A sharpening of the Welch bounds and the existence of real and complex spherical t–designs

Shayne Waldron

Department of Mathematics
University of Auckland
Private Bag 92019, Auckland, New Zealand
e–mail: waldron@math.auckland.ac.nz

September 14, 2016

Abstract

The Welch bounds for a finite set of unit vectors are a family of inequalities indexed by $t = 1, 2, \ldots$, which describe how "evenly spread" the vectors are. They have important applications in signal analysis, where sequences giving equality in the first Welch bound are known as WBE sequences or as unit norm tight frames.

Here we consider sequences of vectors giving equality in the higher order Welch bounds. These are seen to correspond to tight frames for the complex symmetric t–tensors (which we prove always exist). We show that for $t > 1$ the Welch bounds can be sharpened for real vectors, and again, vectors giving equality alway exist. We give a unified treatment of various conditions for equality in both the real and complex cases. In particular, we give an explicit description of the corresponding cubature rules (t–designs). Our results set up a framework for the construction and classification several configurations of vectors of recent interest. These include MUBs (mutually unbiased bases), SICs (complex equiangular lines), spherical half–designs, projective t–designs and minimisers of the higher order frame potential. One interesting consequence is a construction of sets of complex equiangular lines which were previously unknown.

Key Words: Welch bounds, WBE sequences (Welch bound equality sequences), finite tight frames, symmetric tensors, cubature rules for the sphere, spherical t–designs, projective t–designs, MUBs (mutually unbiased bases), SICs (symmetric informationally complete positive operator valued measures), complex equiangular lines

AMS (MOS) Subject Classifications: primary 05B30, 42C15, 65D30, 94A12, secondary 42C15, 42C40

1 Introduction

Let $v_1, \ldots, v_n$ be a finite set of unit vectors in $\mathbb{C}^d$. In [Wel74], Welch gave the following estimate of the maximum cross correlation

$$\left(\max_{j\neq k} |\langle v_j, v_k\rangle|\right)^{2t} \geq \frac{1}{n-1}\left(\frac{n}{\binom{d+t-1}{t}} - 1\right), \qquad t = 1, 2, \ldots, \tag{1.1}$$

which follows from

$$\sum_{j=1}^{n}\sum_{k=1}^{n} |\langle v_j, v_k\rangle|^{2t} \geq \frac{n}{\binom{d+t-1}{t}}, \qquad t = 1, 2, \ldots. \tag{1.2}$$

These *Welch bounds* describe how "evenly spread" the vectors are, and have important applications in signal analysis and quantum information theory [MM93], [EF02], [LL14], where sequences giving equality in the first Welch bound of (1.2) are known as WBE sequences or as unit norm tight frames. Here we consider the inequalities

$$\sum_{j=1}^{n}\sum_{k=1}^{n} |\langle v_j, v_k\rangle|^{2t} \geq \frac{1}{\binom{d+t-1}{t}}\left(\sum_{\ell=1}^{n} \|v_\ell\|^{2t}\right)^2, \qquad t = 1, 2, \ldots, \tag{1.3}$$

(for any $v_1, \ldots, v_n \in \mathbb{C}^d$) from which the Welch bounds follow, and, in particular, when equality is achieved. We will show that

- Equality in (1.3) corresponds to $(v_j^{\otimes t})_{j=1}^n$ being a a tight frame for the space of symmetric t–tensors. This gives a simple proof of (1.3) from the $t = 1$ case (which follows from the Cauchy–Schwartz inequality).

- Equality in (1.3) corresponds to a cubature rule for the complex sphere. From this it follows that sequences of unit vectors giving equality always exist (for some n).

- For $t > 1$, the inequality (1.3) can be sharpened for vectors in $\mathbb{R}^d$ to

$$\sum_{j=1}^{n}\sum_{k=1}^{n} |\langle v_j, v_k\rangle|^{2t} \geq \frac{1\cdot 3\cdot 5\cdots(2t-1)}{d(d+2)\cdots(d+2(t-1))}\left(\sum_{\ell=1}^{n} \|v_\ell\|^{2t}\right)^2. \tag{1.4}$$

  Sequences of vectors in $\mathbb{R}^d$ giving equality in (1.4) correspond to a cubature rule for the real sphere, and so always exist (for some n).

- There are various equivalent conditions for equality in (1.3) and (1.4), which can be treated in a unified way.

The key to our approach is the identification of equality in (1.3) and (1.4) with a suitable cubature rule (spherical design). From this and associated equivalent conditions, the connection with several configurations of vectors of recent interest becomes apparent. These include MUBs (mutually unbiased bases), SICs (complex equiangular lines), spherical half–designs, (weighted) projective t–designs and minimisers of the higher order frame potential. Our unified treatment (for the real and complex cases, and vectors with no restriction on their norms) sets up a framework for the construction and classification of such sets of vectors. To this end, let $\mathbb{F}$ denote $\mathbb{R}$ or $\mathbb{C}$ throughout.

Many special cases of the results presented here have appeared in the literature, and we give references where known. To the best of our knowledge, the case where the vectors have no restriction on their norms has not been considered for $t > 1$, and the sharpened Welch bound (1.4) for real unit vectors is only known implicitly in the area of sphericial t–designs [SW09]. The complex equiangular lines of Example 7.1 are new.

2 Tight frames of symmetric tensors

A sequence of vectors $(f_j)_{j=1}^n$ is a **tight frame** for a Hilbert space $\mathcal{H}$ if (for some $A > 0$)

$$\sum_{j=1}^n |\langle f, f_j\rangle|^2 = A\|f\|^2, \qquad \forall f \in \mathcal{H}, \tag{2.5}$$

or, equivalently (by the polarisation identity)

$$f = \frac{1}{A}\sum_{j=1}^n \langle f, f_j\rangle f_j, \qquad \forall f \in \mathcal{H}. \tag{2.6}$$

These generalisations of orthonormal bases have been extensively used (in the infinite dimensional setting) to construct *wavelets* and *Gabor systems* for $L_2(\mathbb{R}^d)$ with good time–frequency localisation [DS52], [You01], [Chr03], [Grö01]. More recently, they have found applications for finite dimensional spaces [CK13], e.g., signal analysis [GKK01], quantum information theory [RBKSC04] and multivariate orthogonal polynomials [VW05].

The following (see [Wal03]) shows that WBE sequences are unit norm tight frames.

Proposition 2.1 *Let $v_1, \ldots, v_n$ be vectors in $\mathbb{F}^d$, which are not all zero. Then*

$$\sum_{j=1}^n \sum_{k=1}^n |\langle v_j, v_k\rangle|^2 \geq \frac{1}{d}\Big(\sum_{j=1}^n \|v_j\|^2\Big)^2, \tag{2.7}$$

with equality if and only if $(v_j)_{j=1}^n$ is a tight frame for $\mathbb{F}^d$.

