

A characterisation of projective unitary equivalence of finite frames and applications

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February 17, 2015

Abstract

Many applications of finite tight frames, e.g., the use of SICs and MUBs in quantum information theory and harmonic frames for the analysis of signals subject to erasures, depend only on the vectors up to projective unitary equivalence. It is well known that two finite sequences of vectors in inner product spaces are unitarily equivalent if and only if their respective inner products (Gramian matrices) are equal. Here we present a corresponding result for the projective unitary equivalence of two sequences of vectors (lines) in inner product spaces, i.e., that a finite number of (Bargmann) projective (unitary) invariants are equal. This is based on an algorithm to recover the sequence of vectors (up to projective unitary equivalence) from a small subset of these projective invariants. We apply this characterisation to SICs, MUBs and harmonic frames. We also extend our results to the projective similarity of vectors.

Key Words: Projective unitary equivalence, Gramian matrix (Gramian), harmonic frame, equiangular tight frame, SIC-POVM (symmetric informationally complete positive operator valued measure), MUBs (mutually orthogonal bases), triple products, Bargmann invariants, frame graph, cycle space, chordal graph, projective symmetry group

AMS (MOS) Subject Classifications: primary 05C50, 14N05, 14N20, 15A83,
secondary 15A04, 42C15, 81P15, 81P45,

1 Introduction

1.1 Motivation

Finite frames provide redundant and stable expansions, which have numerous applications [1]. These include quantum measurements and codes using SICs [2] [3], [4] and MUBs [5], [6], [7], and signal analysis using harmonic frames which are robust to erasures [8], [9], [10]. Many applications of finite frames, such as those above, depend only on the vectors up to projective unitary equivalence. This point of view naturally leads to the notion of a *fusion frame* (where the vectors are replaced by 1-dimensional orthogonal projections, and more generally d -dimensional orthogonal projections). Such applications motivate our study of when sequences of vectors are projectively unitarily equivalent. In addition, the study of lines in \mathbb{R}^d and \mathbb{C}^d , in particular, equiangular lines [11], is effectively the study of configurations of unit vectors up to projective unitary equivalence.

1.2 Key definitions

Finite sequences of vectors $\Phi = (v_j)$ and $\Psi = (w_j)$ in (real or complex) inner product spaces \mathcal{H}_1 and \mathcal{H}_2 are **unitarily equivalent** if there is a unitary map $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, such that

$$w_j = Uv_j, \quad \forall j,$$

and **projectively unitarily equivalent** if there is a unitary map U and unit scalars c_j , such that

$$w_j = c_j Uv_j, \quad \forall j,$$

equivalently

$$w_j w_j^* = U(v_j v_j^*) U^*, \quad \forall j.$$

A finite spanning sequence of vectors for an inner product space is also called a **finite frame**.

The **Gramian (matrix)** of $\Phi = (v_j)_{j=1}^n$ is

$$\text{Gram}(\Phi) = [\langle v_k, v_j \rangle]_{j,k=1}^n.$$

We take our inner products to be linear in the first variable. It is well known (cf. [12]) that

- Φ and Ψ are unitarily equivalent if and only if

$$\text{Gram}(\Phi) = \text{Gram}(\Psi). \tag{1.1}$$

- Φ and Ψ are projectively unitarily equivalent if and only if

$$\text{Gram}(\Psi) = C \text{Gram}(\Phi) C^*, \tag{1.2}$$

where C is the diagonal matrix with diagonal entries c_j .

Clearly, (1.1) can be used to verify unitary equivalence, as can (1.2) to verify projective unitary equivalence for real inner product spaces (where $c_j \in \{-1, 1\}$, cf. Example 2.3). For complex inner product spaces, we have no knowledge of c_j , other than $|c_j| = 1$, and so (1.2), i.e.,

$$\langle w_j, w_k \rangle = c_j \bar{c}_k \langle v_j, v_k \rangle, \quad \forall j, k,$$

does not provide a practical method for verifying projective unitary equivalence. To our knowledge, projective unitary equivalence has only been calculated in a few cases, which have a very specific structure. These are $\mathcal{H} = \mathbb{R}^d$ by considering all possible $c_j = \pm 1$ in (1.2) or by using two-graphs in the case of equiangular lines [13], for a specific construction of MUBs via symplectic spreads and Hadamard matrices [14], [15], [16], for SICs [17], and for the equal-norm tight frames of n vectors in \mathbb{C}^2 robust to $n - 2$ erasures [10]. Here we present a general method for determining whether **any** two finite sequence of vectors are projectively unitarily equivalent.

Following [18], we define the m -**vertex Bargmann invariants** or m -**products** of a sequence of n vectors $\Phi = (v_j)$ to be

$$\Delta(v_{j_1}, v_{j_2}, \dots, v_{j_m}) := \langle v_{j_1}, v_{j_2} \rangle \langle v_{j_2}, v_{j_3} \rangle \cdots \langle v_{j_m}, v_{j_1} \rangle, \quad 1 \leq j_1, \dots, j_m \leq n. \quad (1.3)$$

In particular, (cf. [17]), we define the **triple products** to be

$$T_{jkl} := \Delta(v_j, v_k, v_l) = \langle v_j, v_k \rangle \langle v_k, v_l \rangle \langle v_l, v_j \rangle. \quad (1.4)$$

We observe that the m -products are *projective unitary invariants*, e.g., for $m = 3$

$$\begin{aligned} \Delta(c_j U v_j, c_k U v_k, c_l U v_l) &= \langle c_j U v_j, c_k U v_k \rangle \langle c_k U v_k, c_l U v_l \rangle \langle c_l U v_l, c_j U v_j \rangle \\ &= c_j \bar{c}_k \langle U v_j, U v_k \rangle c_k \bar{c}_l \langle U v_k, U v_l \rangle c_l \bar{c}_j \langle U v_l, U v_j \rangle \\ &= \langle v_j, v_k \rangle \langle v_k, v_l \rangle \langle v_l, v_j \rangle \\ &= \Delta(v_j, v_k, v_l). \end{aligned} \quad (1.5)$$

We define the **frame graph** (cf. [19]) of a sequence of vectors (v_j) to be the graph with vertices $\{v_j\}$ (or the indices j themselves) and

$$\text{an edge between } v_j \text{ and } v_k, j \neq k \iff \langle v_j, v_k \rangle \neq 0.$$

Clearly, projectively unitarily equivalent frames have the same frame graph.

