left(d^4 - d^2\right)\frac{1}{d+1}}{\left(d^2\right)^2} = \frac{1}{d} = c_1(d, \mathbb{C}),$$

$$\frac{\sum_{f,g\in\Psi} |\langle f, g\rangle|^4}{\left(\sum_{f\in\Psi} \|f\|^4\right)^2} = \frac{d^2 + \left(d^4 - d^2\right)\frac{1}{(d+1)^2}}{\left(d^2\right)^2} = \frac{2}{d(d+1)} = c_2(d, \mathbb{C}). \qquad \square$$

We see in Example 4.3, we have that $d = 2$, the angle set is $\left\{1/\sqrt{3}\right\}$ and we have $4 = 2^2$ vectors.

The following conjecture is widely believed, and there is compelling numerical evidence in support of it [SG10]. However the proof remains an open challenge.

Conjecture 4.10. *For all $d > 2$, there exists a SIC-POVM in $\mathbb{C}^d$ with d^2 vectors and angle set $\left\{1/\sqrt{1+d}\right\}$.*

Theorem 4.11. *[Boh28] Let $\mathbb{S} = \mathbb{S}(\mathcal{H})$ be the unit sphere in $\mathcal{H}$ and $d = \dim(\mathcal{H})$. The local minimisers of the frame potential*

$$\mathcal{F}(f_1, \ldots, f_n) = \sum_{j=1}^{n} \sum_{k=1}^{n} |\langle f_j, f_k \rangle|^2, \quad f_1, \ldots, f_n \in \mathbb{S} \tag{4.4}$$

are global minimisers, which in turn are tight frames for $\mathcal{H}$, or nonspanning orthonormal sequences. In particular, there exist equal norm tight frames of n vectors in $\mathbb{R}^d$ and $\mathbb{C}^d$, for all values of $n \geq d$.

See [Wal11] for a proof that is more intuitive than the one in [Boh28].

Example 4.12. If Ψ is a SIC-POVM Ψ in $\mathbb{C}^d$, then we have

$$\mathcal{F}(\Psi) = \sum_{f,g \in \Psi} |\langle f, g \rangle|^2 = d^2 + (d^4 - d^2)\frac{1}{d+1} = d^3. \tag{4.5}$$

Using the variational inequality, we see that the minimum value of frame potential is

$$\mathcal{F}(\Psi) \geq \frac{1}{d} \left(\sum_{f \in \Psi} \|f\|^2 \right)^2 = d^3. \tag{4.6}$$

Thus we have that Ψ is a local minimiser of frame potential, hence a global minimiser. However, whether the global minimiser is unique up to unitary equivalence remains an open question.

4.4 Summary

In this chapter, we were firstly motivated by the fact that t-designs are good for integrating polynomials. Then we saw by example that sometimes, certain t-designs possess interesting properties such as the equal norm and equiangularity. Later we discussed the motivation for having an equal norm t-design: it is automatically an r-design for $1 \leq r \leq t$. We then looked at MUBs and SIC-POVMs, which are both equal norm 2-designs. The idea of MUBs is to pack orthonormal bases into a space in such a way that the "angles" between any two vectors from **different** orthonormal bases are constant. The idea of SIC-POVMs is similar, but involves packing lines into a space so the angle between any two of them is the same. We suspect that when trying to find a t-design, we will find vector sets that have equal norm and equiangular properties. This provides motivation for us to investigate further, which we do in the next chapter.

Chapter 5

Constructing t-designs

We have seen in Chapter 4 that it is desirable to have weighted t-designs with large t as it allows us to integrate homogeneous polynomials of total even degree up to $2t$. We have seen two special families of tight frames, namely MUBs and SIC-POVMs, that are also weighted 2-designs. We suspect that they can be obtained by trying to find t-designs for $t > 2$: in other words, though they might not be t-designs, they could be "close" to t-designs for $t > 2$. Both MUBs and SIC-POVMs possess the equiangular property that makes them invaluable in quantum computing. The problem lies in their construction. We generated t-designs iteratively by making use of (4.1) (refer to Appendix D for code and comment). In this chapter, we will first explain the methodology we used to construct t-designs. Then we will discuss patterns observed and make some conjectures. Finally we will discuss some interesting examples that showed up in the process.

