Generalised Jacobi Polynomials On A Simplex

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Abstract

A class of mutivariate orthogonal polynomials on a standard simplex in $\mathbb{R}^d$ was investigated. The motivation of this research comes from a study of the three-body problem in quantum physics. The research started by defining a generalised Jacobi weight function over the standard simplex in $\mathbb{R}^d$. Analytic computations of the inner product (integration of the weight function over the simplex) were carried out. The integration was shown to converge under restricted ranges of parameters, the corresponding integrability condition was established. Explicit formulas of orthogonal polynomials in this research were expressed by using the tight frame theory.The calculations were carried out for the space of orthogonal polynomials of degree one, it serves as a future reference for computing the orthogonal polynomials of degree two. It was concluded that a tight frame for the space of orthogonal polynomials of degree one under a special case of the generalised Jacobi weight exists, the corresponding frame scaling factors were computed.

Contents

Chapter 1

Introduction

An orthogonal polynomial sequence is a collection of polynomials such that each pair of distinct polynomials in the sequence are orthogonal. The orthogonality is defined with respect to an inner product associated with a weight function. The weight function is defined on a region in $\mathbb{R}^d$ with non-empty interior, where $d \in \mathbb{N}$ is the number of variables of a polynomial.

The theory of orthogonal polynomials, especially those of several variables, play an prominent role in various branches of modern mathematics and science; they include the approximation theory [1], differential equations [2], quantum physics [3] and statistics [4]. The study of orthogonal polynomials goes back at least as far as Hermite [5]. There has been enumerous developement of the theory since start of the twentieth century. People made significant contributions to the field include Appell, Bernstein and Hahn [5].The study on orthogonal polynomials rely heavily on other branches of mathematics, such as the theory of special functions, complex analysis, tight frame representations and computational mathematics.

In this research we investigated a class of mutivariate orthogonal polynomials on a standard simplex in $\mathbb{R}^d$ (Figures 1.1,1.2). The motivation of this research comes from a study of the three-body problem in quantum physics by Jean [3]. Jean aimed to expand the wave function in terms of orthogonal polynomials, the purpose was to see how the expansion converges. Jean observed that the orthogonal polynomials have a similar but more general weight than that of a classical type Jacobi polynomial over a simplex in $\mathbb{R}^2$. However, no literature on the explicit formulas of these genralised Jacobi polynomials was found by Jean and the author.

The research started by defining a generalised Jacobi weight function over the standard simplex in $\mathbb{R}^d$. Analytic computations of the inner product (integration of the weight function over the simplex) were carried out. The integration was shown to converge under restricted ranges of parameters, however it was difficult to find a closed form of the integral corresponding to the most general case. For so, some results were expressed in terms of incomplete Euler functions. Due to the diffuculty and complexity of integrations, the research focused on a case where some parameters in the generalised Jacobi weight are equal.

Explicit formulas of orthogonal polynomials in this research were expresed by using the frame theory. Shayne [6] ever found a tight frame representation for the space of classical Jacobi polynomials on a standard simplex in $\mathbb{R}^d$. In comparison to the Gram-Schmidt process, the advatage of applying the frame theory is that only the frame scaling factors and the leading term of a polynomial are required to

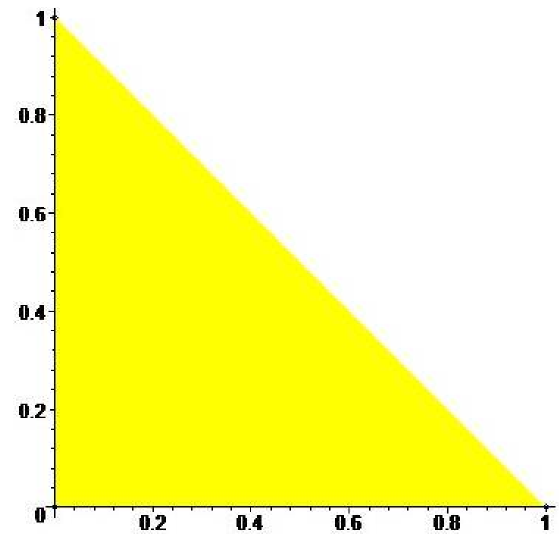

Figure 1.1: Standard simplex in $\mathbb{R}^2$

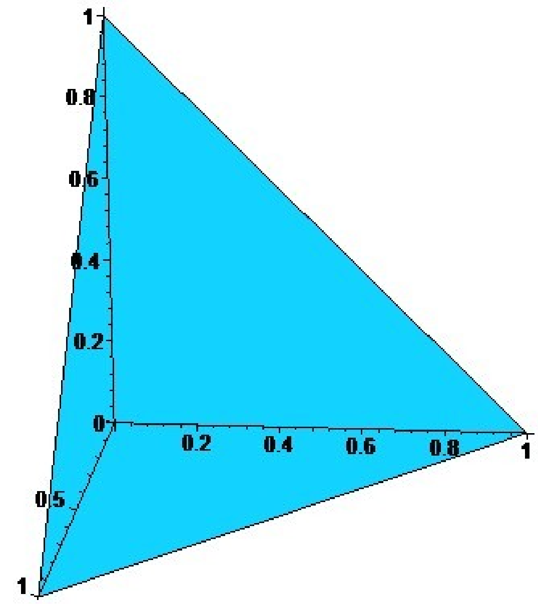

Figure 1.2: Standard simplex in $\mathbb{R}^3$

obtain its explicit formula. Besides, the symmetry of the space of orthogonal polynomials can be clearly depicted by the tight frame representation.

The calculations were carried out for the space of orthogonal polynomials of degree one, which serves as a future reference for computing the orthogonal polynomials of degree two. It was concluded that a tight frame for the space of orthogonal polynomials of degree one under a special case of the generalised Jacobi weight exists, the corresponding frame scaling factors were calculated.

Chapter 2

Euler integrals

In this chapter we introduce a class of special functions known as Euler integrals, they are closely related to the study of Jacobi polynomials. Some contents in this chapter are based on the book ”Special functions” by George E Andrews et al [7] and the book ”Orthogonal polynomials of several variables” by Yuan Xu [5] . Due to the nature of our research, domains of functions in this chapter are restricted to be subsets of $\mathbb{R}$. However, all functions presented have analytic continuations onto subsets of $\mathbb{C}$.

2.1 Complete Euler integrals

2.1.1 The Beta function (Euler integral of the first kind)

Definition 2.1.1. *The* **Beta function** *is the bivariate function*

$$B(p,q) := \int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x,$$

with domain $(0,+\infty)\times(0,+\infty)$.

Remark. To see how the domain arose, one may write the integral as the sum of two integrals

$$B(p,q) := \int_0^{1/2} x^{p-1}(1-x)^{q-1}\mathrm{d}x + \int_{1/2}^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x$$

with an observation that $x^{p-1}(1-x)^{q-1} \sim x^{p-1}$ as $x \to 0$, so the first integral on the right hand side converges if and only if $p > 0$. Similarly, $x^{p-1}(1-x)^{q-1} \sim (1-x)^{q-1}$ as $x \to 1$, thus the second integral on the right hand side converges if and only if $q > 0$. This implies that the function $B(p,q)$ converges on $(0,+\infty)\times(0,+\infty)$.

We next list some properties of the Beta function.

Proposition 2.1.1. *Let* $p,q \in (0,+\infty)$. *The Beta function satisfies the following*
(a) *B(p,q) is continuous on* $(0,+\infty)\times(0,+\infty)$.
(b) (Symmetry) $B(p,q) = B(q,p)$.

(c) (Recursion)

$$B(p,q) = \frac{q-1}{p+q-1} B(p,q-1), \forall\, p>0, q>1 \tag{2.1}$$

$$B(p,q) = \frac{p-1}{p+q-1} B(p-1,q), \forall\, p>1, q>0 \tag{2.2}$$

$$B(p,q) = \frac{(p-1)(q-1)}{(p+q-1)(p+q-2)} B(p-1,q-1), \forall\, p>1, q>1 \tag{2.3}$$

.

(d) $B(p,q) = 2\int_0^{\frac{\pi}{2}} \sin^{2q-1}\varphi \cos^{2p-1}\varphi \mathrm{d}\varphi$.

Proof. (a). Fix $p_0 > 0, q_0 > 0$ and let $p \geq p_0, q \geq q_0$ to get $x^{p-1}(1-x)^{q-1} \leq x^{p_0-1}(1-x)^{q_0-1}, \forall x \in [0,1]$. Since $\int_0^1 x^{p_0-1}(1-x)^{q_0-1}\mathrm{d}x$ converges, by Weierstrass criterion to get $\int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x$ uniformly converges on $[p_0,+\infty) \times [q_0,+\infty)$. Thus $B(p,q)$ is continuous on $[p_0,+\infty) \times [q_0,+\infty)$. Since $p_0 > 0, q_0 > 0$ were chosen arbitrarily, therefore $B(p,q)$ is continuous on $(0,+\infty) \times (0,+\infty)$.

(b). Let $x = 1-t$ to get

$$\begin{aligned} B(p,q) &= \int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x \\ &= \int_0^1 (1-t)^{p-1}t^{q-1}\mathrm{d}t \\ &= B(q,p). \end{aligned}$$

(c).Observe that (2.2) follows from (2.1) by symmetry, (2.3) is a consequence of combining (2.1) and (2.2). Thus it is sufficient to prove (2.1).
Use integration by parts to get

$$\begin{aligned} B(p,q) &= \int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x \\ &= \frac{x^p(1-x)^{q-1}}{p}\Big|_0^1 + \frac{q-1}{p}\int_0^1 x^p(1-x)^{q-2}\mathrm{d}x \\ &= \frac{q-1}{p}\int_0^1 [x^{p-1} - x^{p-1}(1-x)](1-x)^{q-2}\mathrm{d}x \\ &= \frac{q-1}{p}\int_0^1 x^{p-1}(1-x)^{q-2}\mathrm{d}x - \frac{q-1}{p}\int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x \\ &= \frac{q-1}{p}B(p,q-1) - \frac{q-1}{p}B(p,q), \end{aligned}$$

Rearange $B(p,q)$ to the left hand side to get $B(p,q) = \frac{q-1}{p+q-1}B(p,q-1)$.

(d). This follows by making the substitution $x = \cos^2\varphi$ in the Beta function. □

Remark. The Beta function is partial differentiable, its partial derivatives involves Gamma funtions [7] which is introduced next.

2.1.2 The Gamma function (Euler integral of the second kind)

Definition 2.1.2. *The* **Gamma function** *is the function*

$$\Gamma(s) := \int_0^{+\infty} x^{s-1}e^{-x}\mathrm{d}x,$$

with domain $(0, +\infty)$.

Remark. To see how the domain arose, one may write the integral as the sum of two integrals

$$\Gamma(s) = \int_0^1 x^{s-1}e^{-x}\mathrm{d}x + \int_1^{+\infty} x^{s-1}e^{-x}\mathrm{d}x$$

and observe by the convergence criterion for improper integrals that the first integral diverges if $s \leq 0$, and both integrals are convergent if $s > 0$. This implies the domain of the Gamma function is $(0, +\infty)$.

The Gamma function has some nice properties.

Proposition 2.1.2. *Let* $s \in (0, +\infty)$. *The Gamma function satisfies the following*
(a) Γ *is continuous and differentiable on* $(0, +\infty)$.
(b) (Recursion) $\Gamma(s+1) = s\Gamma(s)$.

Proof. (a). Let $[a, b] \subset (0, +\infty)$ with $a < b$. Observe that for each $s \in [a, b]$ one has

$$x^{s-1}e^{-x} \leq x^{a-1}e^{-x}, \forall\, x \in (0, 1].$$

Since $\int_0^1 x^{a-1}e^{-x}\mathrm{d}x$ converges, by Weierstrass criterion to get $\int_0^1 x^{s-1}e^{-x}\mathrm{d}x$ uniformly converges with respect to s on $[a, b]$. On the other hand, for each $s \in [a, b]$ it has $x^{s-1}e^{-x} \leq x^{b-1}e^{-x}, \forall x \in (0, 1]$. By Weierstrass criterion again to get $\int_1^{\infty} x^{s-1}e^{-x}\mathrm{d}x$ uniformly converges with respect to s on $[a, b]$. So $\Gamma(s)$ is uniformly convergent with respect to s on $[a, b]$, therefore Γ is continuous on $[a, b]$.
By using similar arguments one can show

$$\int_0^{+\infty} \frac{\partial}{\partial s}(x^{s-1}e^{-x})\mathrm{d}x = \int_0^{+\infty} x^{s-1}e^{-x}\ln x\mathrm{d}x.$$

converges uniformly on $[a, b]$, thus Γ is differentiable on $[a, b]$. Since $[a, b] \subset (0, +\infty)$ was chosen arbitrarily, so Γ is differentiable on $(0, +\infty)$ with derivative $\Gamma'(s) = \int_0^{+\infty} x^{s-1}e^{-x}\ln x\mathrm{d}x$.

(b).Let $A \in (0, +\infty)$, by using integration by parts to get

$$\begin{aligned}\int_0^A x^s e^{-x}\mathrm{d}x &= -x^s e^{-x}\Big|_0^A + s\int_0^A x^{s-1}e^{-x}\mathrm{d}x \\ &= -A^s e^{-A} + s\int_0^A x^{s-1}e^{-x}\mathrm{d}x.\end{aligned}$$

Let $A \to +\infty$ to get the recursion: $\Gamma(s+1) = s\Gamma(s)$. □

Remark. In fact Γ is C^∞ on $(0,+\infty)$, the proof is by induction and a similar argument used in the proof for part (a). It turns out that $\Gamma^{(n)}(s) = \int_0^{+\infty} x^{s-1}e^{-x}(\ln x)^n \mathrm{d}x$.
For each $n \in \mathbb{N}$ it follows from part (b) that $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1)... = n!\Gamma(1)$. Combine with the fact that $\Gamma(1) = \int_0^{+\infty} e^{-x}\mathrm{d}x = 1$ to get $\Gamma(n+1) = n!$, this illustrates that Γ is a continuation of the factorial fuction.