1.3 Outline

We will show (Theorem 3.2) that a sequence of vectors (v_j) is determined up to projective unitary equivalence by all its m -products. Our proof relies on the fact that certain small subsets of the m -products are sufficient. These depend on which of the m -products are nonzero, which is conveniently encapsulated by the *frame graph*.

In Section 2, generalising the results of [17] for SICs, we show that if the common frame graph of $\Phi = (v_j)$ and $\Psi = (w_j)$ is complete, then they are projectively unitarily equivalent

if and only if their 3-products (triple products) are equal (Theorem 2.2). Later we will show this condition extends (Theorem 4.5), e.g., to the case when the frame graph is chordal. We apply this result to sequences of equiangular lines (including SICs), then give an example to show that the 3-products do not determine projective unitary equivalence in general (Example 2.5).

In Sections 3 and 4, we show that projective unitary equivalence is characterised the m -products (Theorem 3.2). This leads to methods for calculating the projective symmetry group [20]. We show that is sufficient to consider only a small subset of these projective invariants, which can be determined from the frame graph (Theorem 4.3). We give an algorithm for constructing all sequences with given m -products, and consider the classification of MUBs (Theorem 4.7).

In Section 5, we apply our results to the classification of sequences of vectors up to (projective) similarity (Theorem 5.2).

Finally, in Section 6, we consider the classification of harmonic frames up to projective unitary equivalence (Theorem 6.2).

2 Complete frame graphs

A sequence of $n \geq d$ unit vectors (v_j) in \mathbb{C}^d is **equiangular** if for some $C \geq 0$

$$|\langle v_j, v_k \rangle| = C, \quad j \neq k.$$

For $C > 0$, such a sequence has a complete frame graph (no zero inner products), as does a generic sequence of vectors. In [17], it was shown that d^2 equiangular vectors in \mathbb{C}^d are characterised up to projective unitary equivalence by their triple products (3-products). Here we modify the argument to when the frame graph is complete. We then show, by an example, that this result does not extend to a general sequence of vectors.

The **angles** of a sequence of vectors $\Phi = (v_j)$ are the $\theta_{jk} \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ defined by

$$\langle v_j, v_k \rangle = |\langle v_j, v_k \rangle| e^{i\theta_{jk}}, \quad \langle v_j, v_k \rangle \neq 0.$$

Since $\langle v_j, v_k \rangle = \overline{\langle v_k, v_j \rangle}$, these satisfy

$$\theta_{jk} = -\theta_{kj}.$$

A sequence of vectors may have few angles, e.g., an orthogonal basis has no angles.

Lemma 2.1 *Let $\Phi = (v_j)$ and $\Psi = (w_j)$ be finite sequences of vectors in Hilbert spaces, with angles θ_{jk} and θ'_{jk} . Then Φ and Ψ are projectively unitarily equivalent if and only if*

1. *Their Gramians have entries with equal moduli, i.e.,*

$$|\langle w_j, w_k \rangle| = |\langle v_j, v_k \rangle|, \quad \forall j, k.$$

2. Their angles are “gauge equivalent”, i.e., there exist $\phi_j \in \mathbb{T}$ with

$$\theta'_{jk} = \theta_{jk} + \phi_j - \phi_k, \quad \forall j, k.$$

Proof: First suppose that Φ and Ψ are projectively unitarily equivalent, i.e., $w_j = c_j U v_j$, where U is unitary and $c_j = e^{i\phi_j}$. Then

$$\begin{aligned} e^{i\theta'_{jk}} |\langle w_j, w_k \rangle| &= \langle w_j, w_k \rangle = \langle c_j U v_j, c_k U v_k \rangle = c_j \overline{c_k} \langle U v_j, U v_k \rangle = e^{i(\phi_j - \phi_k)} \langle v_j, v_k \rangle \\ &= e^{i(\phi_j - \phi_k)} e^{i\theta_{jk}} |\langle v_j, v_k \rangle| \end{aligned}$$

By equating the moduli and then the arguments we obtain 1 and 2.