Proof: Let S be the frame operator of (v_j), i.e., the $d\times d$ positive semidefinite matrix given by

$$Sf := \sum_{j=1}^n \langle f, v_j\rangle v_j, \qquad \forall f \in \mathbb{F}^d,$$

which satisfies

$$\operatorname{trace}(S) = \sum_j \|v_j\|^2, \qquad \operatorname{trace}(S^2) = \sum_j\sum_k |\langle v_j, v_k\rangle|^2.$$

Let $\lambda_1,\ldots,\lambda_d \geq 0$ be the eigenvalues of S. By the Cauchy–Schwarz inequality

$$\operatorname{trace}(S)^2 = (\sum_j \lambda_j)^2 = \langle (1), (\lambda_j)\rangle^2 \leq \|(1)\|^2 \|(\lambda_j)\|^2 = d \sum_j \lambda_j^2 = d \operatorname{trace}(S^2),$$

which is (2.7), with equality if and only if $\lambda_j = A$, $\forall j$, $A > 0$ i.e., by (2.6),

$$S = AI \quad \iff \quad (v_j) \text{ is a tight frame for } \mathbb{F}^d.$$

Note above, since one of the vectors (v_j) is nonzero, $S \neq 0$, and so $A \neq 0$. □

We now extend (2.7) to an inequality, for which equality gives a tight frame for the symmetric tensors in $\otimes^t \mathbb{F}^d := \mathbb{F}^d \otimes \cdots \otimes \mathbb{F}^d$ (t times). For simplicity, we define the **symmetric tensors** of rank t to be the subspace of $\otimes^t \mathbb{F}^d$ given by

$$\operatorname{Sym}^t(\mathbb{F}^d) := \operatorname{span}\{v^{\otimes t} : v \in \mathbb{F}^d\}, \qquad v^{\otimes t} := v \otimes \cdots \otimes v \quad (t \text{ times}).$$

This Hilbert space has dimension

$$\dim(\operatorname{Sym}^t(\mathbb{F}^d)) = \binom{t+d-1}{t}, \tag{2.8}$$

and its inner product satisfies

$$\langle v^{\otimes t}, w^{\otimes t}\rangle = \langle v, w\rangle^t, \qquad \forall v, w \in \mathbb{F}^d. \tag{2.9}$$

The dual space $(\operatorname{Sym}^t(\mathbb{F}^d))^* = \operatorname{Sym}^t((\mathbb{F}^d)^*)$ contains $\langle \cdot, v\rangle^{\otimes t}$, $v \in \mathbb{F}^d$, and its inner product is given by

$$\langle \langle \cdot, v\rangle^{\otimes t}, \langle \cdot, w\rangle^{\otimes t}\rangle = \langle w, v\rangle^t, \qquad \forall v, w \in \mathbb{F}^d. \tag{2.10}$$

A vector space isomorphism between $\operatorname{Sym}^t((\mathbb{F}^d)^*)$ and the space $\mathcal{L}_t(\mathbb{F}^d)$ of **symmetric** t**–linear maps** $(\mathbb{F}^d)^t \to \mathbb{F}$ is given by

$$\lambda^{\otimes t} \mapsto L, \quad (\lambda \in (\mathbb{F}^d)^*) \qquad L(v_1,\ldots,v_t) := \lambda(v_1)\cdots\lambda(v_t).$$

We define the space of **homogeneous polynomials** on $\mathbb{F}^d$ of degree t to be

$$\Pi_t^\circ(\mathbb{F}^d) := \{\hat{L} : L \in \mathcal{L}_t(\mathbb{F}^d)\}, \qquad \hat{L} : \mathbb{F}^d \to \mathbb{F},\ \hat{L}(v) := L(v,\ldots,v).$$

The map $L \mapsto \hat{L}$ above gives a vector space isomorphism $\mathcal{L}_t(\mathbb{F}^d) \to \Pi_t^\circ(\mathbb{F}^d)$.

The inner product on $\Pi_t^\circ(\mathbb{F}^d)$ induced from that on $(\operatorname{Sym}^t(\mathbb{F}^d))^*$ via the above isomorphisms is the **apolar** (or **Bombieri** or **Fisher**) inner product, which is given by

$$\langle \langle \cdot, v\rangle^t, \langle \cdot, w\rangle^t\rangle_\circ := \langle \langle \cdot, v\rangle^{\otimes t}, \langle \cdot, w\rangle^{\otimes t}\rangle = \langle w, v\rangle^t. \tag{2.11}$$

It follows from (2.11) that the apolar inner product satisfies

$$\langle p, \langle \cdot, w\rangle^t\rangle_\circ = p(w), \qquad \forall p \in \Pi_t^\circ(\mathbb{F}^d), \quad \forall w \in \mathbb{F}^d, \tag{2.12}$$

i.e., $\langle \cdot, w\rangle^t$ is the Riesz representer of point evaluation at w. A calculation (see [Wal16]) shows that the monomials $\{z^\alpha\}_{|\alpha|=t}$ are orthogonal (for $\mathbb{F}$ equal to $\mathbb{R}$ and $\mathbb{C}$), with

$$\langle z^\alpha, z^\alpha\rangle_\circ = \frac{\alpha!}{|\alpha|!}. \tag{2.13}$$

The equation (2.13) is usually used to define the Bombieri inner product.

We are now able to prove the *higher order Welch bounds* (1.3), as an example of Proposition 2.1. The equivalent condition (a) was found independently by [DHC12] (in the case of unit vectors).

Theorem 2.1 *Fix $t \in \{1, 2, \ldots\}$. Let $v_1, \ldots, v_n$ be vectors in $\mathbb{F}^d$, which are not all zero. Then*

$$\sum_{j=1}^n \sum_{k=1}^n |\langle v_j, v_k\rangle|^{2t} \geq \frac{1}{\binom{t+d-1}{t}} \Bigl(\sum_{j=1}^n \|v_j\|^{2t}\Bigr)^2, \tag{2.14}$$

with equality precisely when any of the equivalent conditions holds

(a) $(v_j^{\otimes t})_{j=1}^n$ *is a tight frame for the symmetric tensors* $\mathrm{Sym}^t(\mathbb{F}^d)$.

(b) $(\langle \cdot, v_j\rangle^{\otimes t})_{j=1}^n$ *is a tight frame for* $(\mathrm{Sym}^t(\mathbb{F}^d))^* = \mathrm{Sym}^t((\mathbb{F}^d)^*)$.

(c) $(\langle \cdot, v_j\rangle^t)_{j=1}^n$ *is a tight frame for* $\Pi_t^\circ(\mathbb{F}^d)$ *with the apolar inner product* (2.11).

Proof: Firstly, we observe that $(v_j^{\otimes t})_{j=1}^n$ is a sequence of vectors in $\mathrm{Sym}^t(\mathbb{F}^d)$ which are not all zero, since $v^{\otimes t}$ is zero if and only if $v = 0$.

Thus we may apply Proposition 2.1, using (2.8), to obtain

$$\sum_{j=1}^n \sum_{k=1}^n |\langle v_j^{\otimes t}, v_k^{\otimes t}\rangle|^2 \geq \frac{1}{\binom{t+d-1}{t}} \Bigl(\sum_{j=1}^n \|v_j^{\otimes t}\|^2\Bigr)^2,$$

with equality if and only if (a) holds. By (2.9), the equation above equals (2.14). A similar argument, using (2.10) and (2.11) in place of (2.9), gives (c) and (d), respectively. □

We observe that Theorem 2.1 reduces to Proposition 2.1 for $t = 1$.

Corollary 2.1 *Equality in (2.14) is equivalent to the generalised Bessel and Plancherel identities*

$$\|x\|^{2t} = \frac{\binom{d+t-1}{t}}{\sum_{\ell=1}^n \|v_\ell\|^{2t}} \sum_{j=1}^n |\langle x, v_j\rangle|^{2t}, \qquad \forall x \in \mathbb{F}^d, \tag{2.15}$$

$$\langle x, y\rangle^t = \frac{\binom{d+t-1}{t}}{\sum_{\ell=1}^n \|v_\ell\|^{2t}} \sum_{j=1}^n \langle x, v_j\rangle^t \langle v_j, y\rangle^t, \qquad \forall x, y \in \mathbb{F}^d. \tag{2.16}$$

Proof: By the polarisation identity, the condition that (f_j) be a tight frame for $\mathcal{H}$ is equivalent to the *Plancherel identity*

$$\langle f,g\rangle = \frac{1}{A}\sum_{j=1}^n \langle f,f_j\rangle\langle f_j,g\rangle, \qquad \forall f,g\in\mathcal{H}, \tag{2.17}$$

where (by taking the trace of the frame operator)

$$\dim(\mathcal{H})A = \textstyle\sum_\ell \|f_\ell\|^2. \tag{2.18}$$

Let (f_j) be the tight frame $(v_j^{\otimes t})_{j=1}^n$ for $\mathcal{H}=\mathrm{Sym}^t(\mathbb{F}^d)$. Then taking $f=x^{\otimes t}$, $g=y^{\otimes t}$ in (2.5) and (2.17), and using (2.8), (2.9) and (2.18), gives (2.15) and (2.16). □

3 Cubature on the real and complex spheres

Here we show that inequalities of the form (1.3) and (1.4), i.e.,

$$\sum_{j=1}^n\sum_{k=1}^n |\langle v_j,v_k\rangle|^{2t} \ge C\left(\sum_{\ell=1}^n \|v_\ell\|^{2t}\right)^2, \tag{3.19}$$

are closely related to cubature formulas for polynomials on the real and complex spheres.