Lemma 2.1.1. *The domain of* Γ *can be extended onto* $\mathbb{R} - \mathbb{Z}_{\leq 0}$*, where* $\mathbb{Z}_{\leq 0} := \{0, -1, -2, \cdots\}$.

Proof. By Proposition 2.1.2 (b) to get $\Gamma(s) = \frac{\Gamma(s+1)}{s}$, observe that its right hand side is a real number for $s \in (-1,0)$. Thus $\Gamma(s)$ on $(-1,0)$ can be defined according to $\frac{\Gamma(s+1)}{s}$. Applying the same argument inductively to extend the domain of Γ onto $(-\infty,+\infty) - \mathbb{Z}_{\leq 0}$. □

We now establish the connection between the Beta and Gamma functions, this connection will be used extensively in Chaper 3 and 4.

Theorem 2.1.1. *For each* $p, q \in (0,+\infty)$*,the Beta and Gamma functions have the following relation*

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

.

Proof. Observe by making the substitution $x = t^2$ in the Gamma function that

$$\Gamma(p) = 2\int_0^{+\infty} t^{2p-1}e^{-t^2}\mathrm{d}t, \;\; \Gamma(q) = 2\int_0^{+\infty} t^{2q-1}e^{-t^2}\mathrm{d}t.$$

Let $\Omega := \{(s,t) | s \in [0,+\infty) \text{ and } t \in [0,+\infty)\}$. One has

$$\Gamma(p)\Gamma(q) = 4\int_0^{+\infty} s^{2p-1}e^{-s^2}\mathrm{d}s \int_0^{+\infty} t^{2q-1}e^{-t^2}\mathrm{d}t = 4\iint\limits_{\Omega} s^{2p-1}e^{-s^2}t^{2q-1}e^{-t^2}\mathrm{d}s\mathrm{d}t.$$

Make the substitutions $s = r\cos\theta$, $t = r\sin\theta$ and by Proposition 2.1.1 (d) to get

$$\begin{aligned}
\Gamma(p)\Gamma(q) &= 4\int_0^{+\infty}\int_0^{\frac{\pi}{2}} r^{2(p+q)-1}e^{-r^2}\cos^{2p-1}\theta\sin^{2q-1}\theta\mathrm{d}\theta\mathrm{d}r \\
&= (2\int_0^{\frac{\pi}{2}}\cos^{2p-1}\theta\sin^{2q-1}\theta\mathrm{d}\theta)(2\int_0^{+\infty} r^{2(p+q)-1}e^{-r^2}\mathrm{d}r) \\
&= B(p,q)\Gamma(p+q).
\end{aligned}$$

□

Example 2.1.1. *Let* $T^2 := \{(x,y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, x+y \leq 1\}$ *be the standard simplex in* $\mathbb{R}^2$*. Given* $m, n, p \in (0,+\infty)$ *as constants. Calculate the integral*

$$\iint\limits_{T^2} x^{m-1}y^{n-1}(1-x-y)^{p-1}\mathrm{d}x\mathrm{d}y.$$

Solution. Let $T^3 := \{(x,y,z) \in \mathbb{R}^3 | x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1\}$ Observe that

$$\begin{aligned}(p-1)\iiint\limits_{T^3} x^{m-1}y^{n-1}z^{p-2}\mathrm{d}z\mathrm{d}y\mathrm{d}x &= \iint\limits_{T^2}\left(\int_0^{1-x-y}(p-1)x^{m-1}y^{n-1}z^{p-2}\,dz\right)\mathrm{d}y\mathrm{d}x \\ &= \iint\limits_{T^2} x^{m-1}y^{n-1}z^{p-1}\Big|_0^{1-x-y}\,\mathrm{d}y\mathrm{d}x \\ &= \iint\limits_{T^2} x^{m-1}y^{n-1}(1-x-y)^{p-1}\mathrm{d}y\mathrm{d}x.\end{aligned}$$

Make the substitution of variables $\begin{cases} x = u^2 \\ y = v^2 \\ z = w^2 \end{cases}$ and $\begin{cases} u = r\sin\varphi\cos\theta \\ v = r\sin\varphi\sin\theta \\ w = r\cos\varphi \end{cases}$. By Proposition 2.1.1 (d) and Theorem 2.1.1 to get

$$\begin{aligned}&\iint\limits_{T^2} x^{m-1}y^{n-1}(1-x-y)^{p-1}\mathrm{d}x\mathrm{d}y \\ &= (p-1)\iiint\limits_{T^3} x^{m-1}y^{n-1}z^{p-2}\mathrm{d}z\mathrm{d}y\mathrm{d}x \\ &= 8(p-1)\int_0^{\frac{\pi}{2}}\sin^{2n-1}\theta\cos^{2m-1}\theta\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\sin^{2m+2n-1}\varphi\cos^{2p-3}\varphi\mathrm{d}\varphi\int_0^1 r^{2m+2n+2p-3}\mathrm{d}r \\ &= \frac{B(m,n)B(p-1,m+n)}{m+n+p-1} \\ &= \frac{(p-1)\Gamma(m)\Gamma(n)\Gamma(p-1)\Gamma(m+n)}{(m+n+p-1)\Gamma(m+n)\Gamma(m+n+p-1)} \\ &= \frac{\Gamma(m)\Gamma(n)\Gamma(p)}{\Gamma(m+n+p)}.\end{aligned}$$

Remark. Above example is the integral of the classical Jacobi weight function on the standard simplex T^2 in $\mathbb{R}^2$. The general result for the d-dimensional case will be presented in Chapter 4. As we shall see, the complexity of computing a class of orthogonal polynomials is primarily governed by the complexity of integrating the weight function.

2.2 Incomplete Beta functions

Definition 2.2.1. *The* **Pochhammer symbol**, *also known as the falling factorial, is defined for all* $x \in \mathbb{R}$ *by*

$$(x)_n = \begin{cases} 1 & n = 0; \\ \prod_{i=1}^n (x+i-1) & n \in \mathbb{N} = \{1,2,\cdots\}. \end{cases}$$

Definition 2.2.2. *Let* $a, b \in \mathbb{R}$ *and let* $c \in \mathbb{R} - \mathbb{N}_{\leq 0}$*. A* **hypergeometric function** *is defined for* $x \in \mathbb{R}$ *with* $|x| < 1$ *by the power series*

$$ {}_2F_1(a, b; c; x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}. $$

Next we present the definition of an incomplete Beta function and its relation to the hypergeometric function.

Definition 2.2.3. *Let* $x \in [0, 1]$*. An* **incomplete Beta function** *is the bivariate function*

$$ B_x(p, q) := \int_0^x t^{p-1}(1-t)^{q-1} \mathrm{d}x, $$

with domain $(0, +\infty) \times (0, +\infty)$.

Remark. It's clear that an incomplete Beta function generalises the Beta function. The adjective 'incomplete' reflects the fact that the upper limit x in an Euler's integral of the first kind is allowed to be less than the value of unity. Unlike the (complete) Beta function, the incompleteness prevents the interchangeablility of p and q, this is precised in the following proposition.

Proposition 2.2.1. *Let* $x \in [0, 1]$ *and let* $p, q \in (0, +\infty)$*. An incomplete Beta function satisfies the following*
(a) (Symmetry) $B_x(p, q) = B(p, q) - B_{1-x}(q, p)$.
(b) (Recursion)

$$ B_x(p+1, q) = \frac{p}{q} B_x(p, q+1) - \frac{x^p(1-x)^q}{q} \tag{2.4} $$

$$ B_x(p, q+1) = \frac{q}{p} B_x(p+1, q) + \frac{x^p(1-x)^q}{p} \tag{2.5} $$

$$ B_x(p, q) = B_x(p+1, q) + B_x(p, q+1) \tag{2.6} $$

.
(c) $B_x(p, q) = \left(\frac{x^p}{p}\right) {}_2F_1(p, 1-q; 1+p; x)$.

Proof. (a). By definition to get

$$ \begin{aligned} B(p, q) &= \int_0^1 t^{p-1}(1-t)^{q-1} \mathrm{d}t \\ &= \int_0^x t^{p-1}(1-t)^{q-1} \mathrm{d}t + B(p, q) \int_x^1 t^{p-1}(1-t)^{q-1} \mathrm{d}t \\ &= B_x(p, q) + \int_x^1 t^{p-1}(1-t)^{q-1} \mathrm{d}t. \end{aligned} \tag{2.7} $$

Make the substitution $t = 1 - u$ to get

$$\begin{aligned}\int_x^1 t^{p-1}(1-t)^{q-1}\mathrm{d}t &= -\int_{t=x}^{t=1}(1-u)^{p-1}u^{q-1}\mathrm{d}u \\ &= \int_0^{1-x} u^{q-1}(1-u)^{p-1}\mathrm{d}u \\ &= B_{1-x}(q,p). \end{aligned} \tag{2.8}$$

Substitute (2.7) to (2.8) and rearrange to get $B_x(p,q) = B(p,q) - B_{1-x}(q,p)$.

(b). Use integration by parts to get

$$\frac{p}{q}\int_0^x t^{p-1}(1-t)^q\mathrm{d}t = \frac{t^p(1-t)^q}{q}\Big|_0^x + \int_0^x t^p(1-t)^{q-1}.$$

Rearrange and by definition to get $B_x(p+1,q) = \frac{p}{q}B_x(p,q+1) - \frac{x^p(1-x)^q}{q}$. (2.5) follows from the (2.4) and part(a). (2.6) follows by combining (2.4) and (2.5). □

(c). See [7]

Chapter 3

Orthogonal polynomials of one variable

This chapter reviews some basic knowledge on orthogonal polynomials of one variable. Among all classical weights of orthogonal polynomials, the classical Jacobi weight includes a variety of other weights as special cases [5], it is thus desirable to focus on the study of Jacobi polynomials. Properties of Jacobi polynomials of one variable are investigated, they include the leading coefficient, structural constant, symmetry and the three term relations. It is worth mention that in this research we adopt another approach to study orthogonal polynomials of several variables, the procedure will be presented in Chapter 5.

3.1 A glance of orthogonal polynomials

3.1.1 General definitions

Definition 3.1.1. *Let X be a non-empty interval in $\mathbb{R}$ and let μ be a probability measure on X. Given the space of μ-measurable functions $L^2(X,\mu)$, define $\langle f,g\rangle := \int_X fg\mathrm{d}\mu$ to be the inner product of f and g, where f and g are polynomial functions in $L^2(X,\mu)$.*

Remark. For classical type orthogonal polynomials, the probability measure has the form $\mathrm{d}\mu(x) = cw(x)\mathrm{d}x$, with $w(x) > 0$ on X. The function $w(x)$ is called the weight function and the constant $c := \left(\int_X w(x)\mathrm{d}x\right)^{-1}$ is the corresponding normalization factor. As most researches on orthogonal polynomials do, we will use the weight function as the characteristic of a class of orthogonal polynomials.

The orthogonality is defined with respect to above inner product. Following we assume X and μ are pre-defined region and measure.

Definition 3.1.2. *A sequence of non-zero polynomials $\{P_n(x) : n \in \mathbb{N}_0 = \{0,1,\cdots\}\}$ in $L^2(X,\mu)$ is an* **orthogonal basis of polynomials** *if it satisfies the following*

- $\{P_n(x) : n \in \mathbb{N}_0\}$ *is a basis of polynomials in $L^2(X,\mu)$;*

- *$P_n(x)$ has degree n;*
- *$\langle P_n, x_j \rangle = 0, \forall j \in \mathbb{N}$ with $j < n$.*

Remark. The squared norm $\int_X P_n(x)^2 \mathrm{d}\mu(x) = h_n$ is called the structural constant. Moreover, we denote $p_n(x) = \pm h_n^{-\frac{1}{2}} P_n(x)$, with the sign determined by the the sign of the leading coefficient of $P_n(x)$.

A common way to obtain a sequence of orthogonal polynomials is to apply the Gram-Schmidt algorithm to a basis of polynomials, for example to the basis $\{x^i : i \in \mathbb{N}_0\}$. The Gram-Schmidt algorithm preserves the linear independence of a basis, thus it outputs an orthogonal basis of polynomials. For conveniency one sometimes needs an orthonormal basis of polynomials, it can be obtained by normalising the output of the Gram-Schmidt algorithm. It was shown that a sequence of monic orthonormal polynomials is uniquely determined by the region X and the weight function $w(x)$ [8].

3.1.2 Three-term recurrence

A major disadvantage of applying Gram-Schmidt algorithm to compute orthogonal polynomials is that it acquires a large amount of computations. Besides, it is usually difficult to see the intrinsic relation between orthogonal polynomials based on the outcome of Gram-Schmidt algorithm. As a way to improve the computations, a recurssion relation between three consecutive orthogonal polynomials may be used instead. As we shall see in Chapter 5, a similar approach may be adopted for the study of orthogonal polynomials of several variables, where one uses the tight frame representations.