Conversely, suppose that 1 and 2 hold. Let $\tilde{v}_j := e^{i\phi_j} v_j$. Then

$$\begin{aligned} \langle \tilde{v}_j, \tilde{v}_k \rangle &= \langle e^{i\phi_j} v_j, e^{i\phi_k} v_k \rangle = e^{i(\phi_j - \phi_k)} \langle v_j, v_k \rangle = e^{i(\phi_j - \phi_k)} e^{i\theta_{jk}} |\langle v_j, v_k \rangle| \\ &= e^{i\theta'_{jk}} |\langle w_j, w_k \rangle| = \langle w_j, w_k \rangle. \end{aligned}$$

Thus (w_j) is unitarily equivalent to (\tilde{v}_j) , which is projectively unitarily equivalent to (v_j) , and so Ψ and Φ are projectively unitarily equivalent. \square

We observe that $|\langle v_j, v_k \rangle|$ can be calculated from the triple products of (1.4), since

$$T_{jjj} = \langle v_j, v_j \rangle^3 = \|v_j\|^6, \quad T_{jjk} = \langle v_j, v_j \rangle |\langle v_j, v_k \rangle|^2 = T_{jjj}^{\frac{1}{3}} |\langle v_j, v_k \rangle|^2. \quad (2.6)$$

Theorem 2.2 (Characterisation) *Let $\Phi = (v_j)_{j \in J}$ and $\Psi = (w_j)_{j \in J}$ be finite sequences of vectors in Hilbert spaces. Then*

1. Φ and Ψ are unitarily equivalent if and only if their Gramians are equal, i.e.,

$$\langle v_j, v_k \rangle = \langle w_j, w_k \rangle, \quad \forall j, k.$$

2. If the frame graphs of Φ and Ψ are complete, then they are projectively unitarily equivalent if and only if their triple products are equal, i.e.,

$$\langle v_j, v_k \rangle \langle v_k, v_\ell \rangle \langle v_\ell, v_j \rangle = \langle w_j, w_k \rangle \langle w_k, w_\ell \rangle \langle w_\ell, w_j \rangle, \quad \forall j, k, \ell.$$

Proof: The condition for unitary equivalence is well known. It is included in the theorem only for the purpose of comparison. We now prove 2.

First suppose that Φ and Ψ are projectively unitarily equivalent, i.e., $w_j = c_j U v_j$. Then by (1.5) their triple products are equal.

Conversely, suppose that Φ and Ψ have the same triple products, and their common frame graph is complete, i.e., all the triple products are nonzero.

It follows from (2.6) that their Gramians have entries with equal moduli, i.e.,

$$|\langle v_j, v_k \rangle| = |\langle w_j, w_k \rangle|, \quad \forall j, k.$$

Let θ_{jk} and θ'_{jk} be the angles of Φ and Ψ . Since the triple products have the polar form

$$T_{jkl} = \langle v_j, v_k \rangle \langle v_k, v_\ell \rangle \langle v_\ell, v_j \rangle = e^{i(\theta_{jk} + \theta_{k\ell} + \theta_{\ell j})} |\langle v_j, v_k \rangle \langle v_k, v_\ell \rangle \langle v_\ell, v_j \rangle|,$$

we obtain

$$\theta_{jk} + \theta_{k\ell} + \theta_{\ell j} = \theta'_{jk} + \theta'_{k\ell} + \theta'_{\ell j}.$$

Fix ℓ , and rearrange this, using $\theta_{k\ell} = -\theta_{\ell k}$ and $\theta'_{k\ell} = -\theta'_{\ell k}$, to get

$$\theta'_{jk} = \theta_{jk} + (\theta_{\ell j} - \theta'_{\ell j}) + (\theta_{k\ell} - \theta'_{k\ell}) = \theta_{jk} + (\theta_{\ell j} - \theta'_{\ell j}) - (\theta_{\ell k} - \theta'_{\ell k}) = \theta_{jk} + \phi_j - \phi_k,$$

where $\phi_j := \theta_{\ell j} - \theta'_{\ell j}$, i.e., the angles of Φ and Ψ are gauge equivalent. Since the conditions of Lemma 2.1 hold, it follows that Φ and Ψ are projectively unitarily equivalent. \square

The real case is closely connected with the theory of *two-graphs* (cf. [13]) as follows.

Example 2.3 (*Equiangular lines in \mathbb{R}^d*). Suppose that $\Phi = (v_j)$ is a sequence of $n > d$ equiangular unit vectors (lines) in \mathbb{R}^d , i.e., there is an $\alpha > 0$ with

$$\langle v_j, v_k \rangle = \pm\alpha, \quad j \neq k.$$

Then the Gramian matrix has the form

$$G_\Phi = \text{Gram}(\Phi) = I + \alpha S_\Phi,$$

where $S = S_\Phi$ is a **Seidel matrix**, i.e., S is symmetric, with zero diagonal, and off diagonal entries ± 1 . Moreover, each Seidel matrix is associated with a sequence of equiangular lines. Each Seidel matrix S is in turn associated with the graph $\text{gr}(S)$ which has an edge between $j \neq k$ if and only if $S_{jk} = -1$. Let \mathcal{C} be the diagonal matrices with diagonal entries ± 1 . Then the projective unitary equivalence class of Φ is uniquely determined by all the possible Gramian matrices of its members, i.e.,

$$\mathcal{G} := \{CG_\Phi C^* : C \in \mathcal{C}\},$$

and hence all the possible Seidel matrices

$$\mathcal{S} := \{CG_\Phi C^* : C \in \mathcal{C}\},$$

and in turn the corresponding graphs $\text{gr}(\mathcal{S})$. The set of graphs $\text{gr}(\mathcal{S})$ is called the **switching class** of $\text{gr}(S_\Phi)$, or a **two-graph**. Since the frame graph of Φ is complete, Theorem 2.2 gives that projective unitary equivalence class of Φ (equivalently \mathcal{G} , \mathcal{S} or $\text{gr}(\mathcal{S})$) is in 1–1 correspondence with the triple products of Φ . It suffices to consider only those triple products with distinct indices, since if an index is repeated twice or thrice, then by (2.6) the triple products are depend only on α . In this way, the two-graph is in 1–1 correspondence with the triple products

$$\{T_{jkl} = \pm\alpha^3 : j, k, \ell \text{ are distinct}\}.$$

Since these triple products take only two values, which are independent of the ordering of the indices, they can be described by giving the collection of the subsets $\{j, k, \ell\}$ where they take one of these values. This association leads to the equivalent definition of a two-graph as a set of (unordered) triples chosen from a finite vertex set X , such that every unordered quadruple from X contains an even number of triples of the two-graph.