Let $\mathbb{S}=\mathbb{S}_\mathbb{F}$ denote the unit sphere in $\mathbb{F}^d$, and σ be the normalised surface area on $\mathbb{S}$. The invariance of surface area measure under unitary maps (and a standard computation) shows that

$$\int_\mathbb{S} |\langle x,y\rangle|^{2t}\,d\sigma(y) = \|x\|^{2t}c_t(d,\mathbb{F}), \qquad \forall x\in\mathbb{F}^d, \tag{3.20}$$

where

$$c_t(d,\mathbb{C}) = \frac{1}{\binom{d+t-1}{t}}, \qquad c_t(d,\mathbb{R}) = \frac{1\cdot 3\cdot 5\cdots(2t-1)}{d(d+2)\cdots(d+2(t-1))}. \tag{3.21}$$

We observe that $c_t(d,\mathbb{R})\ge c_t(d,\mathbb{C})$, with strict inequality when $t,d>1$. Suppose that all the vectors (v_j) lie on $\mathbb{S}$. Then integrating (3.19) with respect to $d\sigma(v_k)$ and using (3.20), gives

$$\sum_j\sum_k \|v_j\|^{2t}c_t(d,\mathbb{F}) \ge Cn^2 \quad\Longrightarrow\quad c_t(d,\mathbb{F})\ge C.$$

Let $\Pi_k(\mathbb{R}^d)$ denote the polynomials $\mathbb{R}^d\to\mathbb{R}$ of degree at most k, and $\Pi_k^\circ(\mathbb{R}^d)$ the subspace of homogeneous polynomials of degree k. Of interest to us is the space of polynomials $\mathbb{F}^d\to\mathbb{F}$ given by

$$\Pi_{t,t}^\circ(\mathbb{F}^d) = \mathrm{Hom}(t,t) := \mathrm{span}\{z\mapsto z^\alpha\overline{z}^\beta : |\alpha|=|\beta|=t\}, \tag{3.22}$$

which are homogeneous of degree t in z and in $\overline{z}$. Equivalently

$$\Pi_{t,t}^\circ(\mathbb{F}^d) = \mathrm{span}\{z\mapsto |\langle z,v\rangle|^{2t} : v\in\mathbb{F}^d\}. \tag{3.23}$$

We note that $\Pi_{t,t}^\circ(\mathbb{R}^d)=\Pi_{2t}^\circ(\mathbb{R}^d)$. Denote the restriction of a polynomial space P to the unit sphere by $P(\mathbb{S})$. Recall that a homogeneous polynomial f of degree $2t$ is uniquely determined by its values on $\mathbb{S}$ by $f(x)=\|x\|^{2t}f(x/\|x\|)$, $x\ne 0$.

Definition 3.1 *A sequence* $(v_j)_{j=1}^n$ *of vectors in* $\mathbb{F}^d$ *is a* **cubature rule** *for a space* P *of homogeneous polynomials of degree* $2t$*, such as* $\Pi_{t,t}^\circ(\mathbb{F}^d)$*, if*

$$\int_{\mathbb{S}} p(x)\,d\sigma(x) = \frac{1}{\sum_k \|v_k\|^{2t}} \sum_{j=1}^n p(v_j) = \sum_{\substack{j=1\\ v_j\neq 0}}^n \frac{\|v_j\|^{2t}}{\sum_k \|v_k\|^{2t}} p(\frac{v_j}{\|v_j\|}), \qquad \forall p \in P.$$

A cubature rule for which the vectors (v_j) have equal norms gives an *unweighted* cubature rule for the integration of P over $\mathbb{S}$. The following generalisation of the integral form of the mean value theorem implies that equal weight cubature rules exist.

Theorem 3.1 ([SZ84]) *Let* X *be a path–connected topological space, and* μ *be a finite (positive) measure on* X*, defined on the open sets, with full support, i.e.,* $\mu(U) > 0$ *for every nonempty open set* $U \subset X$*. For a continuous integrable function* $f : X \to \mathbb{R}^m$*, there exists a finite set of samples* $A \subset X$ *for which*

$$\frac{1}{\mu(X)} \int_X f\,d\mu = \frac{1}{|A|} \sum_{a\in A} f(a).$$

Here $|A|$*, the size of* A*, can be any number with a finite number of exceptions.*

Let $\mu = \sigma$ (normalised surface area on $\mathbb{S}$), and the coordinates of $f = (f_1, \ldots, f_n)$ be the real and complex parts of spanning set for P. Then Theorem 3.1 implies that equal weight cubature rules for P exist.

4 The sharpened Welch bounds

We now prove the main result: the Welch bound (1.3) and its sharpened form (1.4) for real vectors, together with conditions for equality. Let $c_t(d, \mathbb{F})$ be given by (3.21).

Theorem 4.1 *Fix* $t \in \{1, 2, \ldots\}$*. Let* $v_1, \ldots, v_n$ *be vectors in* $\mathbb{F}^d$*, not all zero. Then*

$$\sum_{j=1}^n \sum_{k=1}^n |\langle v_j, v_k\rangle|^{2t} \geq c_t(d, \mathbb{F}) \Bigl(\sum_{\ell=1}^n \|v_\ell\|^{2t}\Bigr)^2, \tag{4.24}$$

with equality when any of the following equivalent conditions hold

(a) The generalised Bessel identity

$$c_t(d, \mathbb{F})\|x\|^{2t} = \frac{1}{\sum_{\ell=1}^n \|v_\ell\|^{2t}} \sum_{j=1}^n |\langle x, v_j\rangle|^{2t}, \qquad \forall x \in \mathbb{F}^d. \tag{4.25}$$

(b) The generalised Plancherel identity

$$c_t(d, \mathbb{F})\langle x, y\rangle^t = \frac{1}{\sum_{\ell=1}^n \|v_\ell\|^{2t}} \sum_{j=1}^n \langle x, v_j\rangle^t \langle v_j, y\rangle^t, \qquad \forall x, y \in \mathbb{F}^d. \tag{4.26}$$

(c) The cubature rule for $\Pi_{t,t}(\mathbb{F}^d)$

$$\int_{\mathbb{S}} p(x)\, d\sigma(x) = \frac{1}{\sum_{\ell=1}^n \|v_\ell\|^{2t}} \sum_{j=1}^n p(v_j), \qquad \forall p \in \Pi_{t,t}^\circ(\mathbb{F}^d), \tag{4.27}$$

or, equivalently, for $\Pi_{t,t}(\mathbb{S})$

$$\int_{\mathbb{S}} p(x)\, d\sigma(x) = \sum_{\substack{j=1\\ v_j\neq 0}}^n \frac{\|v_j\|^{2t}}{\sum_{\ell=1}^n \|v_\ell\|^{2t}} p\Big(\frac{v_j}{\|v_j\|}\Big), \qquad \forall p \in \Pi_{t,t}^\circ(\mathbb{S}). \tag{4.28}$$

(d) The tensor product integration formula

$$\int_{\mathbb{S}} x^{\otimes t} \otimes \overline{x}^{\otimes t}\, d\sigma(x) = \frac{1}{\sum_{\ell=1}^n \|v_\ell\|^{2t}} \sum_{j=1}^n v_j^{\otimes t} \otimes \overline{v_j}^{\otimes t}. \tag{4.29}$$