Proposition 3.1.1. *Let $\{P_n(x) : n \in \mathbb{N}_0 \cup \{-1\}\}$ be a sequence of orthogonal polynomials with $P_{-1} = 0$. There exists sequences $(A_n)_{n\geq 0}, (B_n)_{n\geq 0}, (C_n)_{n\geq 0}$ such that*

$$P_{n+1}(x) = (A_n x + B_n)P_n(x) - C_n P_{n-1}(x),$$

where $A_n = \frac{k_{n+1}}{k_n}, B_n = -\frac{k_{n+1}}{k_n h_n} \int_X x P_n(x)^2 \mathrm{d}\mu(x), C_n = \frac{k_{n+1} k_{n-1} h_n}{k_n^2 h_{n-1}}$ and k_n is the leading coefficient of $P_n(x)$.

Proof. Since $\{P_n(x) : n \in \mathbb{N}_0\}$ is a basis of polynomials and $xP_n(x)$ is of degree $n+1$, there exists a sequence of constants $\{a_i : i \in \mathbb{N}_0\}$ such that $xP_n(x) = \sum_{i=0}^{n+1} a_i P_i(x)$. For each $j \leq n+1$ one has

$$\begin{aligned}
\int_X P_n(x) P_j(x) \mathrm{d}\mu(x) &= \int_X \left(\sum_{i=0}^{n+1} a_i P_i(x) \right) P_j(x) \mathrm{d}\mu(x) \\
&= a_j \int_X P_j(x)^2 \mathrm{d}\mu(x) + \sum_{i \neq j} \int_X a_i P_i(x) P_j(x) \mathrm{d}\mu(x) \\
&= a_j h_j.
\end{aligned}$$

Note that the last line of above computation used the orthogonality relation $\langle P_i(x), P_j(x) \rangle = 0, \forall i \neq j$. It implies $a_j = \frac{1}{h_j} \int_X x P_j(x) P_n(x) \mathrm{d}\mu(x)$. Since $xP_j(x)$ has degree $j+1$, so $a_j = 0$ if $|n-j| > 1$. Therefore

$$xP_n(x) = a_{n-1} P_{n-1}(x) + a_n P_n(x) + a_{n+1} P_{n+1}(x). \tag{3.1}$$

Observe that the leading coefficients of $xP_n(x)$ and $a_{n+1}P_{n+1}(x)$ are equal, thus $k_n = a_{n+1}k_{n+1}$, this implies $a_{n+1} = \frac{k_n}{k_{n+1}}$. Moreover, $a_n = \frac{1}{h_n}\int_X xP_n(x)^2 \mathrm{d}\mu(x)$ and

$$\begin{aligned}
a_{n-1} &= \frac{1}{h_{n-1}}\int_X xP_{n-1}(x)P_n(x)\mathrm{d}\mu(x)\\
&= \frac{1}{h_{n-1}}\int_X \left(\frac{k_{n-1}}{k_n}P_n(x) + \text{lower order terms}\right)P_n(x)\mathrm{d}\mu(x)\\
&= \frac{1}{h_{n-1}}\int_X \frac{k_{n-1}}{k_n}P_n(x)^2\mathrm{d}\mu(x)\\
&= \frac{k_{n-1}h_n}{h_{n-1}k_n}.
\end{aligned}$$

Now rearrange (3.1) to get

$$\begin{aligned}
P_{n+1}(x) &= \left(\frac{x}{a_{n+1}} - \frac{a_n}{a_{n+1}}\right)P_n(x) - \frac{a_{n-1}}{a_{n+1}}P_{n-1}(x)\\
&= \left(\frac{k_{n+1}}{k_n}x - \frac{k_{n+1}}{k_n h_n}\int_X xP_n(x)^2\mathrm{d}\mu(x)\right)P_n(x) - \frac{k_{n+1}k_{n-1}h_n}{h_{n-1}k_n^2}P_{n-1}(x). \qquad (3.2)
\end{aligned}$$

According to (3.2) to get $A_n = \frac{k_{n+1}}{k_n}, B_n = -\frac{k_{n+1}}{k_n h_n}\int_X xP_n(x)^2\mathrm{d}\mu(x), C_n = \frac{k_{n+1}k_{n-1}h_n}{k_n^2 h_{n-1}}$. □

Corollary 3.1.1. *Let the leading coefficient of $P_n(x)$ be k_n and let $b_n = \int_X xp_n(x)^2\mathrm{d}\mu(x)$, for each $n \in \mathbb{N}$ one has*

$$xP_n(x) = \frac{k_n}{k_{n+1}}P_{n+1}(x) + b_nP_n(x) + \frac{k_{n-1}h_n}{k_nh_{n-1}}P_{n-1}(x).$$

Proof. This follows by rearranging the three-term relation. Observe that $-\frac{B_n}{A_n}P_n = \frac{1}{h_n}\int_X xP_n(x)^2\mathrm{d}\mu(x) = \int_X x\frac{P_n(x)^2}{h_n}\mathrm{d}\mu(x) = b_n$ (cf. *Remark* after Definition 3.1.2) . □

As an application of the three-term relation, one may derive the Christoffel-Darboux formula.

Proposition 3.1.2. *Let k_n be the leading coefficient of p_n (cf. remark after Definition 3.1.2), for each $n \in \mathbb{N}$ one has the following*

$$\sum_{j=0}^{n} p_j(x)p_j(y) = \frac{k_n}{k_{n+1}}\frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y}. \qquad (3.3)$$

$$\sum_{j=0}^{n} p_j(x)^2 = \frac{k_n}{k_{n+1}}\left(p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x)\right). \qquad (3.4)$$

Proof. According to Corollary 3.1.1, for each $j \in \mathbb{N}$ one has

$$\begin{aligned}
&(x-y)p_j(x)p_j(y) \\
&= (xp_j(x))\,p_j(y) - (yp_j(y))\,p_j(x) \\
&= \left(\frac{k_j}{k_{j+1}}p_{j+1}(x)p_j(y) + b_j p_j(x)p_j(y) + \frac{k_{j-1}}{k_j}p_{j-1}(x)p_j(y)\right) \\
&- \left(\frac{k_j}{k_{j+1}}p_{j+1}(y)p_j(x) + b_j p_j(y)p_j(x) + \frac{k_{j-1}}{k_j}p_{j-1}(y)p_j(x)\right) \\
&= \frac{k_j}{k_{j+1}}\left[p_{j+1}(x)p_j(y) - p_j(x)p_{j+1}(y)\right] + \frac{k_{j-1}}{k_j}\left[p_{j-1}(x)p_j(y) - p_j(x)p_{j-1}(y)\right].
\end{aligned}$$

Denote $p_1(x) = k_1 x - \alpha$, sum above equations over $0 \le j \le n$ to get

$$\begin{aligned}
&\sum_{j=0}^{n}(x-y)p_j(x)p_j(y) \\
&= (x-y)p_0(x)p_0(y) + \sum_{j=1}^{n}(x-y)p_j(x)p_j(y) \\
&= (x-y)k_0^2 + \left(\frac{k_1}{k_2}\left(p_2(x)p_1(y) - p_1(x)p_2(y)\right) + \frac{k_0}{k_1}\left(k_0(k_1 y - \alpha) - k_0(k_1 x - \alpha)\right)\right) \\
&+ \left(\frac{k_2}{k_3}\left(p_3(x)p_2(y) - p_2(x)p_3(y)\right) + \frac{k_1}{k_2}\left(p_1(x)p_2(y) - p_1(y)p_2(x)\right)\right) \\
&+ \ldots + \left(\frac{k_n}{k_{n+1}}\left(p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)\right) + \frac{k_{n-1}}{k_n}\left(p_{n-1}(x)p_n(y) - p_n(x)p_{n-1}(y)\right)\right).
\end{aligned}$$

Observe that the above sum telescopes, it is equal to $\frac{k_n}{k_{n+1}}\left(p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)\right)$. Rearrange the result to get (3.3).
One may next view x as a constant and take the limit as $y \to x$ in (3.3) to get

$$\begin{aligned}
\sum_{j=0}^{n} P_j(x)^2 &= \lim_{y\to x}\frac{k_n}{k_{n+1}}\frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y} \\
&= \frac{k_n}{k_{n+1}}\lim_{y\to x}\frac{p_{n+1}(x)p_n'(y) - p_n(x)p_{n+1}'(y)}{-1} \\
&= \frac{k_n}{k_{n+1}}\lim_{y\to x}\left(p_n(x)p_{n+1}'(y) - p_{n+1}(x)p_n'(y)\right) \\
&= \frac{k_n}{k_{n+1}}\left(p_n(x)p_{n+1}'(x) - p_{n+1}(x)p_n'(x)\right).
\end{aligned}$$

Observe that the L-Hospital's rule was applied at the second step above. □

3.2 Classical Jacobi polynomials of one variable

Throughout this section we let $\alpha, \beta \in (-1, +\infty)$ to be constants.

Definition 3.2.1. *For each $n \in \mathbb{N}_0$, let*

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right). \tag{3.5}$$

Define $\{P_n^{(\alpha,\beta)}(x) : n \in \mathbb{N}_0\}$ to be the sequence of Jacobi polynomials.

Remark. The Jacobi weight function is $(1-x)^\alpha(1+x)^\beta$ on the interval $(-1,1)$.The corresponding normalization constant is computed to be $2^{-\alpha-\beta-1}B(\alpha+1,\beta+1)^{-1}$.

(3.5) can be expressed in terms of a hypergeometric function (cf. Definition 2.2.2).

Proposition 3.2.1. *For each $n \in \mathbb{N}$,*

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{(\alpha+1)_n}{n!}\left(\frac{1+x}{2}\right)^n {}_2F_1(-n,-n-\beta;\alpha+1;\frac{x-1}{x+1}) \\ &= \frac{(\alpha+1)_n}{n!}{}_2F_1(-n,n+\alpha+\beta+1;\alpha+1;\frac{1-x}{2}). \end{aligned}$$

Proof. Let $f(x) = (1-x)^{\alpha+n}$ and $g(x) = (1+x)^{\beta+n}$. Apply the Leibniz rule for derivatives to get

$$\begin{aligned} &\frac{d^n}{dx^n}\left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right) \\ &= \sum_{j=0}^{n}\binom{n}{j}f^{(n-j)}(x)g^{(j)}(x) \\ &= \sum_{j=0}^{n}\binom{n}{j}(-n-\alpha)_{n-j}(1-x)^{\alpha+j}(-n-\beta)_j(-1)^j(1+x)^{\beta+n-j} \\ &= \sum_{j=0}^{n}\frac{(-n)_j(-1)^j}{j!}\frac{(-1)^{n-j}(\alpha+1)_n}{(\alpha+1)_j}(-n-\beta)_j(-1)^j(1-x)^{\alpha+j}(1+x)^{\beta+n-j}. \end{aligned} \tag{3.6}$$

Combine Definition 3.2.1 and (3.6) to get

$$\begin{aligned} &P_n^{(\alpha,\beta)}(x) \\ &= \frac{(-1)^n}{2^n n!}\sum_{j=0}^{n}\frac{(-n)_j(-1)^j}{j!}\frac{(-1)^{n-j}(\alpha+1)_n}{(\alpha+1)_j}(-n-\beta)_j(-1)^j(1-x)^j(1+x)^{n-j} \\ &= \frac{(1+x)^n}{2^n n!}\sum_{j=0}^{n}\frac{(-n)_j(-1)^{j+n}}{j!}\frac{(-1)^{n-j}(\alpha+1)_n}{(\alpha+1)_j}(-n-\beta)_j(-1)^{2j}\left(\frac{x-1}{x+1}\right)^j \\ &= \frac{1}{n!}\left(\frac{1+x}{2}\right)^n\sum_{j=0}^{n}\frac{(-n)_j(\alpha+1)_n(-n-\beta)_j}{j!(\alpha+1)_j}\left(\frac{x-1}{x+1}\right)^j \\ &= \frac{(\alpha+1)_n}{n!}\left(\frac{1+x}{2}\right)^n\sum_{j=0}^{n}\frac{(-n)_j(-n-\beta)_j}{j!(\alpha+1)_j}\left(\frac{x-1}{x+1}\right)^j \\ &= \frac{(\alpha+1)_n}{n!}\left(\frac{1+x}{2}\right)^n {}_2F_1(-n,-n-\beta;\alpha+1;\frac{x-1}{x+1}). \end{aligned}$$

Applying the relation ${}_2F_1(a, b; c; x) = (1-x)^{-a}{}_2F_1(a, c-b; c; \frac{x}{x-1})$ [5] to the result above to get an alternative expression of $P_n^{(\alpha,\beta)}(x)$ stated in the proposition. $\square$

The orthogonality relations and structural constants of Jacobi polynomials can be computed explicitly.

Proposition 3.2.2. *Let $\{P_n^{(\alpha,\beta)}(x) : n \in \mathbb{N}\}$ be the sequence of Jacobi polynomials. It has following properties*
(a) *The leading coefficient of $P_n^{(\alpha,\beta)}(x)$ is*

$$k_n = \frac{(n+\alpha+\beta+1)_n}{2^n n!}.$$

(b)

$$\int_{-1}^{1} q(x)P_n^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}\mathrm{d}x = \frac{1}{2^n n!}\int_{-1}^{1}\frac{d^n}{dx^n}\left(q(x)\right)(1-x)^{\alpha+n}(1+x)^{\beta+n}\mathrm{d}x,$$

for any polynomial $q(x)$.
(c)

$$\int_{-1}^{1} x^m P_n^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}\mathrm{d}x = 0, \forall\, 0 \le m < n.$$

(d)

$$h_n = \frac{1}{2^{(\alpha+\beta+1)}B(\alpha+1,\beta+1)}\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)^2(1-x)^{\alpha}(1+x)^{\beta}\mathrm{d}x = \frac{(\alpha+1)_n(\beta+1)_n(\alpha+\beta+n+1)}{n!(\alpha+\beta+2)_n(\alpha+\beta+2n+1)}.$$

Proof. (a). The previous proposition implies

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{(\alpha+1)_n}{n!}{}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}) \\ &= \frac{(\alpha+1)_n}{n!}\sum_{j=0}^{n}\frac{(-n)_j(n+\alpha+\beta+1)_j}{(\alpha+1)_j j!}\left(\frac{x-1}{2}\right)^j(-1)^j. \end{aligned}$$

Isolate the term involves x^n and observe $(-n)_n = (-1)^n n!$ to get

$$\begin{aligned} &\frac{(\alpha+1)_n}{n!}\frac{(-n)_n(n+\alpha+\beta+1)_n}{(\alpha+1)_n n!}\frac{(-1)^n(x-1)^n}{2^n} \\ &\qquad = \frac{(n+\alpha+\beta+1)_n(x-1)^n}{2^n n!}. \end{aligned}$$

So the leading coefficient is $k_n = \frac{(n+\alpha+\beta+1)_n}{2^n n!}$.