Example 2.4 (Equiangular lines in \mathbb{C}^d) If Φ is a sequence of n equiangular unit vectors (lines) in \mathbb{C}^d , with $C > 0$, then up to projective unitary equivalence Φ is determined by its triple products. This result was given in [17] for the special case $n = d^2$. Such a configuration has $C = \frac{1}{\sqrt{d+1}}$, and is known as a **SIC** or **SIC-POVM** (symmetric informationally complete positive operator valued measure).

We now give an example to show that projective unitary equivalence is not always characterised by the triple products if the frame graph is not complete. We observe that the m -products are closed under complex conjugation, i.e.,

$$\overline{\Delta(v_{j_1}, v_{j_2}, \dots, v_{j_m})} = \Delta(v_{j_m}, \dots, v_{j_2}, v_{j_1}). \quad (2.7)$$

Example 2.5 (n -cycle) Let (e_j) be the standard basis vectors in \mathbb{C}^n . Fix $|z| = 1$, and let

$$v_j := \begin{cases} e_j + e_{j+1}, & 1 \leq j < n, \\ e_n + ze_1, & j = n. \end{cases}$$

Then the frame graph of (v_j) is the n -cycle (v_1, \dots, v_n) , and so the only nonzero m -products for distinct vectors are

$$\Delta(v_j) = \|v_j\|^2 = 2, \quad 1 \leq j \leq n, \quad (2.8)$$

$$\Delta(v_j, v_{j+1}) = |\langle v_j, v_{j+1} \rangle|^2 = 1, \quad 1 \leq j < n, \quad (2.9)$$

$$\Delta(v_1, v_2, \dots, v_n) = z, \quad (2.10)$$

and their complex conjugates. Therefore different choices of z give projectively inequivalent frames. Thus, for $n > 3$, the vectors (v_j) are not defined up to projective unitary equivalence by their triple products.

3 Characterisation of projective unitary equivalence

We now show that a sequence of n vectors is determined up to projective unitary equivalence by its m -products for $1 \leq m \leq n$. This is done by constructing a sequence of vectors (w_j) with given m -products $\Delta(v_{j_1}, \dots, v_{j_m})$, which amounts to finding all the possible Gramian matrices $G = [\langle w_k, w_j \rangle]$ with these m -products, because of the following.

Remark 3.1 Given a Gramian matrix G , there are many ways to construct a sequence of vectors $(v_j)_{j=1}^n$ with $G = [\langle v_k, v_j \rangle]$, i.e., $G = V^*V$ where $V = [v_1, \dots, v_n]$. For example, since

the Gramian matrix G of a sequence of n vectors which span a vector space of dimension d is positive semidefinite of rank d , it is unitarily diagonalisable

$$G = U^* \Lambda U, \quad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \quad \lambda_1, \dots, \lambda_d > 0, \quad \lambda_{d+1}, \dots, \lambda_n = 0,$$

and so we may take $V = [v_1, \dots, v_n] = \Lambda^{\frac{1}{2}} U$. This gives vectors (v_j) in \mathbb{C}^n which are zero in the last $n - d$ components, and so can be identified with vectors in \mathbb{C}^d . Similarly, one could take a Cholesky decomposition $G = V^* V$. In the special case when G is an orthogonal projection matrix P , and one can take $V = P$, since $G = P = P^* P$.

The diagonal entries of the Gramian matrices $G = [\langle w_k, w_j \rangle]$ are given by the 1-products, and the moduli of its off diagonal entries by the 2-products with distinct arguments, i.e.,

$$\Delta(v_j) = \|v_j\|^2, \quad \Delta(v_j, v_k) = |\langle v_j, v_k \rangle|^2, \quad j \neq k. \quad (3.11)$$

It therefore remains to choose arguments for the nonzero off diagonal entries of G , which are consistent with the m -products for $m \geq 3$. By choosing C in (1.2), some of these can be taken to be arbitrary. Once this is done to the full extent (spanning tree argument), we show the remaining arguments are then given by the m -products (completing cycles).

Theorem 3.2 (*Characterisation*) *Two sequences (v_j) and (w_j) of n vectors are projectively unitarily equivalent if and only if their m -products are equal, i.e.,*

$$\Delta(v_{j_1}, v_{j_2}, \dots, v_{j_m}) = \Delta(w_{j_1}, w_{j_2}, \dots, w_{j_m}), \quad 1 \leq j_1, \dots, j_m \leq n, \quad 1 \leq m \leq n.$$

Proof: It suffices to find a Gramian matrix

$$G = [\langle w_k, w_j \rangle] = C \text{Gram}(\Phi) C^*$$

by using only the m -products of $\Phi = (v_j)$. By (3.11), we know the modulus of each entry of G , and in particular the frame graph of Φ . We therefore need only determine the arguments of the (nonzero) inner products, which correspond to edges of the frame graph. This we do on each connected component Γ of the frame graph of Φ .

Spanning tree argument. Find a spanning tree \mathcal{T} of Γ with root vertex r . By working out from the root r , we can multiply the vertices $v \in \Gamma \setminus \{r\}$ by unit scalars so that the arguments of the inner products corresponding to the edges of Γ take arbitrarily assigned values.