5.1 Methodology

Recall that a set of vectors Φ forms a t-design in $\mathbb{F}^d$ precisely when the variation inequality (4.1) attains equality. Thus, with rearrangement we can define the error functional $\operatorname{err}\colon \ell_2(J) \to \mathbb{R}$ to be

$$\operatorname{err}(\Phi) := \sum_{f,g\in\Phi} |\langle f, g\rangle|^{2t} - c_t(d,\mathbb{F}) \left(\sum_{f\in\Phi} \|f\|^{2t} \right)^2, \quad \Phi \in \ell_2(J). \tag{5.1}$$

Then the variational inequality is equivalent to having $\operatorname{err}(\Phi) \geq 0$ and $\operatorname{err}(\Phi) = 0$ precisely when Φ is a t-design. We can think of the error value as a measure of how close we are to a t-design, which is the idea we used in constructing t-designs.

Armed with the error functional, we can construct t-designs in the following way in Matlab.

- Create a random pre-frame operator (in the form of a matrix), calculate the error value, save the pre-frame operator as V_m, and have the error value as $errm$.
- Repeat the following steps.
  - Make small changes to the V_m, where the size of the change is dependent on $errm$ and call the result V. Calculate the error value for V. Call the error value err.
  - If $errm \leq err$, do nothing. Otherwise, let $V_m = V$.

If a t-design does exist, then the $errm$ value should eventually approach 0 and the t-design is the columns of our final V_m. The idea of this algorithm is simple but the number of iterations needed might be extremely large. This is especially true when we are trying to find a t-design with large number of vectors, or a t-design for large t.

5.2 Discussion

From now on we will work in complex Hilbert space by default, we shall refer to the dimension of the space by d, the number of vectors by n, and by the "first" case of a t-design we mean the t-design for a fixed d with the smallest n.
The motivation for this project was the following theorem by Seymour and Zaslavsky.

Theorem 5.1. *[SZ84] There is a number $N(d,t,)$ such that for every $N \geq N(d,t)$, there exists a t-design of N vectors in dimension d.*

Our goal at the very beginning was to find for a fixed d and t the smallest n that will provide us with a t-design, and maybe come up with a formula for n. Due to computer power constraints, that goal now seems overly ambitious. However, as in much mathematical experimentation, we observed interesting patterns along the way, and came up with a few conjectures. In this section, we will list and discuss the conjectures made based on the numerical evidence. The commented code used to obtain the numerical evidence can be found in Appendix D.
We first suspected that "for a fixed t and d, the first case of t-design has the equal-norm

property". The motivation for this suspection is the fact that all orthonormal bases are tight frames, and are the smallest possible tight frames. Thus, for any d, the first case of obtaining a 1-design will always be equal norm. This conjecture also seems to be true for $t = 2$. However, for $t > 2$, we have obtained several counter examples.
Then, we suspected that "the first case of a t-design is more likely to be equal-norm". Unfortunately, numerical evidence shows that this only seems to hold for a small t. For larger value of t it seems that we will almost always get a t-design that is not equal-norm.

Conjecture 5.2. *A 2-design with $n = d^2$ elements always exists, and is a SIC-POVM.*

We have tried this out for a large number of d's ($1 \leq d \leq 20$), and it seems to always be the case. There are also overwhelming numerical evidence in support of this conjecture from Scott and Grassl [SG10].

Conjecture 5.3. *Theorem 4.11 also holds for 2-design. That is, the local minimisers of*

$$\sum_{j=1}^{n} \sum_{k=1}^{n} |\langle f_j, f_k \rangle|^4, \quad f_1, \ldots, f_n \in \mathbb{S} \tag{5.2}$$

are global minimisers.

This is true for every $t = 2$ case we tried ($1 \leq d \leq 20, 2 \leq n \leq 70$). It seems like it could be true for $t = 3$, and difference in minimisers we observed could be due to rounding error. It is definitely not true for $t \geq 4$.