(b). A consequence of applying integration by parts n times. At each time $q(x)$ is differentiated and $(1-x)^{\alpha+n}(1+x)^{\beta+n}$ is integrated.

(c). A trivial consequence of part (b) after substituting $q(x) = x^m$.

(d). Recall that the Jacobi weight function is $(1-x)^\alpha(1+x)^\beta$ and the corresponding normalization constant is $2^{-\alpha-\beta-1}B(\alpha+1,\beta+1)^{-1}$. Apply part (a) and (b) to get

$$\begin{aligned}
h_n &= \frac{1}{2^{\alpha+\beta+1}B(\alpha+1,\beta+1)}\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)^2(1-x)^\alpha(1+x)^\beta \mathrm{d}x \\
&= \frac{1}{2^{\alpha+\beta+n+1}B(\alpha+1,\beta+1)n!}\int_{-1}^{1} \frac{d^n}{dx^n}\left(P_n^{(\alpha,\beta)}(x)\right)(1-x)^{\alpha+n}(1+x)^{\beta+n}\mathrm{d}x \\
&= \frac{1}{2^{\alpha+\beta+n+1}B(\alpha+1,\beta+1)n!}\int_{-1}^{1} \frac{(\alpha+\beta+n+1)_n}{2^n}(1-x)^{\alpha+n}(1+x)^{\beta+n}\mathrm{d}x \\
&= \frac{(\alpha+\beta+n+1)_n}{2^{\alpha+\beta+2n+1}B(\alpha+1,\beta+1)n!}\int_{0}^{2} y^{\beta+n}(2-y)^{\alpha+n}\mathrm{d}y \\
&= \frac{(\alpha+\beta+n+1)_n}{2^{\alpha+\beta+2n+1}B(\alpha+1,\beta+1)n!}2^{\alpha+\beta+2n+1}B(\alpha+n+1,\beta+n+1) \\
&= \frac{(\alpha+\beta+n+1)_n}{n!}\frac{B(\alpha+n+1,\beta+n+1)}{B(\alpha+1,\beta+1)} \\
&= \frac{(\alpha+1)_n(\beta+1)_n(\alpha+\beta+n+1)}{n!(\alpha+\beta+2)_n(\alpha+\beta+2n+1)}.
\end{aligned}$$

□

Proposition 3.2.3. *For each $n \in \mathbb{N}$, the three-term recurrence of Jacobi polynomials is*

$$\begin{aligned}
P_{n+1}^{(\alpha,\beta)}(x) = &\frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)}xP_n^{(\alpha,\beta)}(x) \\
&+\frac{(2n+\alpha+\beta+1)(\alpha^2-\beta^2)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}P_n^{(\alpha,\beta)}(x) \\
&-\frac{(\alpha+n)(\beta+n)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}P_{n-1}^{(\alpha,\beta)}(x).
\end{aligned}$$

Proof. cf. Proposition 3.1.1. The values of $A_n = \frac{k_{n+1}}{k_n}$ and $C_n = \frac{k_{n+1}k_{n-1}h_n}{k_n^2 h_{n-1}}$ can be directly computed. For the computation of B_n see [5]. □

Chapter 4

A generalised Jacobi weight function over a simplex

A sequence of orthogonal polynomials of several variables is similar to those of one variable, except that there is more than one polynomial with degree $n \in \mathbb{N}$ in the sequence. The main task of studying orthogonal polynomials of several variables is to compute an orthogonal basis for each space of polynomials with the same degree.
We reviewed the classical Jacobi polynomials of several variables on the standard simplex in $\mathbb{R}^d$, a more generalised Jacobi weight was defined and investigated thereafter. The author spent a lot of time on integrating the generalized Jacobi weight function, the closed forms of the integration under restricted parameters are presented. It is worth mention that a closed form correponding to the integral of the most general Jacobi weight has not been found so far.
Throughout this research, we shall use the standard multi-index notation. A multi-index is usually denoted by α, i.e. $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}_0^d$ where $d \in \mathbb{N}$. We define the muti-index factorial of α to be $\alpha! = \alpha_1! \cdots \alpha_d!$ with size $|\alpha| = \alpha_1 + .. + \alpha_d$.

4.1 Orthogonal polynomials of several variables

We start by reviewing some basic definitions and properties of multivariate orthogonal polynomials. The author assumes the reader has familarity in the basic theory of multivariate polynomials, such as the content introduced in Chapter 1 of [9].

Definition 4.1.1. *Let k be an algebraically closed field and let $\langle \cdot, \cdot \rangle$ be an inner product defined on the polynomial ring $k[x_1, ..., x_d]$. Two polynomials $P(\boldsymbol{x})$ and $Q(\boldsymbol{x})$ in $k[x_1, ..., x_d]$ are said to be mutually orthogonal with respect to the inner product if $\langle P(\boldsymbol{x}), Q(\boldsymbol{x}) \rangle = 0$.*

Definition 4.1.2. *A polynomial $P(\boldsymbol{x}) \in k[x_1, ..., x_d]$ is called an* **orthogonal polynomial** *if $P(\boldsymbol{x})$ is orthogonal to all polynomials with lower degrees; that is $\langle P(\boldsymbol{x}), Q(\boldsymbol{x}) \rangle = 0, \forall\, Q(\boldsymbol{x}) \in k[x_1, ..., x_d]$ with* $\deg Q(\boldsymbol{x}) < \deg P(\boldsymbol{x})$.

Remark. The inner product is usually given in terms of a weight function W, that is

$$\langle P(\boldsymbol{x}), Q(\boldsymbol{x})\rangle = \int_{\Omega} P(\boldsymbol{x})Q(\boldsymbol{x})W(\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$

where Ω is a region with non-empty interior in $\mathbb{R}^d$. We say that orthogonal polynomials are orthogonal with respect to the weight function W. Denote by V_n^d the space of orthogonal polynomials of degree exactly n. i.e.

$$V_n^d = \{P \in \Pi_n^d : \deg P(\boldsymbol{x}) = n, \langle P(\boldsymbol{x}), Q(\boldsymbol{x})\rangle = 0, \forall\, Q(\boldsymbol{x}) \in \Pi_{n-1}^d\},$$

where Π_n^d is the subspace of $k[x_1, ..., x_d]$ consists of polynomials of degree at most n.

Lemma 4.1.1. *The dimension of* V_n^d *is* $\binom{n+d-1}{n}$.

Proof. Denote the space of homogeneous polynomials of degree n in d variables by P_n^d, that is $P_n^d = \{P(\boldsymbol{x}) : P(\boldsymbol{x}) = \sum_{|\alpha|=n} c_\alpha x^\alpha, c_\alpha \in k\}$.
The space of orthogonal polynomials can be obtained by applying the Gram-Schmidt algorithm to the basis of monomials $\{x^\alpha : |\alpha| = n, n \in \mathbb{N}_0\}$ arranged by the lexicographic order, thus each monic monomial in P_n^d corresponds to a unique monic polynomial in V_n^d (with its leading term equals to the monic monomial). Conversely, each monic polynomial in V_n^d corresponds to a unique monic monomial in P_n^d. So $\dim V_n^d = \dim P_n^d$.
Let $r_n^d = \dim P_n^d$, it is clear that $r_n^d = |\{\alpha \in \mathbb{N}_0^d : |\alpha| = n\}|$. By the expansion of a geometric series to get

$$\frac{1}{(1-t)^d} = \prod_{i=1}^{d} \sum_{\alpha_i=0}^{\infty} t^{\alpha_i} = \sum_{n=0}^{\infty} \left(\sum_{|\alpha|=n} 1 \right) t^n = \sum_{n=0}^{\infty} r_n^d t^n.$$

On the other hand, expand $\frac{1}{(1-t)^d}$ by using hypergeometric function to get

$$\frac{1}{(1-t)^d} = \sum_{n=0}^{\infty} \frac{(d)_n t^n}{n!} = \sum_{n=0}^{\infty} \binom{n+d-1}{n} t^n.$$

Equating the coefficients of t^n in the two expressions above to get $r_n^d = \binom{n+d-1}{n}$.
Therefore $\dim V_n^d = \binom{n+d-1}{n}$. □

Remark. The goal of the study of a class of orthogonal polynomials of several variables is to find a basis for V_n^d. Because the orthogonality is defined for polynomials with distinct degrees, certain results can be formulated in terms of $V_0^d, V_1^d, \cdots, V_n^d, \cdots$ themselves instead of a sequence of representatives from each of the V_n^d's.

Definition 4.1.3. *Let* $\{P_\alpha : |\alpha| = n\}$ *be a basis of* V_n^d. *Order indices in the set* $\{\alpha \in \mathbb{N}_0^d : |\alpha| = n\}$ *by* $\alpha^{(1)} < \alpha^{(2)} < ... < \alpha^{(r_n^d)}$*according to the lexicographic order. Denote* $\mathbb{P}_n(x) := \left(P_{\alpha^{(1)}}(\boldsymbol{x}), ..., P_{\alpha^{(r_n^d)}}(\boldsymbol{x})\right)^T = \sum_{i=1}^{r_n^d} G_i x^i$ *where* G_i *is a matrix of size* $r_n^d \times r_i^d$.

Remark. Observe that $\int_\Omega (x_i P_\alpha(\boldsymbol{x}))\, P_\beta(\boldsymbol{x})W(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0$ for all α and β with $|\alpha| = n$ and $|\beta| \le n-2$. This allows one to extend the three-term recurrence relation for orthogonal polynomials of one variable to the those of several variables.

Proposition 4.1.1. *For each $n \in \mathbb{N}_0$ there exists matrices $A_{n,i}$ of dimension $r_n^d \times r_{n+1}^d$, $B_{n,i}$ of dimension $r_n^d \times r_n^d$ and $C_{n,i}$ of dimension $r_n^d \times r_{n-1}^d$ such that*

$$x_i P_n(\boldsymbol{x}) = A_{n,i} P_{n+1}(\boldsymbol{x}) + C_{n,i} P_{n-1}(\boldsymbol{x}), \forall\, i \in \{1, \cdots, d\}.$$

where we define $P_{-1} = 0$ and $C_{-1,i} = 0$.

Proof. See [5]. □

Example 4.1.1. *Let $\boldsymbol{a} = (a_1, \cdots, a_d)$ and $\boldsymbol{b} = (b_1, \cdots, b_d)$ be two multi-indices in $\mathbb{R}^d$ with $a_i, b_i \in (-1, +\infty), \forall\, i \in \{1, \cdots, d\}$. Define*

$$\mathrm{W}_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{x}) = \prod_{i=1}^{d} (1 - x_i)^{a_i} (1 + x_i)^{b_i}$$

to be the multiple Jacobi weight function on the cube $[-1, 1]^d$ of $\mathbb{R}^d$. An orthogonal basis for the space V_n^d is the set $\{P(W_{\boldsymbol{a},\boldsymbol{b}}; \boldsymbol{x}) = \prod_{i=1}^{d} P_{\alpha_i}^{(a_i,b_i)}(x_i) : |\alpha| = n\}$, in which $P_{\alpha_i}^{(a_i,b_i)}(x_i)$ is a classical Jacobi polynomial of one variable defined by (3.5).

Remark. It is evident from Proposision 3.2.2 that the given basis for V_n^d is orthogonal. If we use the orthonormal Jacobi polynomials $p_{\alpha_i}^{(a_i,b_i)}$ in the product instead, then the basis would become orthonormal. The multiple Jacobi weight function is a typical example of the product type weight functions, they are the simplest multivariate weight functions.

We next review the classical Jacobi weight function over a simplex, in Section 4.2 we will define a generalised Jacobi weight.

4.1.1 Classical Jacobi polynomials over a simplex

Let T^d denote the standard simplex in $\mathbb{R}^d$, that is $T^d = \{(x_1, \cdots, x_d) \in \mathbb{R}^d : x_1 \geq 0, \cdots, x_d \geq 0,\ \sum_{i=1}^{d} x_i \leq 1\}$.

Definition 4.1.4. *Let V be the set of $d + 1$ vertices of a d-simplex in $\mathbb{R}^d$. The sequence $\xi = (\xi_v)_{v \in V}$ is called the barycentric coordinates of V if it satisfies*

$$\sum_{v \in V} \xi_v(\boldsymbol{x}) = 1, \sum_{v \in V} \xi_v(\boldsymbol{x}) v = \boldsymbol{x}, \forall\, \boldsymbol{x} \in \mathbb{R}^d.$$

.