Completing cycles. The only entries of the Gramian matrix G which are not yet defined are those given by the edges of Γ which are not in \mathcal{T} . Since \mathcal{T} is a spanning tree, adding each such edge to \mathcal{T} gives an m -cycle. The corresponding nonzero m -product has all inner products already determined, except the one corresponding to the added edge, which is therefore uniquely determined by the m -product. \square

We now illustrate Theorem 3.2, by constructing all the possible Gramian matrices G for a sequence of vectors (w_j) which is projectively unitarily equivalent to a given sequence $\Phi = (v_j)$, by using only the m -products of Φ .

Example 3.3 Let $\Phi = (e_j)$ be an orthonormal basis for \mathbb{C}^3 , which has Gramian matrix

$$\text{Gram}(\Phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here the frame graph is totally disconnected (see Fig. 1), and so each possible G is determined by the 1-products and 2-products using (3.11), i.e.,

$$\langle w_j, w_j \rangle = \langle v_j, v_j \rangle = 1, \quad |\langle w_k, w_j \rangle|^2 = |\langle v_k, v_j \rangle|^2 = 0, \quad j \neq k \implies \langle w_k, w_j \rangle = \delta_{jk}.$$

Thus there is a unique Gramian matrix G corresponding to the 1-products and 2-products. Alternatively, by (1.2), one has that all G are given by $C^* \text{Gram}(\Phi)C = \text{Gram}(\Phi)$.

Now we give an example where the spanning tree and cycle completing arguments are not trivial.

Example 3.4 Let $\Phi = (v_j)$ be three equally spaced unit vectors in \mathbb{R}^2 , viewed as vectors in \mathbb{C}^2 . These have Gramian matrix

$$\text{Gram}(\Phi) = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix},$$

and the frame graph is complete (see Fig. 1). A spanning tree with root v_1 is given by the path v_1, v_2, v_3 , working out from the root $v_1 = w_1$, we may scale v_2 then v_3 to $w_j = c_j v_j$, so that the arguments of $\langle w_1, w_2 \rangle$ and $\langle w_2, w_3 \rangle$ are arbitrary, say a and b , $|a| = |b| = 1$. The only inner product which is not yet determined is $\langle w_1, w_3 \rangle$, which is given by completing the 3-cycle v_1, v_2, v_3, v_1 , i.e.,

$$\Delta(w_1, w_2, w_3) = \Delta(v_1, v_2, v_3) \implies \left(\frac{1}{2}a\right)\left(\frac{1}{2}b\right)\left(\frac{1}{2}\langle w_3, w_1 \rangle\right) = \left(\frac{1}{2}\right)^3 \implies \langle w_1, w_3 \rangle = ab.$$

Thus all the Gramian matrices G of vectors (w_j) which are projectively unitarily equivalent to $\Phi = (v_j)$ are given by

$$G = \begin{pmatrix} 1 & \frac{1}{2}\bar{a} & \frac{1}{2}\bar{a}\bar{b} \\ \frac{1}{2}a & 1 & \frac{1}{2}\bar{b} \\ \frac{1}{2}ab & \frac{1}{2}b & 1 \end{pmatrix}, \quad |a| = |b| = 1.$$

This can be checked using (1.2)

$$G = C^* \text{Gram}(\Phi)C = \begin{pmatrix} 1 & \frac{1}{2}\bar{c}_1 c_2 & \frac{1}{2}\bar{c}_1 c_3 \\ \frac{1}{2}\bar{c}_2 c_1 & 1 & \frac{1}{2}\bar{c}_2 c_3 \\ \frac{1}{2}\bar{c}_3 c_1 & \frac{1}{2}\bar{c}_3 c_2 & 1 \end{pmatrix}.$$



Figure 1: The frame graph of an orthonormal basis for \mathbb{C}^3 (Example 3.3), and the frame graph for three equiangular vectors in \mathbb{C}^2 (Example 3.4).

Example 3.5 Let $\Phi = (v_j)$ be the “two mutually unbiased bases” for \mathbb{C}^2 given by

$$\Phi = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right\}, \quad \text{Gram}(\Phi) = \begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 1 \end{pmatrix}.$$

This Φ has frame graph the 4-cycle v_1, v_3, v_2, v_4, v_1 , and hence no nonzero triple products with distinct indices. A spanning tree with root v_1 is given by the path v_1, v_3, v_2, v_4 . Fix $v_1 = w_1$, and then scale in order v_3, v_2, v_4 to $w_j = c_j v_j$, so that

$$\langle w_1, w_3 \rangle = \frac{a}{\sqrt{2}}, \quad \langle w_3, w_2 \rangle = \frac{\bar{b}}{\sqrt{2}}, \quad \langle w_2, w_4 \rangle = \frac{c}{\sqrt{2}}.$$

Then $\langle w_4, w_1 \rangle = \frac{1}{\sqrt{2}} \bar{z}$ is determined by completing the 4-cycle, i.e.,

$$\langle w_1, w_3 \rangle \langle w_3, w_2 \rangle \langle w_2, w_4 \rangle \langle w_4, w_1 \rangle = \langle v_1, v_3 \rangle \langle v_3, v_2 \rangle \langle v_2, v_4 \rangle \langle v_4, v_1 \rangle \implies \bar{a} b c \bar{z} = -1$$

Thus the Gramian matrices which match all the m -products of Φ have the form

$$G = \begin{pmatrix} 1 & 0 & \frac{\bar{a}}{\sqrt{2}} & \frac{\bar{z}}{\sqrt{2}} \\ 0 & 1 & \frac{\bar{b}}{\sqrt{2}} & \frac{\bar{c}}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} & \frac{b}{\sqrt{2}} & 1 & 0 \\ \frac{z}{\sqrt{2}} & \frac{c}{\sqrt{2}} & 0 & 1 \end{pmatrix}, \quad |a| = |b| = |c| = 1, \quad z := -\frac{ac}{b}.$$

In this example, Φ is in fact determined up to projective unitary equivalence by its 1-products and 2-products, since Sylvester’s criterion for the G above to be positive semidefinite gives

$$\det(G) = -\frac{1}{4} \frac{(bz + ac)^2}{abcz} = -\frac{1}{4} \left| \frac{bz}{ac} + 1 \right|^2 \geq 0 \implies \frac{bz}{ac} + 1 = 0 \implies z = -\frac{ac}{b}.$$

By way of contrast, the Φ with frame graph a 4-cycle in Example 2.5 is not determined up to projective unitary equivalence by its 1-products and 2-products.