Conjecture 5.4. *When we attempt to find $d(d+1)$ vectors in $\mathbb{C}^d$ that minimise the error for $t = 3$, we get MUBs, when such MUBs exist.*

This seem to always be the case except for $d = 6$, though for big d, the number of iterations needed is large and we have some rounding errors.

Conjecture 5.5. *When trying to find a weighted t-design, the set of vectors that minimises that error but are not t-design sometimes form an r-design for $r < t$.*

Quite often, this seems to be the case.

We shall now list the result from our numerical exercises, for each d and t, we will list under n the smallest n that provides us with a t-design. If we are unsure of the actual value, we will give a bound. In the next column, we will list under k the size of its angle set. Under n_w we will list the smallest n that provides us with a weighted t-design, and similarly for the next column k_w. Note that the maximal value of k or k_w are $\binom{n}{2}$. When calculation limits of Matlab are encountered the missing results are left blank.

t	d	n	k	$\binom{n}{2}$	n_w	k_w	$\binom{n_w}{2}$
1	d	d	1	$\binom{d}{2}$	d	1	$\binom{d}{2}$
2	d	d^2	1	$\binom{d^2}{2}$	d^2	1	$\binom{d^2}{2}$
3	2	6	2	15	6	2	15
3	3	22	60	231	27	280	351
3	4	40	2	780	40	2	780
3	5	$100 < n$			$100 < n_w$		
4	2	10	39	45	12	8	66
4	3	47	513	1081	$47 \leq n_w$		
4	4	$85 < n$			$85 < n_w$		
5	2	13	34	78	16	11	120
5	3	$70 < n$			$70 < n_w$		
6	2	19	128	171	26	249	325
6	3	$81 < n$			$81 < n_w$		
7	2	24	157	276	$24 < n_w$		
7	3	$65 < n$			$65 < n_w$		
8	2	37	403	666	$37 \leq n_w$		
8	3	$80 < n$			$80 < n_w$		
9	2	44	546	946	$44 \leq n_w$		
9	3	$80 < n$			$80 < n_w$		
10	2	$60 < n$			$60 < n_w$		

5.3 Examples

Example 5.6. For $t = 4$, $d = 2$, $n = 8$, minimising the error function gave the union of two equal norm tight frames each with four vectors. However the norms of the two tight frames are different. One thing to be careful about is that there are two minimisers for this situation, and this particular Example corresponds to the global minimiser. The converse is not true: putting two tight frames with four vectors together doesn't seem to get the same minimiser back no matter what is tried.

Example 5.7. For $d = 4$, $n = 40$ we seem to get a 3-design that is similar to a MUBs. For the design to be a MUBs the angle set would have to be $\{0, 1/\sqrt{4}\}$, but this example has angle set $\{0, 1/\sqrt{3}\}$. Each of its vectors is orthogonal to 12 other vectors and forms angle $1/\sqrt{3}$ with 27 other vectors. What is even more interesting is it is the first case of a 3-design for $d = 4$.

Chapter 6

Conclusions

The original goal of this project was to find the number $N(d,t)$ in Seymour and Zaslavsky's theorem regarding the existence of t-design. We wrote code to find t-designs by means of an iterative procedure. Although we did not achieve our original goal, we made interesting discoveries along the way, and they produced conjectures that could be motivation for further work. We saw that the first case of t-designs is hardly ever equal-norm when our t is large, even though the equal norm property of finite case t-design seems to hold for $t = 1$ and $t = 2$. It seems plausible that we can always get a SIC-POVM regardless of dimension, but how do we prove it? Is it possible to prove that the local minimisers are global minimisers in the case when $t = 2$? When $n = d(d+1)$, will minimising the error for $t = 3$ always give us MUBs when such MUBs exist?

Section 5.3 presented some interesting cases and raises many unresolved questions. Future research could consider issues like: Did we obtain such cases purely by coincidence? Or is there a family of those objects which hasn't yet been discovered? If so, what can we use them for?