Remark. The sequence $\xi = (\xi_v)_{v \in V}$ is unique due to the fact that points in V are affinely independent. For the standard simplex T^d, its vertices are $\{\boldsymbol{0} = (0, \cdots, 0), \boldsymbol{e_1} = (1, \cdots, 0), \cdots, \boldsymbol{e_d} = (0, \cdots, 1)\}$.The corresponding barycentric coordinates are

$$\xi_{\boldsymbol{0}}(\boldsymbol{x}) = 1 - x_1 - \cdots - x_d,\ \xi_{\boldsymbol{e_i}}(\boldsymbol{x}) = x_i, \forall\, i \in \{1, \cdots, d\}.$$

Lemma 4.1.2. *For each $h \in (0, +\infty)$, the volume of the d-simplex $\Delta_h^d := \{(x_1, \cdots, x_d) \in \mathbb{R}^d : x_1 \geq 0, \cdots, x_d \geq 0, \sum_{i=1}^{d} x_i \leq h\}$ is $\frac{h^d}{d!}$.*

Proof. We proceed by induction. The volume of Δ_h^2 is

$$\iint_{\Delta_h^2} \mathrm{d}x\mathrm{d}y = \int_0^h \mathrm{d}x \int_0^{h-x} \mathrm{d}y = \int_0^h (h-x)\mathrm{d}x = \frac{h^2}{2!}.$$

The volume of Δ_h^3 is thus

$$\iiint_{\Delta_h^3} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^h \mathrm{d}x \iint_{\Delta_{h-x}^2} \mathrm{d}y\mathrm{d}z = \int_0^h \frac{(h-x)^2}{2}\mathrm{d}x = \frac{h^3}{3!}.$$

Suppose the volume of Δ_h^{d-1} is $\frac{h^{n-1}}{(n-1)!}$ to get the volume of Δ_h^d is

$$\int_{\Delta_h^d} \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_d = \int_0^h \mathrm{d}x_1 \int_{\Delta_{h-x_1}^{d-1}} \mathrm{d}x_2\mathrm{d}x_3\cdots\mathrm{d}x_d = \int_0^h \frac{(h-x_1)^{d-1}}{(d-1)!}\mathrm{d}x_1 = \frac{h^d}{d!}.$$

. □

Definition 4.1.5. *The classicial Jacobi polynomials on T^d are orthogonal with respect to the weight function*

$$W_{\boldsymbol{\nu}}(\boldsymbol{x}) = \xi^{\boldsymbol{\nu}-\mathbf{1}} = x_1^{\nu_1-1}\cdots x_d^{\nu_d-1}(1-x_1-\cdots-x_d)^{\nu_0-1},$$

where $\nu_i \in (0,+\infty), \forall i \in \{0,\cdots,d\}$.

Let us now calculate the *normalization constant* of $W_{\boldsymbol{\nu}}$, namely $\int_{T^d} W_{\boldsymbol{\nu}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$. The next Proposition is a generalisation of Example 2.1.1.

Proposition 4.1.2. *For each $\boldsymbol{\nu} \in \mathbb{R}_{>0}^{d+1}$ one has*

$$\begin{aligned}\int_{T^d} W_{\boldsymbol{\nu}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} &= \frac{\Gamma(\nu_0)\Gamma(\nu_1)\cdots\Gamma(\nu_d)}{\Gamma(\nu_0+\nu_1\cdots+\nu_d)} \\ &= \frac{\Gamma(\boldsymbol{\nu})}{\Gamma(|\boldsymbol{\nu}|)}.\end{aligned}$$

Proof. For each $a \in \mathbb{R}-\{0\}$ and $m,n \in (0,+\infty)$ one has

$$\begin{aligned}\frac{1}{a^{m+n-1}}\int_0^a y^{m-1}(a-y)^{n-1}\mathrm{d}y &= \frac{1}{a^{m+n-1}}\int_0^a a^{m+n-2}\left(\frac{y}{a}\right)^{m-1}\left(1-\frac{y}{a}\right)^{n-1}\mathrm{d}y \\ &= \frac{1}{a}\int_0^a \left(\frac{y}{a}\right)^{m-1}\left(1-\frac{y}{a}\right)^{n-1}\mathrm{d}y \\ &= \int_0^1 x^{m-1}(1-x)^{n-1}\mathrm{d}x.\end{aligned}$$

Thus

$$\int_0^a y^{m-1}(a-y)^{n-1}\mathrm{d}y = a^{m+n-1}B(m,n).$$

With above observation we can integrate $x_1^{\nu_1-1}\cdots x_d^{\nu_d-1}(1-x_1-\cdots-x_d)^{\nu_0-1}$ over T^d by integrating out variables one at each step:

$$\int_{T^d} x_1^{\nu_1-1}\cdots x_d^{\nu_d-1}(1-x_1-\cdots-x_d)^{\nu_0-1}\mathrm{d}\boldsymbol{x}$$
$$=\int_0^1\int_0^{1-x_1}\cdots\int_0^{1-x_1-\cdots-x_{d-1}} x_1^{\nu_1-1}x_2^{\nu_2-1}\cdots x_d^{\nu_d-1}(1-x_1-\cdots x_d)^{\nu_0-1}\mathrm{d}x_d\cdots\mathrm{d}x_2\mathrm{d}x_1$$
$$=\int_0^1 x_1^{\nu_1-1}\int_0^{1-x_1} x_2^{\nu_2-1}\cdots\int_0^{1-x_1-\cdots-x_{d-1}} x_d^{\nu_d-1}(1-x_1-\cdots x_d)^{\nu_0-1}\mathrm{d}x_d\cdots\mathrm{d}x_2\mathrm{d}x_1$$
$$=B(\nu_0,\nu_d)\int_0^1 x_1^{\nu_1-1}\int_0^{1-x_1} x_2^{\nu_2-1}\cdots\int_0^{1-x_1-\cdots-x_{d-2}} x_{d-1}^{\nu_{d-1}-1}(1-x_1-\cdots x_{d-1})^{\nu_0+\nu_d-1}\mathrm{d}x_{d-1}\cdots\mathrm{d}x_2\mathrm{d}x_1$$
$$=B(\nu_{d-1},\nu_0+\nu_d)B(\nu_0,\nu_d)\int_0^1 x_1^{\nu_1-1}\cdots\int_0^{1-x_1\cdots-x_{d-2}} x_{d-2}^{\nu_{d-2}-1}(1-x_1-\cdots-x_{d-2})^{\nu_0+\nu_{d-1}+\nu_d-1}$$
$$\mathrm{d}x_{d-2}\cdots\mathrm{d}x_1$$
$$=\cdots$$
$$=B(\nu_1,\nu_0+\nu_d+\nu_{d-1}+\cdots+\nu_2)B(\nu_2,\nu_0+\nu_d+\nu_{d-1}+\cdots+\nu_3)\cdots B(\nu_{d-1},\nu_0+\nu_d)B(\nu_0,\nu_d)$$
$$=\frac{\Gamma(\nu_0)\Gamma(\nu_1)\cdots\Gamma(\nu_d)}{\Gamma(\nu_0+\nu_1+\cdots+\nu_d)}$$
$$=\frac{\Gamma(\boldsymbol{\nu})}{\Gamma(|\boldsymbol{\nu}|)}.$$

□

Proposition 4.1.3. *For each $j\in\{1,\cdots,d\}$ let $\boldsymbol{x}_j=(x_1,\cdots,x_j)$, $\alpha^j=(\alpha_j,\cdots,\alpha_d)$ and $\nu^j=(\nu_j,\cdots,\nu_{d+1})$. The set of polynomials*

$$\{P_\alpha(W_{\boldsymbol{\nu}};x)=[h_\alpha]^{-1}\prod_{j=1}^d\left(\frac{1-|\boldsymbol{x}_j|}{1-|\boldsymbol{x}_{j-1}|}\right)^{2|\alpha^{j+1}|}p_{\alpha_j}^{(a_j,b_j)}\left(\frac{2x_j}{1-|\boldsymbol{x}_{j-1}|}-1\right),|\alpha|=n\}$$

forms an orthonormal basis for V_n^d of the classical Jacobi polynomials on T^d, where $a_j=2|\alpha^{j+1}|+|\nu^{j+1}|-(d-j)$, $b_j=\nu_j-1$ and the constants $[h_\alpha]^2$ are given by

$$[h_\alpha]^2=\frac{(|\nu|)_{|\alpha|}}{\prod_{j=1}^d(2|\alpha^{j+1}|+|\nu^j|)_{2\alpha_j}}.$$

Proof. See [5]. □

Remark. Explicit formulas for orthogonal polynomials usually appear to be complex. It is diffcult to see the structure of a space of orthogonal polynomials through their explicit expressions, thus it is desirable to find a more sensible representation of these polynomials.

4.2 The generalised Jacobi weight function

We now generalise the weight function in Definition 4.1.5. Observe that the classical Jacobi weight function consists of powers of barycentric coordinates of T^d, that is $\xi^{\boldsymbol{\nu}-\mathbf{1}}$. In some applications one may encounter a more general weight function which includes the classical weight along with powers of $(1-\xi)$ [3]. The general weight function has the form $(1-\xi)^{\boldsymbol{\mu}}\xi^{\boldsymbol{\nu}}$.
No research on this generalised Jacobi weight was carried out in the past, thus our research is original.

4.2.1 The integrability condition of the generalised weight function

Consider a normalised weight function over T^d: $\frac{1}{d!}(1-\xi)^{\boldsymbol{\mu}}\xi^{\boldsymbol{\nu}}$, where $\frac{1}{d!}$ is the normalisation constant that is equal to the volume of T^d (cf. Lemma 4.1.2).

Lemma 4.2.1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a Riemann integrable function. For each $h\in(0,+\infty)$ one has*

$$\int_{\Delta_h^d} f(x_1+\cdots+x_d)\mathrm{d}\boldsymbol{x} = \int_0^h f(x)\frac{x^{d-1}}{(d-1)!}\mathrm{d}x. \tag{4.1}$$

where Δ_h^d is the d-simplex defined in Lemma 4.1.2.

Proof. Make the substitutions $\begin{cases} y_i = x_i,\ \forall i\in\{1,2,\cdots,d-1\} \\ y_d = x_1+\cdots+x_d \end{cases}$. By Lemma 4.1.2 to get

$$\int_{\Delta_h^d} f(x_1+\cdots+x_d)\mathrm{d}\boldsymbol{x} = \int_0^h f(y_d)\int_{\Delta_{y_d}^{d-1}}\mathrm{d}\boldsymbol{y} = \int_0^h f(y_d)\frac{y_d^{d-1}}{(d-1)!}\mathrm{d}y_d = \int_0^h f(x)\frac{x^{d-1}}{(d-1)!}\mathrm{d}x.$$

□

We next establish the condition for integrability of the generalised weight function.

Lemma 4.2.2. *Let $d>1, d\in\mathbb{N}$ and let $\xi=(\xi_j)_{j=0}^d$ be the barycentric coordinates of the standard d-simplex T^d. The integral*

$$\frac{1}{d!}\int_{T^d}(1-\xi)^{\boldsymbol{\mu}}\xi^{\boldsymbol{\nu}} = \int_0^1\int_0^{1-x_1}\cdots\int_0^{1-x_1-\cdots-x_{d-1}}\left(\prod_{j=0}^d(1-x_j)^{\mu_j}x_j^{\nu_j}\right)\mathrm{d}x_d\cdots\mathrm{d}x_2\mathrm{d}x_1, \tag{4.2}$$

where $x_0:=1-x_1-\cdots-x_d$, converges if and only if

$$\nu_j>-1,\ \ \mu_j+\sum_{k\neq j}\nu_k>-d,\ \forall j\in\{0,\cdots,d\}.$$

For $\mu_k=0,\forall k\neq j$, we have the explicit formula

$$\frac{1}{d!}\int_{T^d}(1-\xi_j)^{\mu_j}\xi^{\boldsymbol{\nu}} = \frac{\Gamma(\boldsymbol{\nu}+\mathbf{1})}{\Gamma(\mu_j+|\boldsymbol{\nu}|+d+1)}\frac{\Gamma(\mu_j+\sum_{k\neq j}v_k+d)}{\Gamma(\sum_{k\neq j}\nu_k+d)}.$$

Proof. Since $0 \leq \xi_j \leq 1$ on the simplex T^d, the only singularities are when $\mu_j < 0$ (at the vertex $\boldsymbol{e_j}$) and $\nu_j < 0$ (on the opposite face of the vertex $\boldsymbol{e_j}$). When just one barycentric coordinate ξ_j is involved, the formula (cf.formula (4.1))

$$\frac{1}{d!}\int_{T^d} f(\xi_j) = \frac{1}{(d-1)!}\int_0^1 f(x)(1-x)^{d-1}\mathrm{d}x,$$

gives the Beta function

$$\frac{1}{d!}\int_{T^d}(1-\xi_j)^{\mu_j}(\xi_j)^{\nu_j} = \frac{1}{(d-1)!}\int_0^1 (1-x)^{\mu_j}x^{\nu_j}(1-x)^{d-1}\mathrm{d}x,$$

which is finite if and only if $\nu_j > -1, \quad \mu_j > -d$. Thus, the condition $\nu_j > -1$ is necessary and sufficient for the integrand to be integrable over a region which excludes neighbourhoods of the vertices. We now estimate the integral over a neighbourhood of a vertex, say the first one. This is bounded above and below by the (possibly infinite) iterated integral

$$I(\mu_1,\nu_0,...,\nu_d) := \frac{1}{d!}\int_{T^d}(1-\xi_1)^{\mu_1}\xi^{\boldsymbol{\nu}}$$
$$= \int_0^1\int_0^{1-x_1}\cdots\int_0^{1-x_1-\cdots-x_{d-1}}(1-x_1)^{\mu_1}\left(\prod_{j=1}^{d}x_j^{\nu_j}\right)(1-x_1-\cdots-x_d)^{\nu_0}\mathrm{d}x_d\cdots\mathrm{d}x_2\mathrm{d}x_1.$$