(e) The integration formula

$$\int_{\mathbb{S}} \langle \cdot, x^{\otimes t}\rangle x^{\otimes t}\, d\sigma(x) = \frac{1}{\sum_{\ell=1}^n \|v_\ell\|^{2t}} \sum_{j=1}^n \langle \cdot, v_j^{\otimes t}\rangle v_j^{\otimes t}. \tag{4.30}$$

(f) For all univariate polynomials $g \in \Pi_t(\mathbb{R})$, we have

$$\int_{\mathbb{S}}\int_{\mathbb{S}} g\big(|\langle x,y\rangle|^2\big)\, d\sigma(y)\, d\sigma(x) = \sum_{\substack{j=1\\ v_j\neq 0}}^n \sum_{\substack{k=1\\ v_k\neq 0}}^n \frac{\|v_j\|^{2t}\|v_k\|^{2t}}{(\sum_\ell \|v_\ell\|^{2t})^2} g\Big(|\langle \frac{v_j}{\|v_j\|}, \frac{v_k}{\|v_k\|}\rangle|^2\Big). \tag{4.31}$$

Proof: Let $C := \sum_{\ell=1}^n \|v_\ell\|^{2t}$. Define a tensor $\xi \in \mathrm{Sym}^t(\mathbb{F}^d) \otimes \mathrm{Sym}^t(\overline{\mathbb{F}^d})$ and a self adjoint operator Q on $\mathrm{Sym}^t(\mathbb{F}^d)$ by

$$\xi := \int_{\mathbb{S}} x^{\otimes t} \otimes \overline{x}^{\otimes t}\, d\sigma(x) - \frac{1}{C} \sum_{j=1}^n v_j^{\otimes t} \otimes \overline{v_j}^{\otimes t},$$

$$Q := \int_{\mathbb{S}} \langle \cdot, x^{\otimes t}\rangle x^{\otimes t}\, d\sigma(x) - \frac{1}{C} \sum_{j=1}^n \langle \cdot, v_j^{\otimes t}\rangle v_j^{\otimes t}.$$

Equip $\mathrm{Sym}^t(\mathbb{F}^d) \otimes \mathrm{Sym}^t(\overline{\mathbb{F}^d})$ with the apolar inner product, and the linear operators on $\mathrm{Sym}^t(\mathbb{F}^d)$ with the Frobenius inner product. Then a simple calculation using (2.9) and (3.20) shows that

$$\langle \xi, \xi\rangle_\circ = \langle Q, Q\rangle_F = \frac{1}{C^2} \sum_j \sum_k |\langle v_j, v_k\rangle|^{2t} - c_t(d,\mathbb{F}) \geq 0,$$

which is (4.24). Moreover, equality in (4.24) is equivalent to (d) or to (e). By the polarisation identity and (2.10), (a) and (b) are equivalent.

We now complete the proof by showing

$$(\mathrm{d}) \implies (\mathrm{c}) \implies (\mathrm{a}), (\mathrm{f}) \implies \text{equality in (4.24)}.$$

(d)$\Longrightarrow$(c): Expand $x^{\otimes t} \otimes \overline{x}^{\otimes t}$ in terms of the coordinates of x. Since

$$x^{\otimes t} = \sum_{k_1=1}^{d} x_{k_1} e_{k_1} \otimes \cdots \otimes \sum_{k_t=1}^{d} x_{k_t} e_{k_t} = \sum_{k \in \{1,\ldots,d\}^t} p_k(x) \eta_k,$$

$$p_k(x) := x_{k_1} x_{k_2} \cdots x_{k_t}, \qquad \eta_k := e_{k_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_t},$$

we obtain

$$x^{\otimes t} \otimes \overline{x}^{\otimes t} = \sum_{k,\ell} p_k(x) p_\ell(\overline{x})\, \eta_k \otimes \eta_\ell.$$

Thus (d) can be written as

$$\int_{\mathbb{S}} \sum_{k,\ell} p_k(x) p_\ell(\overline{x})\, \eta_k \otimes \eta_\ell \, d\sigma(x) = \frac{1}{C} \sum_{j=1}^{n} \sum_{k,\ell} p_k(v_j) p_\ell(\overline{v_j})\, \eta_k \otimes \eta_\ell.$$

Since the tensors $\eta_k \otimes \eta_\ell$ are linearly independent, equating their coefficients gives the cubature rule for all the polynomials $x \mapsto p_k(x) p_\ell(\overline{x})$, and hence for $\Pi^\circ_{t,t}(\mathbb{F}^d)$.
(c)$\Longrightarrow$(a): Let $p = |\langle x, \cdot \rangle|^{2t} \in \Pi^\circ_{t,t}(\mathbb{F}^d)$ in (4.27) and use (3.20) to obtain

$$c_t(d, \mathbb{F}) \|x\|^{2t} = \int_{\mathbb{S}} |\langle x, y \rangle|^{2t} \, d\sigma(y) = \frac{1}{C} \sum_j |\langle x, v_j \rangle|^{2t}.$$

(c)$\Longrightarrow$(f): Let $p = \|\cdot\|^{2(t-s)} |\langle x, \cdot \rangle|^{2s} \in \Pi^\circ_{t,t}$, $0 \le s \le t$ in (4.28) to get

$$\int_{\mathbb{S}} |\langle x, y \rangle|^{2s} \, d\sigma(y) = \sum_k \frac{\|v_k\|^{2t}}{C} |\langle x, \frac{v_k}{\|v_k\|} \rangle|^{2s}.$$

For $x \in \mathbb{S}$, $|\langle x, \frac{v_k}{\|v_k\|} \rangle|^{2s} = \|x\|^{2(t-s)} |\langle x, \frac{v_k}{\|v_k\|} \rangle|^{2s}$, and so using (4.28) again gives

$$\int_{\mathbb{S}} \int_{\mathbb{S}} |\langle x, y \rangle|^{2s} \, d\sigma(y) \, d\sigma(x) = \sum_j \frac{\|v_j\|^{2t}}{C} \sum_k \frac{\|v_k\|^{2t}}{C} |\langle \frac{v_j}{\|v_j\|}, \frac{v_k}{\|v_k\|} \rangle|^{2s}.$$

Thus (4.31) holds for the monomials $(\cdot)^s$, $0 \le s \le t$, and hence for $\Pi_t(\mathbb{R})$.
(a)$\Longrightarrow$ equality in (4.24): Take $x = v_k$ in (a) then sum over k to obtain the required equality

$$c_t(d, \mathbb{F}) \|v_k\|^{2t} = \frac{1}{C} \sum_j |\langle v_k, v_j \rangle|^{2t},$$

$$c_t(d, \mathbb{F}) C = c_t(d, \mathbb{F}) \sum_k \|v_k\|^{2t} = \frac{1}{C} \sum_k \sum_j |\langle v_k, v_j \rangle|^{2t}.$$

(f)$\Longrightarrow$ equality in (4.24): Take $g = (\cdot)^t$ in (f) to obtain the desired equality

$$c_t(d,\mathbb{F}) = \int_{\mathbb{S}}\int_{\mathbb{S}} |\langle x,y\rangle|^{2t}\, d\sigma(y)\, d\sigma(x) = \sum_{\substack{j=1\\ v_j\neq 0}}^n \sum_{\substack{k=1\\ v_k\neq 0}}^n \frac{\|v_j\|^{2t}\|v_k\|^{2t}}{(\sum_\ell \|v_\ell\|^{2t})^2} |\langle \frac{v_j}{\|v_j\|}, \frac{v_k}{\|v_k\|}\rangle|^{2t}$$

$$= \frac{1}{C^2}\sum_j\sum_k |\langle v_j,v_k\rangle|^{2t}.$$

□

Since $c_t(d,\mathbb{R}) \geq c_t(d,\mathbb{C})$, with strict inequality when $t,d>1$, (4.24) *sharpens* (1.3) to (1.4) for $t>1$ (and $d>1$). The corresponding sharpened form of (1.1) for unit vectors $v_1,\ldots,v_n \in \mathbb{R}^d$ is

$$\Bigl(\max_{j\neq k}|\langle v_j,v_k\rangle|\Bigr)^{2t} \geq \frac{1}{n-1}\left(nc_t(d,\mathbb{R})-1\right), \qquad t=1,2,\ldots. \tag{4.32}$$