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Appendix A

Sensitivity analysis

Recall Section 1.1. The goal is to conserve the position of a point in the Euclidean space. The position of the point is determined by its coordinates with respect to some basis. Further assume the probability of losing each coordinate is p, that the losses are independent, and that we are working with a space with dimension d.

If we keep two copies of the same coordinate, then the probability of begin able to recover the point is $\left(1 - p^2\right)^d$. We call this approach method one.

If we can add α redundant vectors to the basis in such a way that we now have $d + \alpha$ coordinates, and we can lose any α coordinates and still recover the position of the point, then the probability of being able to recover the point is now $\sum_{i=0}^{\alpha} \binom{d+\alpha}{i} p^i (1-p)^{d+\alpha-i}$. We call this approach method two.

We will present the results from our sensitivity analysis over the next four pages. Here, we will explain how to interpret the results.

Consider the output below. The values in this table are fictional; they are only used to illustrate the concept.

	2	3	4	5
0.0005	0.0000	0.0000	0.0034	0.0043
0.001	0.0000	0.0000	0.0038	0.0048
0.0015	0.0033	0.0042	*0.0053*	*0.0064*
0.002	0.0037	0.0047	*0.0058*	*0.0070*

The top row represents the values of the different d at which we are evaluating the probability of recovering the coordinates. The leftmost column contains the values of the coordinate-loss probability p. The other data in the table represents the difference in the probabilities of recovering the point between method one and method two. That is, if method one has higher chance of retaining the point, the number will be positive,

and negative otherwise. The green formatting for the four top left values is used when method two has a greater than or equal probability of retaining the point as method one. The red formatting for the four bottom left cells is used when method one's probability of retaining the point is at least 0.005 greater than that of method two. The other formatting is used when method one has higher probability of retaining the point compared to method two, but not high enough for us to consider it to be practically useful. This is a crude measure of practical significance as otherwise it would involve constructing a utility curve which could be highly subjective in this case.
We will provide four sensitivity tables of this sort in the next two pages, each with a different α. The value d ranges from 2 to 30, the value p ranges from 0.0005 to 0.1 with a 0.0005 increment. We will not consider $p > 0.1$ as it would be absurd to have probability of losing an individual point being that high; in reality p would tend more towards the lower end of the scale. The values in the tables displayed below are not intended to be readable, the information we require is displayed through conditional formatting and we should view them as graphs

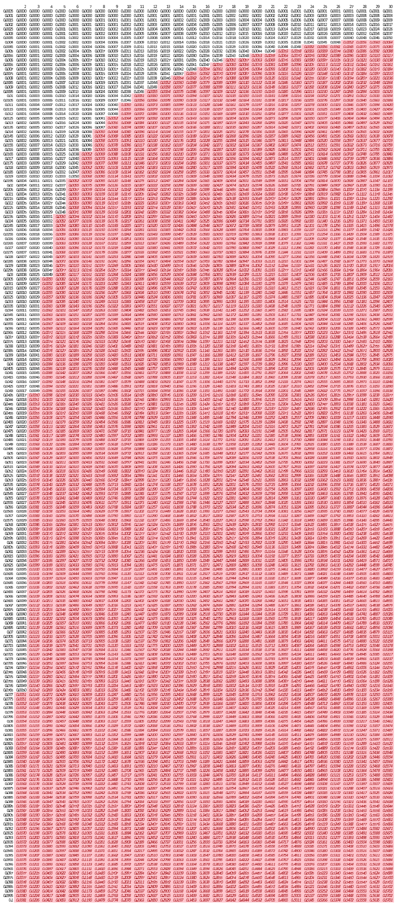


Figure 1: Sensitivity graph for $\alpha = 1$.