Make the substitution $x_d = (1-x_1-\cdots-x_{d-1})t$ to get $\mathrm{d}x_d = (1-x_1-\cdots-x_{d-1})\mathrm{d}t$, one has

$$I(\mu_1,\nu_0,\cdots,\nu_d) = \int_0^1\int_0^{1-x_1}\cdots\int_0^{1-x_1-\cdots-x_{d-2}}\int_0^1(1-x_1)^{\mu_1}\left(\prod_{j=1}^{d-1}x_j^{\nu_j}\right)[(1-x_1-\cdots-x_{d-1})t]^{\nu_d}$$
$$[(1-x_1-\cdots-x_{d-1})(1-t)]^{\nu_0}(1-x_1-\cdots-x_{d-1})\mathrm{d}t\mathrm{d}x_{d-1}\cdots\mathrm{d}x_2\mathrm{d}x_1$$
$$= B(\nu_0+1,\nu_d+1)I(\mu_1,\nu_0+\nu_d+1,\nu_1,\cdots,\nu_{d-1}).$$

Applying this recurrence a further $d-2$ times implies $I(\mu_1,\nu_0,...,\nu_d)$ is

$$\prod_{j=2}^{d}B(\nu_0+\nu_d+\nu_{d-1}+\cdots+\nu_{j+1}+d+1-j,\nu_j+1)I(\mu_1,\nu_0+\nu_d+\nu_{d-1}+\cdots+\nu_2+d-1,\nu_1)$$
$$= \frac{\Gamma(\nu_0+1)\Gamma(\mu_d+1)\cdots\Gamma(\nu_2+1)}{\Gamma(\nu_0+\nu_d+\cdots+\nu_2+d)}I(\mu_1,\nu_0+\nu_d+\nu_{d-1}+\cdots+\nu_2+d-1,\nu_1),$$

Where the condition for the Beta functions to converge is $\nu_0,\nu_d > -1,\nu_{d-1} > 0,...,\nu_2 > 0$. Finally,

$$\begin{aligned}I(\mu_1,\nu_0+\nu_d+\nu_{d-1}+\cdots+\nu_2+d-1,\nu_1) &= \int_0^1(1-x)^{\mu_1}x^{\nu_1}(1-x)^{\nu_0+\nu_d+\nu_{d-1}+\cdots+\nu_2+d-1}\mathrm{d}x\\ &= B(\mu_1+\nu_0+\nu_d+\nu_{d-1}+\cdots+\nu_2+d,\nu_1+1)\\ &= \frac{\Gamma(\mu_1+\nu_0+\nu_d+\nu_{d-1}+\cdots+\nu_2+d)\Gamma(\nu_1+1)}{\Gamma(\mu_1+\nu_0+\nu_1+\cdots+\nu_d+d+1)}\end{aligned}$$

which is finite if and only if $\mu_1+\sum_{k\neq 1}\nu_k+d>0,\quad \nu_1>0$. Thus we obtain the integrability condition, moreover

$$\frac{1}{d!}\int_{T^d}(1-\xi_j)^{\mu_j}\xi^{\boldsymbol{\nu}}=\frac{\Gamma(\boldsymbol{\nu}+\mathbf{1})}{\Gamma(\mu_j+|\boldsymbol{\nu}|+d+1)}\frac{\Gamma(\mu_j+\sum_{k\neq j}v_k+d)}{\Gamma(\sum_{k\neq j}\nu_k+d)}.$$

□

Remark. When more than one μ_j is non-zero, we have been unable to come up with a closed form description of the integral. This is illustrated in the following subsection, in which we compute some explicit forms of the integral under restricted ranges of parameters (power indices $\boldsymbol{\nu}$ and $\boldsymbol{\mu}$). Some results will be used in Chapter 5.

4.2.2 Explicit forms of the integral

To show the complexity of computing a closed form for (4.2), an example is shown below. For clarity, the integrals in this section are not normalised (that is they are not divided by $d!$)

Example 4.2.1. *Let $\mu_1,\mu_2\in(-2,+\infty)$ and all other parameters in (4.2) are zero, for clarity let $x=x_1$ and $y=x_2$ to get*

$$\begin{aligned}\int_0^1\int_0^{1-x}(1-x)^{\mu_1}(1-y)^{\mu_2}\mathrm{d}y\mathrm{d}x&=\int_0^1-\frac{(1-x)^{\mu_1}(1-y)^{\mu_2+1}}{\mu_2+1}\Big|_{y=0}^{y=1-x}\mathrm{d}x\\&=\int_0^1\frac{(1-x)^{\mu_1}(1-x^{\mu_2+1})}{\mu_2+1}\mathrm{d}x\\&=\frac{1}{(\mu_1+1)(\mu_2+1)}-\int_0^1\frac{(1-x)^{\mu_1}x^{\mu_2+1}}{\mu_2+1}\mathrm{d}x\\&=\frac{1}{(\mu_1+1)(\mu_2+1)}-\frac{\Gamma(\mu_1+1)\Gamma(\mu_2+1)}{\Gamma(\mu_1+\mu_2+3)}.\end{aligned}$$

Remark. The calculation presented above is **incomplete**, because the result excludes the cases where μ_1 or/and μ_2 is equal to -1. To obtain a complete answer one needs to investigate this integral further.

Lemma 4.2.3. $\int_0^1\frac{\ln x}{1-x}\mathrm{d}x=-\frac{\pi^2}{6}$.

Proof. For each $i\in\mathbb{N}_0$ one has

$$\begin{aligned}\int_0^1(\ln x)x^i\mathrm{d}x&=\ln x\frac{x^{i+1}}{i+1}\Big|_0^1-\int_0^1\frac{x^i}{i+1}\mathrm{d}x\\&=\Big[\frac{(\ln x)x^{i+1}}{i+1}-\frac{x^{i+1}}{(i+1)^2}\Big]_0^1\\&=\frac{x^{i+1}\left(\ln x(i+1)-1\right)}{(i+1)^2}\Big|_0^1\\&=-\frac{1}{(i+1)^2}.\end{aligned}\tag{4.3}$$

By Dominated Convergence Theorem and (4.3) to get

$$\begin{aligned}\int_0^1 \frac{\ln x}{1-x}\mathrm{d}x &= \lim_{n\to+\infty}\int_0^1\sum_{i=0}^{n}(\ln x)x^i\mathrm{d}x\\ &= \lim_{n\to\infty}\sum_{i=0}^{n}\int_0^1(\ln x)x^i\mathrm{d}x\\ &= -\sum_{i=0}^{\infty}\frac{1}{(i+1)^2} = -\frac{\pi^2}{6}.\end{aligned}$$

□

The following proposition involves complex-valued Euler functions. The aim of formulating the integral over a subset of $\mathbb{C}$ is to apply an analytic continuation, that will allow us to take limits and thus find the complete solution for the previous Example.

Proposition 4.2.1. *Let* $\mu_1, \mu_2 \in (-2, +\infty)$,

$$\int_0^1\int_0^{1-x}(1-x)^{\mu_1}(1-y)^{\mu_2}\mathrm{d}y\mathrm{d}x = \begin{cases}\frac{1}{(\mu_1+1)(\mu_2+1)} - \frac{\Gamma(\mu_1+1)\Gamma(\mu_2+1)}{\Gamma(\mu_1+\mu_2+3)} & ,\mu_1,\mu_2\in(-2,-1)\cup(-1,+\infty);\\ \frac{1}{\mu_j+1}\left(\psi(\mu_i+2)+\gamma\right) & ,\mu_i=-1,\mu_j\neq-1,\{i,j\}=\{1,2\};\\ \frac{\pi^2}{6} & ,\mu_1=\mu_2=-1.\end{cases}$$

, where ψ is the Digamma function and $\gamma = -\psi(1) = 0.577\cdots$ is the Euler-Mascheroni constant.

Proof. Let

$$V := \{(z_1,z_2)\in\mathbb{C}^2 : Re(z_1), Re(z_2)\in(-2,-1)\cup(-1,+\infty)\}$$

and let

$$U := \{(z_1,z_2)\in\mathbb{C}^2 : Re(z_1), Re(z_2)\in(-1,+\infty)\}.$$

Define funtions $g, h : V \to \mathbb{C}$ by

$$g(\mu_1,\mu_2) := \int_0^1\int_0^{1-x}(1-x)^{\mu_1}(1-y)^{\mu_2}\mathrm{d}y\mathrm{d}x$$

and

$$h(\mu_1,\mu_2) := \frac{1}{(\mu_1+1)(\mu_2+1)} - \frac{\Gamma(\mu_1+1)\Gamma(\mu_2+1)}{\Gamma(\mu_1+\mu_2+3)}.$$

From the previous Example to get $g\Big|_U = h\Big|_U$. Since both g and h are analytic (due to they are analytic in each of the variables μ_1 and μ_2), morever U, V are open and connected sets in $\mathbb{C}^2$ with $U \subset V$, so $h = g$ due to the Identity Theorem [10]. That is

$$g(\mu_1,\mu_2) = \frac{1}{(\mu_1+1)(\mu_2+1)} - \frac{\Gamma(\mu_1+1)\Gamma(\mu_2+1)}{\Gamma(\mu_1+\mu_2+3)}, \forall(\mu_1,\mu_2)\in V.$$

Now consider the case $\mu_1 = -1, \mu_2 \neq -1$. Let

$$V^+ := \{(z_1,z_2)\in\mathbb{C}^2 : Re(z_1), Re(z_2)\in(-2,+\infty)\}$$

and

$$K := \{(\mu_1, -1), (-1, \mu_2) : Re(z_1), Re(z_2) \in (-2, +\infty)\}.$$

Observe that $V \subset V^+$ and $V = V^+ - K = V$ is connected. According to Hartog's Theorem [10] to get g can be uniquelly extended onto V^+ analytically (note that the domain of g may incorporate K due to Lemma 4.2.2) .

By the continuity of g at $(-1, \mu_2)$, for each μ_2 with $Re(\mu_2) > -2$ and $\mu_2 \neq -1$ one has

$$\begin{aligned}
g(-1, \mu_2) &= \lim_{\mu_1 \to -1} \left[\frac{1}{(\mu_1+1)(\mu_2+1)} - \frac{\Gamma(\mu_1+1)\Gamma(\mu_2+1)}{\Gamma(\mu_1+\mu_2+3)} \right] \\
&= \lim_{\mu_1 \to -1} \frac{1}{1+\mu_1} \left[\frac{1}{1+\mu_2} - \frac{\Gamma(\mu_1+2)\Gamma(\mu_2+1)}{\Gamma(\mu_1+\mu_2+3)} \right] \\
&= \lim_{h \to 0} \frac{1}{h} \left[\frac{\Gamma(\mu_2+1)}{\Gamma(\mu_2+2)} - \frac{\Gamma(h+1)\Gamma(\mu_2+1)}{\Gamma(\mu_2+h+2)} \right] \\
&= \frac{\partial}{\partial \mu_1}\Big|_{\mu_1=-1} \left[-\frac{\Gamma(\mu_1+2)\Gamma(\mu_2+1)}{\Gamma(\mu_1+\mu_2+3)} \right] \\
&= \left[-B(\mu_1+2, \mu_2+1) \left(\frac{\Gamma'(\mu_1+2)}{\Gamma(\mu_1+2)} - \frac{\Gamma'(\mu_1+\mu_2+3)}{\Gamma(\mu_1+\mu_2+3)} \right) \right]_{\mu_1=-1} \\
&= -B(1, \mu_2+1) \left[\frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'(\mu_2+2)}{\Gamma(\mu_2+2)} \right] \\
&= -\frac{\Gamma(\mu_2+1)}{\Gamma(\mu_2+2)} \left[\psi(1) - \psi(\mu_2+2) \right] \\
&= \frac{1}{\mu_2+1} \left(\psi(\mu_2+2) + \gamma \right).
\end{aligned}$$

By symmetry of μ_1, μ_2 to get

$$g(\mu_1, -1) = \frac{1}{\mu_1+1} \left(\psi(\mu_1+2) + \gamma \right), \forall\, \mu_2 \neq -1, Re(\mu_2) > -2.$$

For the case $\mu_1 = \mu_2 = -1$ by Lemma 4.2.3 to get

$$\begin{aligned}
g(-1,-1) = \int_0^1 \int_0^{1-x} \frac{1}{(1-x)(1-y)} \mathrm{d}y \mathrm{d}x &= \int_0^1 \frac{1}{1-x} \mathrm{d}x \int_0^{1-x} \frac{1}{1-y} \mathrm{d}y \\
&= \int_0^1 \frac{1}{1-x} \left[\ln|1-y| \right]_0^{1-x} \mathrm{d}x \\
&= \int_0^1 \frac{1}{1-x} \left[-\ln|x| \right] \mathrm{d}x \\
&= -\int_0^1 \frac{\ln x}{1-x} \mathrm{d}x \\
&= \frac{\pi^2}{6}.
\end{aligned}$$

□

Remark. The author tried to simplify the result further by trying to combine the three cases, but no closer form was discovered. This result predicts how complex a closed form of (4.2) may be. In fact the author was not able to even integrate for the case where three μ parameters are present in (4.2), this is illustrated in the following observation.