4 Reconstruction from the m -products

It is apparent from the proof of Theorem 3.2 that only a small subset of the m -products is required to determine a sequence of vectors Φ up to projective unitary equivalence. We

call a subset of the m -products (or the corresponding indices) a **determining set** for the m -products if all m -products can be determined from them.

Corollary 4.1 *Let Γ be the frame graph of a sequence of n vectors Φ (this is determined by the 2-products). For each connected component Γ_j of Γ , let \mathcal{T}_j be a spanning tree. Then Φ is determined up to projective unitary equivalence by the following m -products*

- (i) *The 2-products*
- (ii) *The m -products, $3 \leq m \leq n$, used to obtain Γ_j from \mathcal{T}_j by completing m -cycles (these have indices in Γ_j), as detailed in the proof of Theorem 3.2.*

In particular, if M is the number of edges of Γ which are not in any \mathcal{T}_j , then it is sufficient to know all of the 2-products, and M of the m -products, $3 \leq m \leq n$.

In other words, all possible m -products can be determined from those of (i) and (ii), which therefore are a determining set. As indicated by Example 3.5, the m -products of (ii) may not be a minimal such subset, though it is a minimal subset from which the m -products can be determined using only the proof of Theorem 3.2.

Example 4.2 *Let $\Phi = (v_j)$ be four equiangular vectors with $C > 0$. The frame graph of Φ is complete, and $M = 6 - 3 = 3$. Spanning trees (see Fig. 2) include*

$$\begin{aligned} \mathcal{T}_p &:= \text{the path } v_1, v_2, v_3, v_4, \\ \mathcal{T}_s &:= \text{the star graph internal vertex } v_1 \text{ and leaves } v_2, v_3, v_4. \end{aligned}$$

For \mathcal{T}_p we can complete either the 4-cycle v_1, v_2, v_3, v_4, v_1 followed by any one of the four 3-cycles (which also completes another 3-cycle) and then one of the remaining two 3-cycles, so, e.g., a determining set is given by the 2-products, and

$$\Delta(v_1, v_2, v_3, v_4), \quad \Delta(v_1, v_2, v_3), \quad \Delta(v_1, v_2, v_4). \quad (4.12)$$

For \mathcal{T}_s , adding any edge completes a 3-cycle containing v_1 , there are then two edges which can be added to complete a 3-cycle, after which adding the other edge completes the remaining two 3-cycles, so, e.g., a determining set is given by the 2-products, and

$$\Delta(v_1, v_2, v_3), \quad \Delta(v_1, v_2, v_4), \quad \Delta(v_1, v_3, v_4). \quad (4.13)$$

We observe that the 4-product of (4.12) can be “decomposed” into smaller m -products, e.g.,

$$\Delta(v_1, v_2, v_3, v_4) = \frac{\Delta(v_1, v_2, v_3)\Delta(v_1, v_3, v_4)}{\Delta(v_1, v_3)}, \quad (4.14)$$

and so $\Delta(v_1, v_2, v_3, v_4)$ can be replaced by the 3-cycle $\Delta(v_1, v_3, v_4)$, which gives the determining set of (4.13).

The m -products of (ii) can be taken to be those corresponding to the **fundamental cycles**, i.e., the unique cycle completed by adding an edge to Γ_j . These fundamental cycles form a basis for the **cycle space**, i.e., the \mathbb{Z}_2 -formal combinations of cycles. Using this terminology, we can generalise Theorem 4.3.

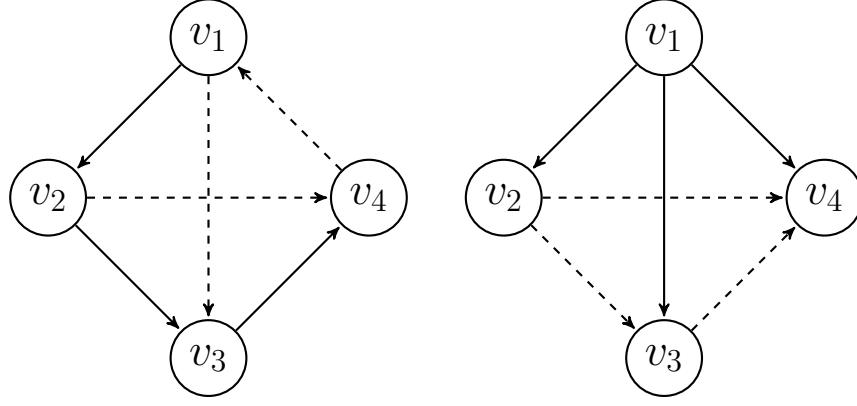


Figure 2: The spanning trees \mathcal{T}_p and \mathcal{T}_s (and cycle completions) of Example 4.2.