Example 4.1 *For $d\geq 2$ and $t>1$, there is no tight frame $(v_j^{\otimes t})$ for $\mathrm{Sym}^t(\mathbb{R}^d)$. If there was, then Proposition 2.1 gives*

$$\sum_j\sum_k |\langle v_j^{\otimes t}, v_k^{\otimes t}\rangle|^2 = \sum_j\sum_k |\langle v_j,v_k\rangle|^{2t} = \frac{1}{\binom{t+d-1}{t}}\Bigl(\sum_j \|v_j\|^{2t}\Bigr)^2,$$

which violates the sharpened Welch bound (1.4).

We adapt the notation of [RS14] (for the cubature rule).

Definition 4.1 *A sequence of vectors (v_j) in $\mathbb{F}^d$ giving equality in Theorem 4.1, i.e.,*

$$\sum_{j=1}^n\sum_{k=1}^n |\langle v_j,v_k\rangle|^{2t} = c_t(d,\mathbb{F})\Bigl(\sum_{\ell=1}^n \|v_\ell\|^{2t}\Bigr)^2, \tag{4.33}$$

is called a (**spherical**) (t,t)**–design** *for* $\mathbb{F}^d$.

In view of condition (c), it follows from Theorem 3.1 that unit norm (t,t)–designs always exist (for some n). We now seek to place Theorem 4.1 within the literature.

For $t=1$,

$$c_1(d,\mathbb{R}) = c_1(d,\mathbb{C}) = \frac{1}{d},$$

and the generalised Bessel identity (4.25) implies that (v_j) is a tight frame for $\mathbb{F}^d$ (see [Wal03]). In [BF03], the left hand side of (4.24) for unit vectors (v_j) is identified as the potential corresponding to a *frame force*, and its minimisers identified as the unit norm tight frames.

For $t>1$ (and $d>1$) the circle of ideas embodied in Theorem 4.1 seem to first appear via cubature rules for the sphere, which are widely known as *spherical designs*. Up until recently, cubature rules for the real sphere were the most studied (see [BB09]). Here the space of polynomials integrated is usually $\Pi_t(\mathbb{R}^d)$, which gives a *(real) spherical*

t–*design*. Cubature rules which integrate $\Pi_t^\circ(\mathbb{R}^d)$ for t *even*[1] were considered by [Sei01]. These were also called spherical t–designs, but the term *(real) spherical half–design of order* t is now in common use [KP11]. In view of (4.28) and $\Pi_{t,t}^\circ(\mathbb{R}^d) = \Pi_{2t}^\circ(\mathbb{R}^d)$, unit vectors (v_j) in $\mathbb{R}^d$ are a (t,t)–design if and only if they are a spherical half–design of order $2t$. In [Sei01], (4.24) for unit vectors is attributed to Sidel'nikov, and the equivalent conditions (a) and (d) are developed: they are called Definitions III (*Waring formula*) or IV (*isometry condition*) and II (*tensor*). The condition (f) is used by [Lev98] to define weighted t–designs in the very general setting of a metric space endowed with a measure.

Cubature rules for the complex sphere (including projective versions) were introduced in the 1970's [DGS77] and various estimates given for their (minimum) size. There has been a recent resurgence of interest [RS07] [RS14], motivated by the application of SICs and MUBs in quantum information theory (see §7). The conditions (c), (e) and (f) are developed in [RS07] and are presented in terms of *weighted complex projective* t–*designs* (see §6).

5 Weighted spherical (t,t)–designs

For a given spherical (t,t)–design (v_j), the cubature rule (4.28) can be written as

$$\int_{\mathbb{S}} p(x)\,d\sigma(x) = \sum_{\substack{j=1\\ v_j\neq 0}}^n w_j p(\phi_j), \qquad \forall p\in\Pi_{t,t}^\circ(\mathbb{S}), \tag{5.34}$$

where

$$\phi_j = \frac{v_j}{\|v_j\|}, \qquad w_j = \frac{\|v_j\|^{2t}}{\sum_\ell \|v_\ell\|^{2t}}. \tag{5.35}$$

For $q\in\Pi_{r,r}^\circ(\mathbb{F}^d)$, $\|\cdot\|^{2(t-r)}q\in\Pi_{t,t}^\circ(\mathbb{F}^d)$, $1\le r\le t$, and so we obtain

$$\Pi_{r,r}^\circ(\mathbb{S})\subset\Pi_{t,t}^\circ(\mathbb{S}), \qquad 0\le r\le t. \tag{5.36}$$

Combining these observations gives:

Proposition 5.1 *Fix $t\ge 1$. If $(v_j)_{j=1}^n$ is a (t,t)–design for $\mathbb{F}^d$, then $(\|v_j\|^{t/r-1}v_j)$ is an (r,r)–design for $\mathbb{F}^d$, $1\le r\le t$, i.e.,*

$$\sum_{j=1}^n\sum_{k=1}^n |\langle v_j,v_k\rangle|^{2r}\|v_j\|^{2(t-r)}\|v_k\|^{2(t-r)} = c_r(d,\mathbb{F})\Bigl(\sum_{\ell=1}^n \|v_\ell\|^{2t}\Bigr)^2. \tag{5.37}$$

Proof: Let $g_j := \|v_j\|^{t/r-1}v_j$ and $q\in\Pi_{r,r}^\circ(\mathbb{S})$. Since $p := \|\cdot\|^{2(t-r)}q\in\Pi_{t,t}^\circ(\mathbb{S})$, we have

$$\sum_{\substack{j=1\\ g_j\neq 0}}^n \frac{\|g_j\|^{2r}}{\sum_{\ell=1}^n\|g_\ell\|^{2r}} q\Bigl(\frac{g_j}{\|g_j\|}\Bigr) = \sum_{\substack{j=1\\ v_j\neq 0}}^n \frac{\|v_j\|^{2t}}{\sum_{\ell=1}^n\|v_\ell\|^{2t}} p\Bigl(\frac{v_j}{\|v_j\|}\Bigr) = \int_{\mathbb{S}} p\,d\sigma = \int_{\mathbb{S}} q\,d\sigma,$$

and so, by (4.28), (g_j) is an (r,r)–design. Substituting into (4.33) gives (5.37). □

[1] [Sei01] does not say explicitly that t must be even. However, if t is odd, then the *distribution t–tensor* $\int_{\mathbb{S}} x^{\otimes t}\,d\sigma(x)$ is zero, and the results of [Sei01] do not hold.

Example 5.1 *An equal norm (t,t)–design for $\mathbb{F}^d$ is an (r,r)–design, $1 \le r \le t$.*

Example 5.2 *If (v_j) is a (t,t)–design, $t \ge 1$, then $(\|v_j\|^{t-1}v_j)$ is tight frame.*

If the norms of (v_j) are not all equal, then the properties (5.37) for $1 \le r \le t$ of a (t,t)–design and the corresponding equivalent conditions given by Theorem 4.1 are most naturally described in terms of *weighted* (t,t)-designs.