Observation. Let $\mu_1, \mu_2, \mu_3 \in (-2, +\infty)$, make substitutions $t = x + y$ and $s = \frac{t}{1+x}$, by Proposition 2.2.1.(a) to get

$$
\begin{aligned}
&\int_0^1 \int_0^{1-x} (1-x)^{\mu_1}(1-y)^{\mu_2}(x+y)^{\mu_3}\mathrm{d}y\mathrm{d}x \\
=&\int_0^1 (1-x)^{\mu_1}\mathrm{d}x \int_0^{1-x} (1-y)^{\mu_2}(x+y)^{\mu_3}\mathrm{d}y \\
=&\int_0^1 (1-x)^{\mu_1}\mathrm{d}x \int_{t=x}^{t=1} [(1+x)-t]^{\mu_2}\, t^{\mu_3}\mathrm{d}t \\
=&\int_0^1 (1-x)^{\mu_1}\mathrm{d}x(1+x)^{\mu_2+\mu_3} \int_{t=x}^{t=1} \left[1-\frac{t}{1+x}\right]^{\mu_2}\left(\frac{t}{1+x}\right)^{\mu_3}\mathrm{d}t \\
=&\int_0^1 (1-x)^{\mu_1}\mathrm{d}x(1+x)^{\mu_2+\mu_3+1} \int_{s=\frac{x}{1+x}}^{s=\frac{1}{1+x}} (1-s)^{\mu_2}s^{\mu_3}\mathrm{d}s \\
=&\int_0^1 (1-x)^{\mu_1}\mathrm{d}x(1+x)^{\mu_2+\mu_3+1} \left[\int_{s=0}^{s=\frac{1}{1+x}} (1-s)^{\mu_2}s^{\mu_3}\mathrm{d}s - \int_{s=0}^{s=\frac{x}{1+x}} (1-s)^{\mu_2}s^{\mu_3}\mathrm{d}s\right] \\
=&\int_0^1 (1-x)^{\mu_1}(1+x)^{\mu_2+\mu_3+1}\left[B_{\frac{1}{1+x}}(\mu_3+1,\mu_2+1) - B_{\frac{x}{1+x}}(\mu_3+1,\mu_2+1)\right]\mathrm{d}x \\
=&\int_0^1 (1-x)^{\mu_1}(1+x)^{\mu_2+\mu_3+1}\left[B(\mu_3+1,\mu_2+1) - B_{\frac{x}{1+x}}(\mu_2+1,\mu_3+1) - B_{\frac{x}{1+x}}(\mu_3+1,\mu_2+1)\right]\mathrm{d}x \\
=&\int_0^1 (1-x)^{\mu_1}\left[B(\mu_3+1,\mu_2+1) - \int_0^x \frac{w^{\mu_2}+w^{\mu_3}}{(1+w)^{\mu_2+\mu_3+2}}\mathrm{d}w\right]\mathrm{d}x.
\end{aligned}
$$

The author attempted to simplify the result further, but any further substitutions made became circular. The difficulty of this computation is to integrate the two incomplete Beta functions arose in the second to the last line above.

We end this section by presenting another computation which is needed in Chapter 5.

Proposition 4.2.2. *Let $\boldsymbol{\mu} = (1, 1, \cdots, 1)$ and let $\boldsymbol{\nu} \in \mathbb{N}_0^{d+1}$, a closed form for* (4.2) *(non-normalised) is*

$$
\sum_{m_0=0}^{1}\sum_{m_1=0}^{1}\cdots\sum_{m_d=0}^{1}(-1)^{m_0+\cdots+m_d}\frac{\prod_{j=0}^d \Gamma(\nu_j+m_j+1)}{\Gamma\left(\sum_{j=0}^d(\nu_j+m_j+1)\right)}. \tag{4.4}
$$

Proof. The result is obtained by expanding $(1-\xi)^{\boldsymbol{\mu}}$ followed by applying Proposition 4.1.2. $\square$

Chapter 5

Tight frames of generalised Jacobi polynomials

5.1 A glance of the tight frame theory

Over the last twenty years there has been renewed interest in frame representations due to their applications in physics and engineering, such as in the wavelet theory [11]. Shayne Waldron discovered a tight frame representation of the classical Jaocobi polynomials over the standard simplex in $\mathbb{R}^d$ [6]. In comparison to the presentation of a basis of V_n^d, the advantage of a tight frame representation is that it depicts the intrinsic structure of V_n^d. Via the tight frame representation of V_n^d one may compute an orthogonal polynomial of degree n with a particular leading term easily, this process is much more efficient than applying the Gram-Schmidt algorithm.
In this Section we present some knowledge of tight frame theory, they are selected from Shayne's paper [6]. We use H to denote a Hilbert space over $\mathbb{R}$ with an inner product $\langle\cdot,\cdot\rangle$.

Lemma 5.1.1. *Let $\varPhi_j \in H$ and c_j be scalers. Then there exists a representation*

$$f = \sum_j c_j \langle f, \varPhi_j\rangle \varPhi_j, \quad \forall f \in H. \tag{5.1}$$

if and only if

$$||f||^2 = \sum_j c_j |\langle f, \varPhi_j\rangle|^2, \quad \forall f \in H. \tag{5.2}$$

Proof. $\Longrightarrow$: Suppose (5.1) holds to get

$$||f||^2 = \langle f, f\rangle = \sum_j \langle c_j \langle f, \varPhi_j\rangle \varPhi_j, f\rangle = \sum_j c_j \langle f, \varPhi_j\rangle\langle \varPhi_j, f\rangle = \sum_j c_j |\langle f, \varPhi_j\rangle|^2, \ \forall f \in H.$$

$\Longleftarrow$: By using the polarisation identity to get

$$\begin{aligned}\langle f, \Phi_j \rangle &= \frac{1}{4}\left(||f+\Phi_j||^2 - ||f-\Phi_j||^2\right)\\ &= \frac{1}{4}\left(\sum_k c_k|\langle f+\Phi_j, \Phi_k\rangle|^2 - \sum_k c_k|\langle f-\Phi_j, \Phi_k\rangle|^2\right)\\ &= \frac{1}{4}\sum_k c_k\left(|\langle f+\Phi_j, \Phi_k\rangle|^2 - |\langle f-\Phi_j, \Phi_k\rangle|^2\right)\\ &= \frac{1}{2}\sum_k c_k\left(\langle f, \Phi_k\rangle\langle \Phi_k, \Phi_j\rangle + \langle \Phi_j, \Phi_k\rangle\langle \Phi_k, f\rangle\right)\\ &= \sum_k c_k\langle f, \Phi_k\rangle\langle \Phi_k, \Phi_j\rangle.\end{aligned}$$

Therefore $f = \sum_k c_j\langle f, \Phi_k\rangle\Phi_k, \quad \forall f \in H.$ □

Remark. Condition (5.2) can be rewritten as $||f||^2 = \sum_j \sigma_j|\langle f, \psi_j\rangle|^2$, where $\sigma_j := \text{sign}(c_j)$, and $\psi_j := \sqrt{|c_j|}\Phi_j$.

Definition 5.1.1. *A family (ψ_j) in H is called a* **signed frame** *with signature $\sigma = (\sigma_j), \sigma_j \in \{-1, 1\}$ if there exists $A, B > 0$ with*

$$A||f||^2 \le \sum_j \sigma_j|\langle f, \psi_j\rangle|^2 \le B||f||^2, \forall\, f \in H,$$

and (ψ_j) is a Bessel set, i.e., there exists $C > 0$ with

$$\sum_j |\langle f, \psi_j\rangle|^2 \le C||f||^2, \forall f\ \in H.$$

If $A = B$ then (ψ_j) is called a **tight** *signed frame.*

Recall from Section 4.1 that the classical Jacobi polynomials on T^d are associalted with the inner product $\langle f, g\rangle = \int_{T^d} fg\xi^{\boldsymbol{\nu}}$ where $\boldsymbol{\nu} > -\mathbf{1}$. Let $\alpha \in \mathbb{N}_0^{d+1}$ with $|\alpha| = n$, denote the orthgonal projection of ξ^α onto the space V_n^d by p_{ξ^α}. A tight signed frame representation of V_n^d has the form [6]

$$f = \sum_{|\alpha|=n} c_\alpha \frac{\langle f, p_{\xi^\alpha}\rangle}{\langle p_{\xi^\alpha}, p_{\xi^\alpha}\rangle} p_{\xi^\alpha}, \quad \forall f \in V_n^d. \tag{5.3}$$

Observe that $\{p_{\xi^\alpha} : |\alpha| = n\}$ spans V_n^d.
A signed tight frame of classical Jacobi polynomials over T^d was explicitly computed by Shayne [6].
In the next Section we will compute for tight frames of the generalised Jacobi polynomials over T^d .
Due to the complexity of the genralised weight function (cf. Section 4.2), the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are restricted to be integers.

5.2 Generalised Jacobi polynomials of degree one

We now carry out some investigations on the tight frame representation of V_1^d for the generalised Jacobi polynomials. Denote the inner product associated with the non-normalised generalised Jacobi weight function by $\langle \cdot, \cdot \rangle$, that is $\langle f, g \rangle = \int_{T^d} fg(1-\xi)^{\boldsymbol{\mu}} \xi^{\boldsymbol{\nu}}$. We restrict $\boldsymbol{\mu} = (1, 1, \cdots, 1) \in \mathbb{N}_0^{d+1}$.
Recall that the $d+1$ barycentric coordinates of T^d are

$$\xi_0 = 1 - x_1 - \cdots - x_d, \;\; \xi_j = x_j, \;\; \forall j \in \{1, \cdots, d\}.$$

We need to compute for a set of constants $\{c_j : j \in \{0, \cdots, d\}\}$ such that the following representation holds

$$f = \sum_{j=0}^{d} c_j \langle f, p_{\xi_j} \rangle p_{\xi_j}, \; \forall f \in V_1^d. \tag{5.4}$$

Since $\{p_{\xi_i} : i \in \{0, \cdots, d\}\}$ is a basis for V_1^d, so (5.4) is valid if it is valid for each $f = p_{\xi_i}$. Moreover, according to Lemma 5.1.1, an equivalent condition to (5.4) is as follows

$$\langle p_{\xi_i}, p_{\xi_i} \rangle = \sum_{j=0}^{d} c_j \langle p_{\xi_i}, p_{\xi_j} \rangle^2, \; \forall i \in \{0, \cdots, d\}. \tag{5.5}$$

5.2.1 V_1^d for the case $\boldsymbol{\nu}$ is homogeneous

Let $\boldsymbol{\nu} = (\nu, \nu, \cdots, \nu) \in \mathbb{N}_0^{d+1}$. Since both $\boldsymbol{\mu} = (1, 1, \cdots, 1)$ and $\boldsymbol{\nu} = (\nu, \nu, \cdots, \nu)$ are homogeneous and T^d is the standard $d-$simplex , so $\langle \xi_i, 1 \rangle = \langle \xi_j, 1 \rangle, \langle \xi_i, \xi_i \rangle = \langle \xi_j, \xi_j \rangle, \forall \; i, j \in \{0, \cdots, d+1\}$. Similarly, $\langle \xi_i, \xi_j \rangle = \langle \xi_k, \xi_l \rangle, \forall \; i \neq j, k \neq l \in \{0, \cdots, d\}$. Let

$$a := \frac{\langle x_j, 1 \rangle}{\langle 1, 1 \rangle}, b_1 = \frac{\langle x_i, x_i \rangle}{\langle 1, 1 \rangle}, b_2 = \frac{\langle x_i, x_j \rangle}{\langle 1, 1 \rangle}, \forall \; i \neq j \in \{1, \cdots, d\}.$$

Observe by Proposition 4.2.2 that $b_1 \neq b_2$. Since p_{ξ_i} is the orthogonal projection of ξ_i onto V_1^d and $\{1\}$ is a basis for V_0^d, so $p_{\xi_i} = \xi_i - \frac{\langle \xi_i, 1 \rangle}{\langle 1, 1 \rangle}$.The inner products in (5.5) are given by

$$\langle p_{\xi_i}, p_{\xi_j} \rangle = \langle \xi_i - a, \xi_j - a \rangle = \begin{cases} (b_2 - a^2)\langle 1, 1 \rangle & \text{if } i \neq j \in \{1, \cdots, d\} \\ (b_1 - a^2)\langle 1, 1 \rangle & \text{if } i = j \in \{1, \cdots, d\} \end{cases}$$

and

$$\begin{cases} \langle p_{\xi_0}, p_{\xi_0} \rangle = \sum_{1 \leq i, j \leq d} \left[\langle x_i, x_j \rangle - a_i a_j \langle 1, 1 \rangle \right] = \left[db_1 + d(d-1)b_2 - d^2 a^2 \right] \langle 1, 1 \rangle \\ \langle p_{\xi_0}, p_{\xi_j} \rangle = -\sum_{i=1}^{d} \left[\langle x_i, x_j \rangle - a_i a_j \langle 1, 1 \rangle \right] = -\left[(d-1)b_2 + b_1 - da^2 \right] \langle 1, 1 \rangle \end{cases}.$$

By homogeneity of $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ to get $\langle p_{\xi_0}, p_{\xi_0} \rangle = \langle p_{\xi_j}, p_{\xi_j} \rangle, \forall j \in \{1, \cdots, d\}$ and $\langle p_{\xi_0}, p_{\xi_j} \rangle = \langle p_{\xi_i}, p_{\xi_j} \rangle, \forall i \neq j \in \{1, \cdots, d\}$.