Theorem 4.3 (*Characterisation II*) *Two finite sequences $(v_j)_{j \in J}$ and $(w_j)_{j \in J}$ of vectors are projectively unitarily equivalent if and only if their 2-products are equal and their m -products corresponding to a basis for the cycle space of their frame graph are equal.*

Proof: All cycles in the frame graph can be calculated from those in basis, and hence all m -products can be calculated from those corresponding to a basis, as indicated in (4.14). \square

Example 4.4 *The fundamental cycles corresponding to the trees \mathcal{T}_p and \mathcal{T}_s of Example 4.2 give the following m -products*

$$\Delta(v_1, v_2, v_3, v_4), \quad \Delta(v_1, v_2, v_3), \quad \Delta(v_2, v_3, v_4), \quad (4.15)$$

$$\Delta(v_1, v_2, v_3), \quad \Delta(v_1, v_2, v_4), \quad \Delta(v_1, v_3, v_4), \quad (4.16)$$

respectively.

We can now generalise Theorem 2.2.

Theorem 4.5 *Let $\Phi = (v_j)$ be finite sequence of vectors. Then Φ is determined up to projective unitary equivalence by its 3-products if the cycle space of its frame graph is spanned by 3-cycles (and so there is a basis of 3-cycles).*

Example 4.6 (*Chordal graphs*) *A graph is said to be **chordal** (or **triangulated**) if each of its cycles of four or more vertices has a chord, and so the cycle space is spanned by the 3-cycles. Thus, if the frame graph of Φ is chordal, as is the case for equiangular lines, then Φ is determined by its triple products. The extreme cases are a totally disconnected graph (orthogonal bases) where there are no cycles, and the complete graph where all subsets of three vectors lie on a 3-cycle.*

We now give an example (Theorem 4.7) where the cycle space of the frame graph has a basis of 3–cycles, but is not chordal.

A family of orthonormal bases $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_k$ for \mathbb{C}^d is said to be **mutually unbiased** if

$$|\langle v, w \rangle|^2 = \frac{1}{d}, \quad v \in \mathcal{B}_r, \quad w \in \mathcal{B}_j.$$

We call $\mathcal{B}_1, \dots, \mathcal{B}_k$ a sequence of k **MUBs** (**mutually unbiased bases**). The frame graph of two or more MUBs ($d > 1$) is not chordal, because there is a 4–cycle (v_1, w_1, v_2, w_2) , $v_1, v_2 \in \mathcal{B}_r$, $w_1, w_2 \in \mathcal{B}_s$ not containing a chord.

We now show for three or more MUBs the cycle space of the frame graph is spanned by the 3–cycles. This is not case for two MUBs (cf. Example 3.5).

Theorem 4.7 (MUBs) *Let Φ consist of three or more MUBs in \mathbb{C}^d . Then Φ is determined up to projective unitary equivalence by its 3–products.*

Proof: It suffices to show the cycle space of the frame graph Γ of Φ has a basis of 3–cycles. To this end, let \mathcal{B}_j , $j = 1, \dots, k$, be the orthonormal bases for \mathbb{C}^d , so that Γ is a complete k –partite graph (with partite sets \mathcal{B}_j). Fix $v_1 \in \mathcal{B}_1$ and $v_2 \in \mathcal{B}_2$. A spanning tree \mathcal{T} for Γ is given by taking an edge from v_1 to each vertex of \mathcal{B}_j , $j \neq 1$, and an edge from v_2 to each vertex of $\mathcal{B}_1 \setminus v_1$. Each of the remaining edges of $\Gamma \setminus \mathcal{T}$ gives a fundamental cycle. These have two types (see Fig. 3):

1. $\frac{1}{2}d^2(k-1)(k-2)$ edges between vertices in \mathcal{B}_r and \mathcal{B}_s , $r, s \neq 1$, which give fundamental 3–cycles (involving v_1).
2. $(d-1)((k-1)d-1)$ edges between vertices $u \in \mathcal{B}_1 \setminus v_1$ and $w \in \cup_{j \neq 1} \mathcal{B}_j \setminus v_2$, which give fundamental 4–cycles (u, w, v_1, v_2) . These can be written as a union of the 3–cycles (u, w, v_2) and (v_1, v_2, w) .

Thus the cycle space is spanned by 3–cycles. □

The maximal number of MUBs is of interest in quantum information theory. For d a prime, or a power of a prime, the maximal number of MUBs in \mathbb{C}^d is $d + 1$, see [16], [5], [6] for constructions. These have a special (Heisenberg) structure, which has been used to classify them up to projective unitary equivalence, see [7], [15], [16]. Our classification using 3–products does not presuppose any structure on the MUBs.

There exists graphs which are not chordal, with every edge on a 3–cycle (as is the case for the frame graph of three or more MUBs), but for which the cycle space is not spanned by 3–cycles (see Fig. 4).

5 Similarity and m –products for vector spaces

Using the theory of frames for vector spaces [21], one can give analogous results for vector spaces, where the role of unitary equivalence is played by “similarity”, and the role of m –products by “canonical m –products”. This allows the “projective symmetry group” to be defined in a very general setting (see [20]).