Definition 5.1 *Suppose that $\Phi = (\phi_j)_{j=1}^n$ are unit vectors in $\mathbb{F}^d$, and $w = (w_j)_{j=1}^n$ satisfy $w_j \ge 0$, $\sum_j w_j = 1$. Then (Φ, w) is a* **weighted** *(***spherical***)* (t,t)**–design**[2] *if*

$$\sum_{j=1}^n \sum_{k=1}^n w_j w_k |\langle \phi_j, \phi_k \rangle|^{2t} = c_t(d, \mathbb{F}). \tag{5.38}$$

Clearly, there is a 1–1 correspondence between the (t,t)–designs (v_j) and the weighted (t,t)–designs (Φ, w) given by (5.35), where ϕ_j can be any vector and $w_j = 0$ when $v_j = 0$. In this terminology, Theorem 4.1 gives:

Corollary 5.1 *(Weighted version) Let $\Phi = (\phi_j)_{j=1}^n$ be a sequence of unit vectors in $\mathbb{F}^d$, and $w = (w_j)_{j=1}^n$ be nonnegative weights, i.e., $w_j \ge 0$, $\sum_j w_j = 1$. Then*

$$\sum_{j=1}^n \sum_{k=1}^n w_j w_k |\langle \phi_j, \phi_k \rangle|^{2t} \ge c_t(d, \mathbb{F}), \tag{5.39}$$

with equality if and only if (Φ, w) is a weighted (t,t)–design, or, equivalently,

$$\int_{\mathbb{S}} p(x)\, d\sigma(x) = \sum_{j=1}^n w_j p(\phi_j), \qquad \forall p \in \Pi_{t,t}^\circ(\mathbb{S}). \tag{5.40}$$

If (Φ, w) is a weighted (t,t)–design, then it is a weighted (r,r)–design, $1 \le r \le t$.

Proof: Make the substitution (5.35) in Theorem 4.1, and observe that (c) can be written as (5.40). The last assertion follows from this and (5.36). □

Example 5.3 *A weighted (t,t)–design (Φ, w) satisfies*

$$\sum_{j=1}^n \sum_{k=1}^n w_j w_k |\langle \phi_j, \phi_k \rangle|^{2r} = c_r(d, \mathbb{F}), \qquad 1 \le r \le t,$$

which is the weighted version of (5.37).

[2] These are also known as *weighted spherical half–designs* of order t (see [KP11]).

Substituting (5.35) into Theorem 4.1 gives a weighted version of each of the equivalent conditions, e.g., condition (a) becomes

$$c_t(d,F)\|x\|^{2t} = \sum_{j=1}^{n} w_j |\langle x, \phi_j\rangle|^{2t}, \qquad \forall x \in \mathbb{F}^d,$$

or, equivalently

$$\sum_{j=1}^{n} w_j |\langle x, \phi_j\rangle|^{2t} = c_t(d,\mathbb{F}), \qquad \forall x \in \mathbb{S}. \tag{5.41}$$

A naive numerical search (see [Bra11]) for weighted spherical (t,t)–designs in $\mathbb{C}^d$ suggests that in some cases, e.g., $t=4$, $d=4$ and $t=3$, $d=3$, those with the minimal number of vectors do not have constant weights. There are few known constructions of such weighted (t,t)–designs. A very general construction is given in [RS07].

6 Weighted complex projective (t,t)–designs

The equality (4.33) defining (t,t)–designs is invariant under multiplying the vectors by unit scalars, and so (t,t)–designs can be extended to a projective setting. This has been done not only for $\mathbb{R}$ and $\mathbb{C}$, but also the quaternians $\mathbb{H}$ and the octonians $\mathbb{O}$ (see [Hog82]). We will focus on the *complex projective sphere*, as this is currently being intensively studied, particularly in quantum physics.

The **complex projective sphere** $\mathbb{C}P^{d-1}$ can be viewed variously as

- The complex sphere $\mathbb{S}(\mathbb{C}^d)$ with points z and az, $|a|=1$ identified.
- The 1–dimensional subspaces of $\mathbb{C}^d$ (the complex lines through 0).
- The rank 1 orthogonal projections on $\mathbb{C}^d$.

The polynomials on $\mathbb{S}(\mathbb{C}^d)$ which carry over to this space, i.e., those with

$$p(z) = p(az), \qquad \forall z, \quad \forall a \in \mathbb{F}, |\alpha| = 1$$

are precisely those in $\Pi^\circ_{0,0}(\mathbb{F}^d) \oplus \Pi^\circ_{1,1}(\mathbb{F}^d) \oplus \Pi^\circ_{2,2}(\mathbb{F}^d) \cdots$.

We will take the elements of $\mathbb{C}P^{d-1}$ to be rank one orthogonal projections. There is a unique unitarily invariant probability measure μ on $\mathbb{F}P^{d-1}$ induced from the area measure σ on the sphere $\mathbb{S}(\mathbb{F}^d)$, via

$$\int_{\mathbb{F}P^{d-1}} f(P)\, d\mu(P) = \int_{\mathbb{F}(\mathbb{C}^d)} f(P_x)\, d\sigma(x), \tag{6.42}$$

where $P_x = \langle \cdot, x\rangle x$ denotes the rank one orthogonal projection onto span$\{x\}$, $\|x\|=1$. The Frobenius inner product between rank one orthogonal projections is

$$\langle P_x, P_y\rangle = \operatorname{trace}(P_x P_y) = |\langle x, y\rangle|^2 \in \mathbb{R}. \tag{6.43}$$

Definition 6.1 *Suppose that* $\mathcal{P} = (P_j)_{j=1}^n$ *are rank one orthogonal projections on* $\mathbb{F}^d$*, and* $w = (w_j)_{j=1}^n$ *satisfy* $w_j \geq 0$, $\sum_j w_j = 1$. *We say* $(\mathcal{P}, w)$ *is a* (**weighted**) **projective** (t,t)–**design**[3] *if*

$$\sum_{j=1}^n \sum_{k=1}^n w_j w_k \langle P_j, P_k \rangle^t = c_t(d, \mathbb{F}).$$

The (t,t)–designs (v_j) (up to multiplication by unit scalars) are in 1–1 correspondence with the projective (t,t)–designs $(\mathcal{P}, w)$, via

$$P_j = \frac{1}{\|v_j\|^2} \langle \cdot, v_j \rangle v_j, \qquad w_j = \frac{\|v_j\|^{2t}}{\sum_\ell \|v_\ell\|^{2t}}. \tag{6.44}$$

This gives the following projective version of Theorem 4.1 (see [RS07], cf. Corollary 5.1).

Corollary 6.1 *(Projective version) Let* $\mathcal{P} = (P_j)_{j=1}^n$ *be rank one orthogonal projections in* $\mathbb{F}^d$*, and* $w = (w_j)_{j=1}^n$ *satisfy* $w_j \geq 0$, $\sum_j w_j = 1$. *Then*

$$\sum_{j=1}^n \sum_{k=1}^n w_j w_k \langle P_j, P_k \rangle^t \geq c_t(d, \mathbb{F}), \tag{6.45}$$

with equality if and only if $(\mathcal{P}, w)$ *is a projective* (t,t)*–design.*
A projective (t,t)*–design is a projective* (r,r)*–design,* $1 \leq r \leq t$.