Let $e := (b_1 - a^2)\langle 1, 1\rangle$ and $w := (b_2 - a^2)\langle 1, 1\rangle$ to get the $(d + 1) \times (d + 2)$ augmented matrix for the system (5.5)

$$\begin{pmatrix} e^2 & w^2 & w^2 & \dots & w^2 & e \\ w^2 & e^2 & w^2 & \dots & w^2 & e \\ w^2 & w^2 & e^2 & \dots & w^2 & e \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ w^2 & w^2 & w^2 & \dots & e^2 & e \end{pmatrix}.$$

Notice that that $e = \langle p_{\xi_0}, p_{\xi_0}\rangle = -d\langle p_{\xi_0}, p_{\xi_j}\rangle = -d\langle p_{\xi_i}, p_{\xi_j}\rangle = -dw, \forall i \neq j \in \{1, \cdots, d\}$. From the second row onwards, each row minus the first row to get

$$\begin{pmatrix} e^2 & w^2 & w^2 & \dots & w^2 & e \\ w^2 - e^2 & e^2 - w^2 & 0 & \dots & 0 & 0 \\ w^2 - e^2 & 0 & e^2 - w^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ w^2 - e^2 & 0 & 0 & \dots & e^2 - w^2 & 0 \end{pmatrix}.$$

Adding $-\frac{w^2}{e^2 - w^2}$ times each row below the first row to the first row to get

$$\begin{pmatrix} e^2 + dw^2 & 0 & 0 & \dots & 0 & e \\ w^2 - e^2 & e^2 - w^2 & 0 & \dots & 0 & 0 \\ w^2 - e^2 & 0 & e^2 - w^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ w^2 - e^2 & 0 & 0 & \dots & e^2 - w^2 & 0 \end{pmatrix}.$$

Forward substitution implies

$$c_j = \frac{e}{e^2 + dw^2} = \frac{e}{e^2 - ew} = \frac{1}{e - w} = \frac{1}{(b_1 - b_2)\langle 1, 1\rangle}, \forall\, j \in \{0, \cdots, d\}.$$

Remark. Above formulation can be used to compute scalings for any case where $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathbb{N}_0^{d+1}$ are both homogeneous.

5.2.2 V_1^2 for the case $\boldsymbol{\nu}$ is arbitrary

Let $\boldsymbol{\nu} = (\nu_0, \nu_1, \cdots, \nu_d) \in \mathbb{N}_0^{d+1}$. We start by formulating a procedure to compute the scalings for V_1^d. Let

$$a_j := \frac{\langle 1, x_j\rangle}{\langle 1, 1\rangle}, b_{ij} := \frac{\langle x_i, x_j\rangle}{\langle 1, 1\rangle}, e_{ij} := (b_{ij} - a_i a_j)\langle 1, 1\rangle, \forall\, i, j \in \{1, \cdots, d\}$$

to get

$$\begin{aligned} p_{\xi_0} &= (1 - x_1 - \cdots - x_d) - \frac{\langle 1 - x_1 - \cdots - x_d, 1\rangle}{\langle 1, 1\rangle} \\ &= (1 - x_1 - \cdots - x_d) - [1 - (a_1 + \cdots + a_d)] \\ &= -(x_1 + \cdots + x_d) + (a_1 + \cdots + a_d) \end{aligned}$$

and $p_{\xi_j} = x_j - a_j, \forall\, j \in \{1, \cdots, d\}$. For each $i, j \in \{1, \cdots, d\}$ one has

$$\begin{aligned}
\langle p_{\xi_0}, p_{\xi_0}\rangle &= \langle -(x_1 + \cdots + x_d) + (a_1 + \cdots + a_d), -(x_1 + \cdots + x_d) + (a_1 + \cdots + a_d)\rangle \\
&= \langle x_1 + \cdots + x_d, x_1 + \cdots + x_d\rangle - 2(a_1 + \cdots + a_d)\langle x_1 + \cdots + x_d, 1\rangle + (a_1 + \cdots + a_d)^2\langle 1, 1\rangle \\
&= \left(\sum_{1\le i,j\le d} b_{ij}\right)\langle 1,1\rangle - 2(a_1 + \cdots + a_d)^2\langle 1,1\rangle + (a_1 + \cdots + a_d)^2\langle 1,1\rangle \\
&= \left[\sum_{1\le i,j\le d} b_{ij} - (a_1 + \cdots + a_d)^2\right]\langle 1,1\rangle \\
&= \left(\sum_{1\le i,j\le d} b_{ij} - \sum_{1\le i,j\le d} a_i a_j\right)\langle 1,1\rangle \\
&= \left(\sum_{1\le i,j\le d} e_{ij}\right).
\end{aligned}$$

$$\begin{aligned}
\langle p_{\xi_0}, p_{\xi_j}\rangle &= -\langle x_1 + \cdots + x_d - (a_1 + \cdots + a_d), x_j - a_j\rangle \\
&= -\langle x_1 + \cdots + x_d, x_j\rangle + a_j\langle x_1 + \cdots + x_d, 1\rangle + (a_1 + \cdots + a_d)\langle x_j, 1\rangle - a_j(a_1 + \cdots + a_d)\langle 1,1\rangle \\
&= -(b_{1j} + \cdots + b_{dj})\langle 1,1\rangle + 2a_j(a_1 + \cdots + a_d)\langle 1,1\rangle - a_j(a_1 + \cdots + a_d)\langle 1,1\rangle \\
&= -(e_{1j} + \cdots + e_{dj}).
\end{aligned}$$

$$\langle p_{\xi_i}, p_{\xi_j}\rangle = \langle x_i - a_i, x_j - a_j\rangle = \langle x_i, x_j\rangle - a_j\langle x_i, 1\rangle - a_i\langle 1, x_j\rangle + a_i a_j\langle 1,1\rangle = (b_{ij} - 2a_i a_j + a_i a_j)\langle 1,1\rangle = e_{ij}.$$

The augmented $(d+1)\times(d+2)$ matrix for the system (5.5) is

$$\begin{pmatrix}
\left(\sum_{1\le i,j\le d} e_{ij}\right)^2 & \left(\sum_{j=1}^d e_{1j}\right)^2 & \left(\sum_{j=1}^d e_{2j}\right)^2 & \cdots & \left(\sum_{j=1}^d e_{dj}\right)^2 & \sum_{1\le i,j\le d} e_{ij} \\
\left(\sum_{j=1}^d e_{1j}\right)^2 & e_{11}^2 & e_{12}^2 & \cdots & e_{1d}^2 & e_{11} \\
\left(\sum_{j=1}^d e_{2j}\right)^2 & e_{21}^2 & e_{22}^2 & \cdots & e_{2d}^2 & e_{22} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\left(\sum_{j=1}^d e_{dj}\right)^2 & e_{d1}^2 & e_{d2}^2 & \cdots & e_{dd}^2 & e_{dd}
\end{pmatrix}.$$

The coefficient determinant is the determinant of the following $(d+1)\times(d+1)$ real symmetric submatrix

$$\begin{vmatrix}
\left(\sum_{1\le i,j\le d} e_{ij}\right)^2 & \left(\sum_{j=1}^d e_{1j}\right)^2 & \left(\sum_{j=1}^d e_{2j}\right)^2 & \cdots & \left(\sum_{j=1}^d e_{dj}\right)^2 \\
\left(\sum_{j=1}^d e_{1j}\right)^2 & e_{11}^2 & e_{12}^2 & \cdots & e_{1d}^2 \\
\left(\sum_{j=1}^d e_{2j}\right)^2 & e_{21}^2 & e_{22}^2 & \cdots & e_{2d}^2 \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\left(\sum_{j=1}^d e_{dj}\right)^2 & e_{d1}^2 & e_{d2}^2 & \cdots & e_{dd}^2
\end{vmatrix}. \tag{5.6}$$

If one can show the coefficient determinant (5.6) is non-zero, then by Cramer's rule the system would have a unique solution. No proper way has been found by the author for computing this determinant in general.
Here we present the computation for the case $d = 2$, note that $e_{12} = e_{21}$. The coefficient determinant (5.6) is

$$\begin{vmatrix} (e_{11}+2e_{12}+e_{22})^2 & (e_{11}+e_{12})^2 & (e_{12}+e_{22})^2 \\ (e_{11}+e_{12})^2 & e_{11}^2 & e_{12}^2 \\ (e_{12}+e_{22})^2 & e_{12}^2 & e_{22}^2 \end{vmatrix} = 2(e_{11}e_{22}-e_{12}^2)^3.$$

It is non-zero according to Proposition 4.2.2, thus the system has a unique solution. By Cramer's rule to get

$$c_0 = \frac{\begin{vmatrix} e_{11}+2e_{12}+e_{22} & (e_{11}+e_{12})^2 & (e_{12}+e_{22})^2 \\ e_{11}+e_{12} & e_{11}^2 & e_{12}^2 \\ e_{12}+e_{22} & e_{12}^2 & e_{22}^2 \end{vmatrix}}{2(e_{11}e_{22}-e_{12}^2)^3} = \frac{-2e_{12}(-e_{12}^2+e_{11}e_{22})^2}{2(e_{11}e_{22}-e_{12}^2)^3} = -\frac{e_{12}}{e_{11}e_{22}-e_{12}^2},$$

Similarly,

$$c_1 = \frac{e_{12}+e_{22}}{e_{11}e_{22}-e_{12}^2},\ c_2 = \frac{e_{11}+e_{12}}{e_{11}e_{22}-e_{12}^2}.$$

Remark. The coefficient determinant (5.6) can not be factorized into "simple" factors when $d > 2$.

5.2.3 An approach for representing V_2^d

The tight frame representation for quadratic generalised Jacobi polynomials will be computed as a sequel of the representation of V_1^d. According to (5.3), the tight frame representation for V_2^d has two scaling factors γ_1, γ_2 with the following form

$$f = \gamma_1 \sum_{j=0}^{d} \langle f, p_{\xi_j^2}\rangle p_{\xi_j^2} + \gamma_2 \sum_{0\le i\ne j\le d} \langle f, p_{\xi_i\xi_j}\rangle p_{\xi_i\xi_j},\ \forall\, f \in V_2^d. \tag{5.7}$$

For each quadratic polynomial g one has

$$p_g = g - \text{the orthogonal projection of } g \text{ onto } V_1^d = g - \sum_{j=0}^{d} c_j\langle g, p_{\xi_j}\rangle p_{\xi_j}.$$

Thus we may use the tight frame representation of V_1^d to compute the polynomials $p_{\xi_j^2}$ and $p_{\xi_i\xi_j}$. According to Lemma 5.1.1, an equivalent condition for (5.7) is as follows

$$\begin{cases} \langle p_{\xi_k^2}, p_{\xi_k^2}\rangle = \gamma_1 \sum_{j=0}^{d}\langle p_{\xi_k^2}, p_{\xi_j^2}\rangle^2 + \gamma_2 \sum_{0\le i\ne j\le d}\langle p_{\xi_k^2}, p_{\xi_i\xi_j}\rangle^2,\ \forall k \in \{0,\cdots,d\} \\ \langle p_{\xi_l\xi_s}, p_{\xi_l\xi_s}\rangle = \gamma_1 \sum_{j=0}^{d}\langle p_{\xi_l\xi_s}, p_{\xi_j^2}\rangle^2 + \gamma_2 \sum_{0\le i\ne j\le d}\langle p_{\xi_l\xi_s}, p_{\xi_i\xi_j}\rangle^2,\ \forall l \ne s \in \{0,\cdots,d\} \end{cases}. \tag{5.8}$$

Due to the time restraint of this research, the computation of scaling factors of V_2^d will be carried out in the future. The next example presents the scaling factors of quadratics corresponding to some specific $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$.

Example 5.2.1. *Denote* $\gamma^{\boldsymbol{\mu},\boldsymbol{\nu}} := (\gamma_1, \gamma_2)$. *Some scaling factors for the cases* $d = 2, 3$ *are*

$$\gamma^{(1,1,1),(0,0,0)} = \left(\frac{63972720}{40843}, \frac{127144080}{40843}\right), \gamma^{(1,1,1),(1,1,1)} = \left(\frac{3168396}{19}, \frac{7076916}{19}\right),$$

$$\gamma^{(1,1,1),(2,2,2)} = \left(\frac{338929390800}{29203}, \frac{767145178800}{29203}\right), \gamma^{(1,1,1),(3,3,3)} = \left(\frac{294756548486400}{456251}, \frac{663685217395200}{456251}\right);$$

$$\gamma^{(1,1,1,1),(0,0,0,0)} = \left(\frac{3624705777719263240080000}{5046592566614819461 73}, \frac{99246690126205396236 0000}{50465925661481946173}\right),$$

$$\gamma^{(1,1,1,1),(1,1,1,1)} = \left(\frac{131963989757414812997519520000}{94776445839660232248 1}, \frac{342862689538086386193645120000}{947764458396602322481}\right),$$

$$\gamma^{(1,1,1,1),(2,2,2,2)} = \left(\frac{2388397700039906369401868736000}{185932448301399208063}, \frac{5906448832671823147207279056000}{185932448301399208063}\right),$$

$$\gamma^{(1,1,1,1),(3,3,3,3)} = \left(\frac{589180238984573932183516252224 00}{6997702198160017783}, \frac{4078006911218832135238481859072000}{20293336374664051570 7}\right).$$

Remark. With observations from the trial computations, we suspect that there exists a unique set of scaling factors for V_2^d in cases where $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathbb{N}_0^{d+1}$ are homogeneous.

Chapter 6

Conclusions

A generalised Jacobi weight function over the standard simplex in $\mathbb{R}^d$ was defined, the integrability condition of the weight function over the simplex was established. The orthogonal polynomials corresponding to the generalised Jacobi weight function were investigated by using the tight frame theory. It was observed that the symmetry of a space of orthogonal polynomials can be clearly depicted by its tight frame representation.

Computations were carried out for the space of orthogonal polynomials of degree one, which serves as a future reference for computing the tight frame representation of orthogonal polynomials of degree two. It was concluded that a tight frame for the space of orthogonal polynomials of degree one under a special case of the generalised Jacobi weight exists and it is unique, the corresponding frame scaling factors were computed.

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