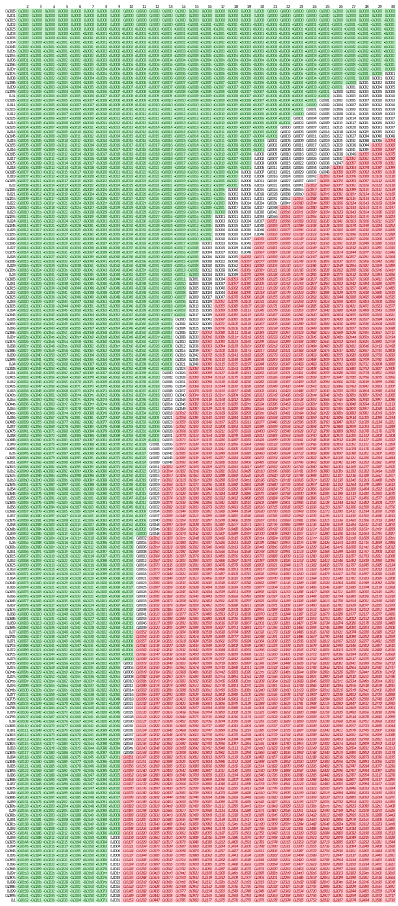


Figure 2: Sensitivity graph for $\alpha = 2$.

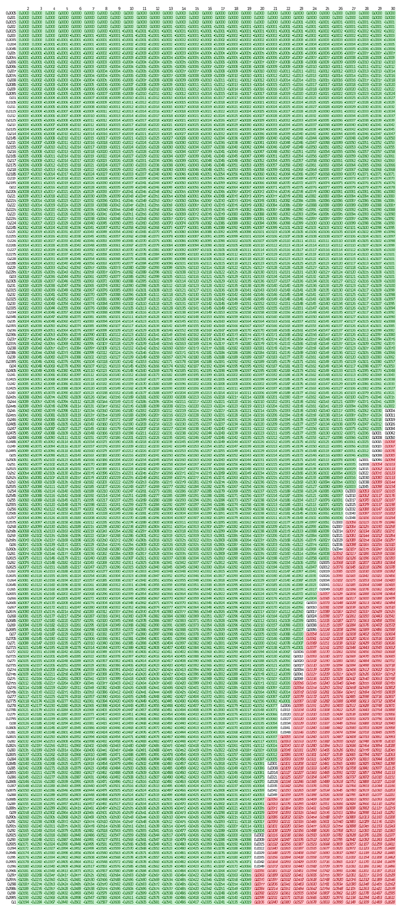


Figure 3: Sensitivity graph for $\alpha = 3$.

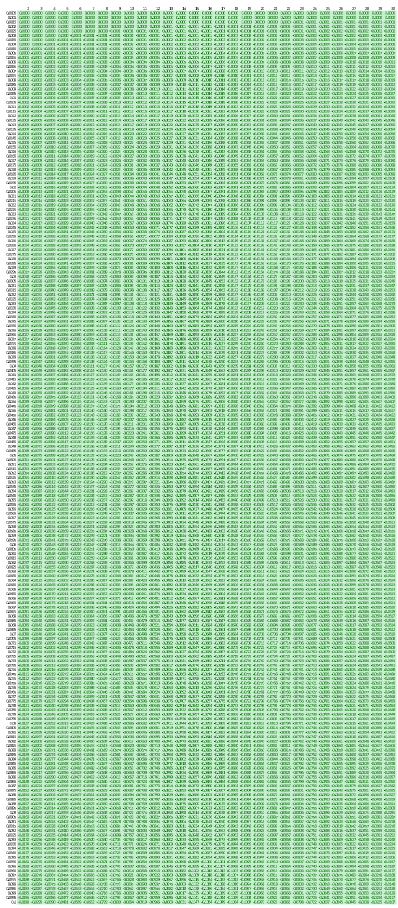


Figure 4: Sensitivity graph for $\alpha = 4$.

We can see above that in Figure 1, where $\alpha = 1$, making two copies of the same coordinate will always yield better probabilities of retaining the point as there are no green cells at all. When the number of redundant vectors added grows however, method two will produce better probabilities of retaining the point for smaller p and d. When $\alpha = 4$, method two always produces better probabilities of retaining the point; even when p and d are both large. This shows that adding redundant vectors is usually more efficient than making two copies of the same coordinate, as we need to keep fewer coordinates and this usually results with a higher probability of retaining the point.
Thus, this sensitivity analysis illustrates the efficiency of tight frames over repeated orthonormal vectors in providing redundancy in information encoding.