Proof: With P_j and w_j given by (6.44), (6.43) gives

$$w_j w_k \langle P_j, P_k \rangle^t = \frac{\|f_j\|^{2t} \|f_k\|^{2t}}{(\sum_\ell \|f_\ell\|^{2t})^2} \left(\frac{|\langle f_j, f_k \rangle|^2}{\|f_j\|^2 \|f_k\|^2} \right)^t = \frac{|\langle f_j, f_k \rangle|^{2t}}{(\sum_\ell \|f_\ell\|^{2t})^2}.$$

Thus, making the substitution (6.44) in Theorem 4.1 gives (6.45), with equality for projective (t,t)–designs. The last part follows from Corollary 5.1 and (6.44). □

Other conditions equivalent to being a projective (t,t)–design can be obtained by substituting (6.44) into Theorem 4.1, e.g., by using (6.43), condition (a) becomes

$$c_t(d, \mathbb{F}) = \sum_{j=1}^n w_j \langle Q, P_j \rangle^t, \qquad \forall Q \in \mathbb{F}P^{d-1}. \tag{6.46}$$

The condition (e) gives Levenshtein's definition [Lev98] of a *weighted* t*–design*

$$\int_{\mathbb{F}P^{d-1}} \int_{\mathbb{F}P^{d-1}} g\big(\langle P, Q \rangle\big)\, d\mu(P)\, d\mu(Q) = \sum_{j=1}^n \sum_{k=1}^n w_j w_k\, g\big(\langle P_j, P_k \rangle\big), \quad \forall g \in \Pi_t(\mathbb{R}). \tag{6.47}$$

The condition (d) becomes

$$\int_{\mathbb{F}P^{d-1}} P^{\otimes t}\, d\mu(P) = \sum_{j=1}^n w_j\, P_j^{\otimes t}. \tag{6.48}$$

[3]Other terms such as **weighted** or **quantum** t–**design** are also commonly used.

7 SICs and MUBs

We now consider *SICs* and *MUBs*, which are of interest in quantum information theory (where they are viewed as projections giving quantum measurements).

Recall (6.43), that the rank one orthogonal projections P_j, P_k corresponding to unit vectors $v_j, v_k \in \mathbb{C}^d$ satisfy
$$\langle P_j, P_k \rangle = |\langle v_j, v_k \rangle|^2.$$
A sequence of equal norm vectors (v_j) is said to be **equiangular** if they have equal cross–correlation, for some constant $C \geq 0$, one has

$$|\langle v_j, v_k \rangle| = C, \qquad j \neq k.$$

An orthonormal basis is an equiangular tight frame, with $C = 0$.

Definition 7.1 *A sequence of d^2 vectors (v_j) in $\mathbb{C}^d$, or the corresponding rank one orthogonal projections (P_j), is said to be a* **SIC** *(***symmetric informationally complete positive operator valued measure***) for $\mathbb{C}^d$ if*

$$\langle P_j, P_k \rangle = |\langle v_j, v_k \rangle|^2 = \frac{1}{d+1}, \qquad j \neq k.$$

It follows that a SIC (v_j) is a $(2,2)$–design by the calculation

$$\sum_j \sum_k |\langle v_j, v_k \rangle|^4 = \big((d^2)^2 - d^2\big)\Big(\frac{1}{d+1}\Big)^2 + d^2 = \frac{2d^3}{d+1} = c_2(2,\mathbb{C})\Big(\sum_j \|v_j\|^4\Big)^2.$$

The existence of a SIC for every dimension d is known as *Zauner's conjecture* (or the *SIC problem*) [Zau10]. There is strong evidence for this conjecture [SG10], which has been established for certain values of d (see Example 7.1).

Since a SIC (v_j) is a $(2,2)$–design, it follows from Theorem 2.1 that $(v_j \otimes v_j)$ is an equiangular tight frame for $\mathrm{Sym}^2(\mathbb{C}^d)$. Thus we have:

Corollary 7.1 *If there is a SIC for $\mathbb{C}^d$, then there are equiangular tight frames of d^2 vectors for $\mathbb{C}^{\frac{1}{2}d(d+1)}$ and for $\mathbb{C}^{\frac{1}{2}d(d-1)}$.*

Proof: Let (v_j) be a SIC. Since (v_j) is a $(2,2)$–design, it follows from Theorem 2.1 that $(v_j^{\otimes 2}) = (v_j \otimes v_j)$ is a tight frame for the space $\mathrm{Sym}^2(\mathbb{C}^d)$ of dimension $\frac{1}{2}d(d+1)$, which has the property

$$|\langle v_j^{\otimes 2}, v_k^{\otimes 2} \rangle| = |\langle v_j, v_k \rangle^2| = \frac{1}{d+1}, \qquad j \neq k.$$

The complement of an equiangular tight frame of n vectors for $\mathbb{C}^d$ is an equiangular tight frame of n vectors for $\mathbb{C}^{n-d}$, and so we also obtain the other equiangular tight frame. □

Complex equiangular tight frames with the above parameters can be constructed for $d \geq 2$ from the $d \times d$ Fourier matrix [BE10], and for $d = 2^a p^b$, p prime and $0 \leq a \leq b$, from Butson–type complex Hadamard matrices of order d with p–th root entries [Szö13]. Thus Corollary 7.1 can't be used to prove Zauner's conjecture is false. On the other hand it does provide some new sets of equiangular lines:

Example 7.1 *The complex equiangular lines (tight frames) obtained from the known SICs by Corollary 7.1 for*

$$d = 4, \ldots, 20, 24, 35, 48 \quad \textit{(published)}, \qquad d = 21, 22, 31, 37, 43 \quad \textit{(unpublished)}$$

are new examples of equiangular lines since their m–products (see [CW16]) are not in a cyclotomic field, as is the case for the known constructions mentioned above. This suggests that in these cases the algebraic variety of equiangular tight frames might have a nonzero dimension (if its dimension was zero, then its study would shed light on Zauner's conjecture).

Definition 7.2 *Orthonormal basis $\mathcal{B}_1, \ldots, \mathcal{B}_m$ for $\mathbb{C}^d$ are* **mutually unbiased** *if*

$$|\langle v, w \rangle|^2 = \frac{1}{d}, \qquad v \in \mathcal{B}_j, \quad w \in \mathcal{B}_k, \quad j \neq k.$$

We call $\mathcal{B}_1, \ldots, \mathcal{B}_m$ a sequence of m **MUBs** *(***mutually unbiased bases***) for $\mathbb{C}^d$.*

The maximal number of MUBs for $\mathbb{C}^d$, which we denote by $\mathcal{M}_d$, is bounded by $d+1$. For d a prime power, there are various constructions giving $d + 1$ MUBs. On the other hand, for $d = 6$, it is only known that $3 \leq \mathcal{M}_6 \leq 7$, and there is ongoing interest in this *MUB problem* (see, e.g., [WF89], [GR09], [MB15], [ABD15]).

A set of $d+1$ MUBs (v_j) for $\mathbb{C}^d$ (known as a *maximal* set of MUBs) is a $(2, 2)$–design by the calculation

$$\sum_j \sum_k |\langle v_j, v_k \rangle|^4 = d(d+1) + d^3(d+1)\Big(\frac{1}{d}\Big)^2 + d(d^2-1)\cdot 0 = 2d(d+1) = c_2(2, \mathbb{C})\Big(\sum_j \|v_j\|^4\Big)^2.$$

The analogue of Corollary 7.1 is the following.

Corollary 7.2 *If there is a maximal set of MUBs for $\mathbb{C}^d$, then there is a unit norm tight frame (u_j) of $d(d+1)$ vectors for $\mathbb{C}^{\frac{1}{2}d(d+1)}$ which has the property that the cross correlation $|\langle u_j, u_k \rangle|$, $j \neq k$, is zero for $\frac{1}{2}d(d^2-1)$ pairs of vectors, and equal to $\frac{1}{d}$ for the remaining $\frac{1}{2}d^3(d+1)$ pairs of vectors.*

Proof: Let (v_j) be the $d(d+1)$ unit vectors given by $d+1$ MUBs for $\mathbb{C}^d$, which is a $(2, 2)$–design. By Theorem 2.1, $(u_j) := (v_j^{\otimes 2}) = (v_j \otimes v_j)$ is a tight frame for the space $\mathrm{Sym}^2(\mathbb{C}^d)$, which has dimension $\frac{1}{2}d(d+1)$. If the vectors v_j, v_k come from the same orthonormal basis, then u_j, u_k are orthogonal, otherwise

$$|\langle u_j, u_k \rangle| = |\langle v_j^{\otimes 2}, v_k^{\otimes 2} \rangle| = |\langle v_j, v_k \rangle^2| = \frac{1}{d}, \qquad j \neq k.$$