Appendix B

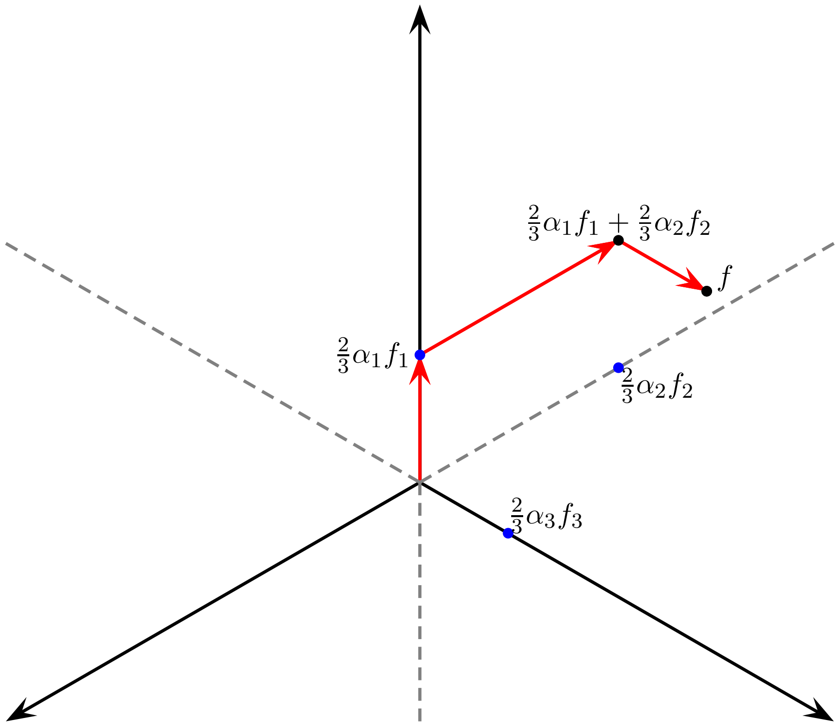
Reconstructing a point when partial information is lost

Recall Example 1.1. We will demonstrate how to recover a point recorded using the Mercedes-Benz frame when one coordinate is lost.

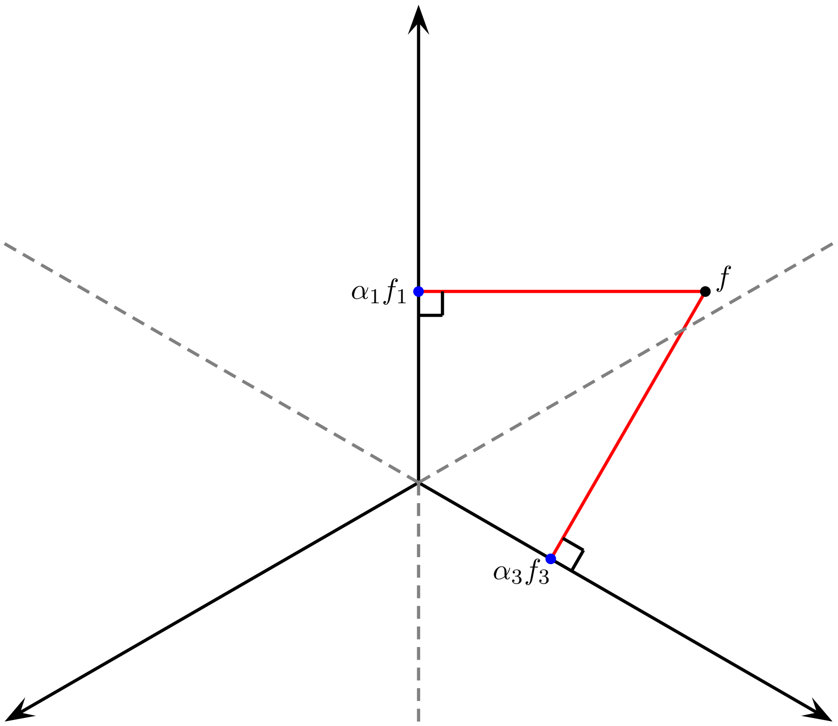
Let f_1, f_2, f_3 be as in Example 1.1, and f be the location of the Angle Man's secret lair that Superman would like to send to Wonder Woman. Superman transmits the coordinates $\alpha_i = \langle f, f_i \rangle$. If Wonder Woman receives all the coordinates, she can compute the location of the Angle Man's lair by