The complement of (u_j) is also an equiangular tight frame for $\mathbb{C}^{\frac{1}{2}d(d+1)}$. □

Example 7.2 *A maximal set of MUBs (v_j) for $\mathbb{C}^2$ is given by*

$$\{e_1, e_2\}, \quad \{\tfrac{1}{\sqrt{2}}(e_1+e_2), \tfrac{1}{\sqrt{2}}(e_1-e_2)\}, \quad \{\tfrac{1}{\sqrt{2}}(e_1+ie_2), \tfrac{1}{\sqrt{2}}(e_1-ie_2)\}.$$

The corresponding tight frame of six vectors $[u_1, \ldots, u_6]$ for $\mathbb{C}^3$ is given by

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2}i & -\frac{1}{2}i \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2}i & -\frac{1}{2}i \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix},$$

where the Kronecker products above lie in the 3–dimensional space $\{x \in \mathbb{C}^4 : x_2 = x_3\}$. This set of MUBs is also a $(3,3)$–design, and so $(v_j^{\otimes 3})$ is a tight frame of six vectors for the 4–dimensional space $\mathrm{Sym}^3(\mathbb{C}^2)$.

References

[ABD15] D. M. Appleby, Ingemar Bengtsson, and Hoan Bui Dang. Galois unitaries, mutually unbiased bases, and MUB-balanced states. *Quantum Inf. Comput.*, 15(15-16):1261–1294, 2015.

[BB09] Eiichi Bannai and Etsuko Bannai. A survey on spherical designs and algebraic combinatorics on spheres. *European J. Combin.*, 30(6):1392–1425, 2009.

[BE10] Bernhard G. Bodmann and Helen J. Elwood. Complex equiangular Parseval frames and Seidel matrices containing pth roots of unity. *Proc. Amer. Math. Soc.*, 138(12):4387–4404, 2010.

[BF03] John J. Benedetto and Matthew Fickus. Finite normalized tight frames. *Adv. Comput. Math.*, 18(2-4):357–385, 2003. Frames.

[Bra11] J. Bramwell. On the existence of spherical (t,t)–designs. Honours Project, University of Auckland, 1 2011.

[Chr03] Ole Christensen. *An introduction to frames and Riesz bases.* Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, 2003.

[CK13] Peter G. Casazza and Gitta Kutyniok, editors. *Finite frames.* Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, New York, 2013. Theory and applications.

[CKL08] Peter G. Casazza, Gitta Kutyniok, and Shidong Li. Fusion frames and distributed processing. *Appl. Comput. Harmon. Anal.*, 25(1):114–132, 2008.

[CW16] Tuan-Yow Chien and Shayne Waldron. A characterization of projective unitary equivalence of finite frames and applications. *SIAM J. Discrete Math.*, 30(2):976–994, 2016.

[DGS77] P. Delsarte, J. M. Goethals, and J. J. Seidel. Spherical codes and designs. *Geometriae Dedicata*, 6(3):363–388, 1977.

[DHC12] S. Datta, S. Howard, and D. Cochran. Geometry of the Welch bounds. *Linear Algebra Appl.*, 437(10):2455–2470, 2012.

[DS52] R. J. Duffin and A. C. Schaeffer. A class of nonharmonic Fourier series. *Trans. Amer. Math. Soc.*, 72:341–366, 1952.

[EF02] Yonina C. Eldar and G. David Forney, Jr. Optimal tight frames and quantum measurement. *IEEE Trans. Inform. Theory*, 48(3):599–610, 2002.

[GKK01] Vivek K. Goyal, Jelena Kovačević, and Jonathan A. Kelner. Quantized frame expansions with erasures. *Appl. Comput. Harmon. Anal.*, 10(3):203–233, 2001.

[GR09] Chris Godsil and Aidan Roy. Equiangular lines, mutually unbiased bases, and spin models. *European J. Combin.*, 30(1):246–262, 2009.

[Grö01] Karlheinz Gröchenig. *Foundations of time-frequency analysis*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, 2001.

[Hog82] S. G. Hoggar. t-designs in projective spaces. *European J. Combin.*, 3(3):233–254, 1982.

[KP11] N. O. Kotelina and A. B. Pevnyi. The Venkov inequality with weights and weighted spherical half-designs. *J. Math. Sci. (N. Y.)*, 173(6):674–682, 2011. Problems in mathematical analysis. No. 55.

[Lev98] Vladimir Levenshtein. On designs in compact metric spaces and a universal bound on their size. *Discrete Math.*, 192(1-3):251–271, 1998. Discrete metric spaces (Villeurbanne, 1996).

[LL14] Junhong Lin and Song Li. Sparse recovery with coherent tight frames via analysis Dantzig selector and analysis LASSO. *Appl. Comput. Harmon. Anal.*, 37(1):126–139, 2014.

[MB15] Andrew S. Maxwell and Stephen Brierley. On properties of Karlsson Hadamards and sets of mutually unbiased bases in dimension six. *Linear Algebra Appl.*, 466:296–306, 2015.

[MM93] James L. Massey and Thomas Mittelholzer. Welch's bound and sequence sets for code-division multiple-access systems. In *Sequences, II (Positano, 1991)*, pages 63–78. Springer, New York, 1993.

[RBKSC04] Joseph M. Renes, Robin Blume-Kohout, A. J. Scott, and Carlton M. Caves. Symmetric informationally complete quantum measurements. *J. Math. Phys.*, 45(6):2171–2180, 2004.

[RS07] Aidan Roy and A. J. Scott. Weighted complex projective 2-designs from bases: optimal state determination by orthogonal measurements. *J. Math. Phys.*, 48(7):072110, 24, 2007.

[RS14] Aidan Roy and Sho Suda. Complex spherical designs and codes. *J. Combin. Des.*, 22(3):105–148, 2014.

[Sei01] J. J. Seidel. Definitions for spherical designs. *J. Statist. Plann. Inference*, 95(1-2):307–313, 2001. Special issue on design combinatorics: in honor of S. S. Shrikhande.

[SG10] A. J. Scott and M. Grassl. Symmetric informationally complete positive-operator-valued measures: a new computer study. *J. Math. Phys.*, 51(4):042203, 16, 2010.

[SW09] Ian H. Sloan and Robert S. Womersley. A variational characterisation of spherical designs. *J. Approx. Theory*, 159(2):308–318, 2009.

[SZ84] P. D. Seymour and Thomas Zaslavsky. Averaging sets: a generalization of mean values and spherical designs. *Adv. in Math.*, 52(3):213–240, 1984.

[Szö13] Ferenc Szöllősi. Complex Hadamard matrices and equiangular tight frames. *Linear Algebra Appl.*, 438(4):1962–1967, 2013.

[VW05] Richard Vale and Shayne Waldron. Tight frames and their symmetries. *Constr. Approx.*, 21(1):83–112, 2005.

[Wal03] Shayne Waldron. Generalized Welch bound equality sequences are tight frames. *IEEE Trans. Inform. Theory*, 49(9):2307–2309, 2003.

[Wal16] Shayne Waldron. *An introduction to finite tight frames.* Birkhäuser/Springer, New York, 2016.

[Wel74] L. R. Welch. Lower bounds on the maximum cross correlation of signals. *IEEE Trans. Inform. Theory*, 20:397–399, 1974.

[WF89] William K. Wootters and Brian D. Fields. Optimal state-determination by mutually unbiased measurements. *Ann. Physics*, 191(2):363–381, 1989.

[You01] Robert M. Young. *An introduction to nonharmonic Fourier series.* Academic Press, Inc., San Diego, CA, first edition, 2001.

[Zau10] Gerhard Zauner. *Quantum Designs: Foundations of a non-commutative Design Theory.* PhD thesis, University of Vienna, University of Vienna, 1 2010. English translation of 1999 Doctorial thesis including a new preface.