$$f = \frac{2}{3} \sum_{i=1}^{3} \alpha_i f_i.$$

See below for a diagram illustrating this.



To make sure the information is safe, Superman put each coordinate into an egg and sent the three eggs separately. Unfortunately, Robin stole an egg. Thus Wonder Woman only received two coordinates, say α_1 and α_3. Nothing is too hard for Wonder Woman. With her in depth knowledge of frames she set about computing the the location of Angle Man's lair the following ways. She ploted $\alpha_1 f_1$ and $\alpha_2 f_3$ on the atlas, draw the lines perpendicular to f_1 at $\alpha_1 f_1$ and perpendicular to f_3 at $\alpha_3 f_3$. The intersection of two lines provides her with location of Angle Man. Below is an illustration of how Wonder Woman found the location of the Angle Man so she can set out to destroy him and bring peace to the world.



Appendix C

Demonstration of the error reduction property of tight frames

Recall Example 1.1, the Mercedes-Benz frame. We will demonstrate how tight frame can reduce error compared with orthonormal bases.

Assume we are back in the situation of Appendix B. This time, Wonder Woman would like to send Superman the location g of Darkseid's secret lair. She knows that her Amazons would do a better job than the lousy postman Superman employs, so all coordinates should reach Superman. She also knows however that Lex Luthor will perturb the coordinates on the way with random error ε_i for each coordinate α_i, where $\varepsilon_i \sim N(0, \sigma^2)$ independently. Wonder Woman set about doing some calculations. If she chooses to send the location using an orthonormal basis, then the mean square error in each coordinate would be σ^2. In the Mercedes-Benz frame situation, she found the error of the reconstruction to be

$$g - \hat{g} = \frac{2}{3}\sum_{j=1}^{n} \langle g, f_j \rangle f_j - \frac{2}{3}\sum_{j=1}^{3} \left(\langle g, f_j \rangle + \varepsilon\right) f_j = -\frac{2}{3}\sum_{j=1}^{3} \varepsilon_j f_j.$$

Hence the average mean-squared error per component would be

$$\mathrm{MSE} = \frac{1}{2}\mathbb{E}\left\|g - \hat{g}\right\|^2 = \frac{1}{2}\mathbb{E}\left\|\frac{2}{3}\sum_{j=1}^{3} \varepsilon_j f_j\right\|^2 = \frac{1}{2}\sigma^2\frac{4}{9}\sum_{j=1}^{3}\left\|f_j\right\|^2 = \frac{2}{3}\sigma^2.$$

That is, the amount of error per component has been reduced using the Mercedes-Benz frame. Let $\hat{g}$ be the coordinates received by Superman. If Wonder Woman sent Superman the coordinates in an orthonormal basis, then if Superman wanders around in a circle of radius $2\sigma^2$ centered at $\hat{g}$, he will have approximately a 95% chance of finding Darkseid's secret lair. If Wonder Woman used the Mercedes-Benz frame coordinates, Superman would only need to wander around in a circle of radius $\frac{4}{3}\sigma^2$ to have approximately a 95% chance of finding Darkseid. Wonder Woman knows that Superman is extremely lazy and will go home to sleep if he doesn't find Darkseid quickly, therefore, sending the Mercedes-Benz frame coordinates is a better option. Wonder Woman could do even better by using the frames in Example 3.8. If she used a large number of equally spaced vectors in $\mathbb{R}^2$, this would make the error per component approach zero, and thus make it extra easy for Superman to be lazy.

Appendix D

Code for finding spherical t-designs

```
function [errm Vst angle Vm VVmm] = findingtdesign(d, n, t, iscomplex,k, Vin)

% d=the dimension we are in
% n=the number of vectors we want
% t= t-design integer
% iscomplex=1 if we are dealing with complex vectors
% this function finds V such that the set of vectors formed by its columns are ...
    closest
to being a t-design
% k=numbers of iteration to we run
% Vin=optional, it allow us to start the iteration at using a certain pre-frame ...
    operator

ct = iscomplex*factorial(t)*factorial(d-1)/factorial(d+t-1) + ...
    (1-iscomplex)*((factorial(2*t-1))/(factorial(t)*(2^t)))/(factorial(d+2*(t-1))/factorial(d-1));
%constant
errn=zeros(k,1);
%errn is current error
errm=zeros(k,1);
%minimal error we found so far
B=zeros(d,n);
%initializer.

if nargin==5
%if we don't have start matrix Vin, create a random one
Vm = (rand(d,n)-1/2)+iscomplex*1i*(rand(d,n)-1/2);
else
Vm=Vin;
%if we have a start matrix Vin, use that
end

Vm=(n/(sum(diag(Vm'*Vm).^t)))^(1/(2*t))*Vm;
%standardise it so norms sum to n which helps with comparability
Vst=Vm;
%save the matrix we start with
```

```
for j=2:k;
%run iterations

if iscomplex==1
B=1i*(rand(d,n)−1/2);
%make a random matrix B if in complex case
end

Vn=Vm+((errm(j−1)+0.1)/n^3)*(rand(d,n)−1/2+B);
%take the error minimising matrix, and make small random change to it, include ...
    complex
case if we are interested in complex design
Vn=(n/(sum(diag(Vn'*Vn).^t)))^(1/(2*t))*Vn;
%standardise it so norms sum to n which helps with comparability
for jj=1:n; w=Vn(:,jj); Vn(:,jj)=w/norm(w);end;
%keep this code if trying to find spherical t−design, comment out otherwise
V=Vn'*Vn; VV=Vm'*Vm;
%creat the Gram matrix for the current matrix and error minimising matrix we saved...
     from previous
iterations
errn(j)= norm(V(:),2*t)^(2*t)−ct*n^2;
%compute erro for current matrix
errm(j)= norm(VV(:),2*t)^(2*t)−ct*n^2;
%compute error for error minimising matrix

if errn(j) < errm(j);
%check if current matrix has smaller error then error minimising matrix
Vm=Vn;
%make current matrix the new error minimising matrix if it does have smaller error
end;
end;

plot(errm(10:k)/(n^2))
%to check that the error converged

A=Vm'*Vm;
angle=abs(A);
VVmm=Vm*Vm';
%out put the angle matrix, Gram matrix to check for angularity
```

Code for checking multiple local minimisers

```
function []=locmin(jj)
%jj=number of iterations we would like to run
res=[];
%create empty array to save minimising errors in
for a=1:jj
    [errm Vst angle Vm VVmm] = findingtdesign(d, n, t, iscomplex,k);
%run findingtdesign
```

```
    res(a)=errm(k);
%save the minimising errors
end;
diff=max(res)-min(res),
% output the difference between the maximum and minimum errors we found
```

Code for checking if something is a t-design

Since our main interest in this project are complex t-designs, this code is only appropriate for the complex case.

```
function [] = istdesign(G, t)
% G=Gram matrix of the vectors we want to check whether it is a t-design
% t= the kind of design we want to check
d=rank(G);
% to find out the dimension those vectors in
v=size(G);
% to find the size of G
n=v(1);
% since G is a square matrix, n x n will give us size of G
ct = factorial(t)*factorial(d-1)/factorial(d+t-1) ;
% calculate constant
sum(sum(abs(G).^(2*t)))-ct*(sum(diag(abs(G)).^t))^2,
%calculate the error value
% if output is close to 0 then it is a t-design
```