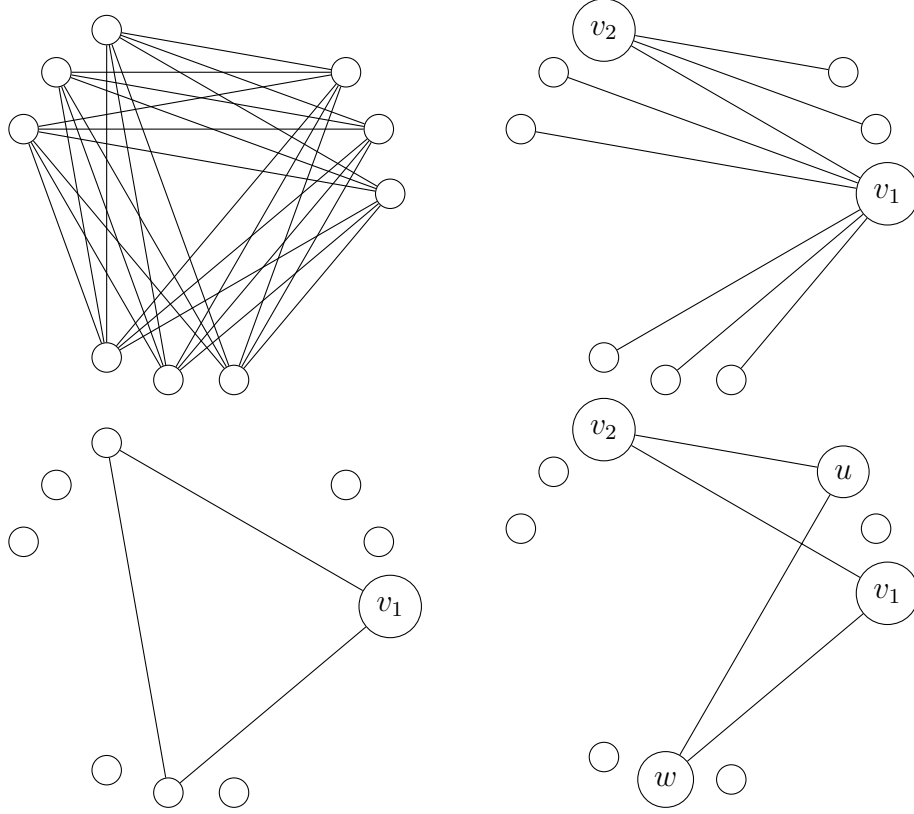


Figure 3: The proof of Theorem 4.7 for MUBs $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ in \mathbb{C}^3 . The frame graph Γ , the spanning tree \mathcal{T} , and fundamental cycles of type 1 and 2.

Let $\Phi = (v_j)$ and $\Psi = (w_j)$ be finite sequences of vectors which span vector spaces X and Y over a subfield \mathbb{F} of \mathbb{C} . We say that Φ and Ψ are **similar** if there is an invertible linear map $Q : X \rightarrow Y$ with

$$w_j = Qv_j, \quad \forall j,$$

and **projectively similar** if there is an invertible linear map $Q : X \rightarrow Y$ and unit scalars c_j with

$$w_j = c_j Qv_j, \quad \forall j.$$

For a finite sequence $\Phi = (v_j)_{j \in J}$ in X the **synthesis map** is

$$V = [v_j]_{j \in J} : \mathbb{F}^J \rightarrow X : a \mapsto \sum_j a_j v_j.$$

The subspace of all linear dependencies between the vectors of Φ is

$$\text{dep}(\Phi) := \ker(V) = \{a \in \mathbb{F}^J : \sum_j a_j v_j = 0\},$$

and we denote the orthogonal projection onto $\text{dep}(\Phi)^\perp$ (orthogonal complement) by P_Φ .

We have following characterisation of similarity in terms of linear dependencies.

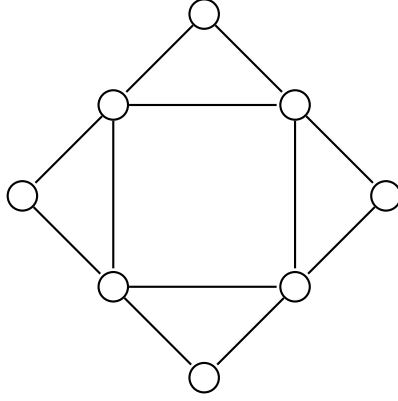


Figure 4: A nonchordal graph for which each edge is on a 3-cycle.

Lemma 5.1 ([21]) *Let $\Phi = (v_j)_{j \in J}$ and $\Psi = (w_j)_{j \in J}$ be spanning sequences for the \mathbb{F} -vector spaces X and Y . Then the following are equivalent*

- (a) Φ and Ψ are similar, i.e., there is an invertible linear map $Q : v_j \mapsto w_j$.
- (b) $\text{dep}(\Phi) = \text{dep}(\Psi)$ (the dependencies are equal).
- (c) $P_\Phi = P_\Psi$ (the associated projections are equal).

The proof of Lemma 5.1 shows that $\Phi = (v_j)$ is similar to columns of $P = P_\Phi$. These columns (Pe_j) span a subspace of \mathbb{F}^J , which inherits the Euclidean inner product. Indeed

$$\langle Pe_j, Pe_k \rangle = P_{jk},$$

i.e., the Gramian of (Pe_j) is $P = P_\Phi$. We will call the m -products of (Pe_j) the **canonical m -products** of (v_j) , and denote them

$$\Delta_C(v_{j_1}, \dots, v_{j_m}) := \Delta(Pe_{j_1}, \dots, Pe_{j_m}) = P_{j_1 j_2} P_{j_2 j_3} \cdots P_{j_m j_1}. \quad (5.17)$$

In this way, we may apply Theorem 2.2.

Theorem 5.2 (Characterisation) *Let $\Phi = (v_j)$ and $\Psi = (w_j)$ be finite sequences of vectors in vector spaces over a subfield \mathbb{F} of \mathbb{C} which is closed under complex conjugation. Then*

1. Φ and Ψ are similar if and only if $P_\Phi = P_\Psi$.
2. Φ and Ψ are projectively similar if and only if their canonical m -products (for a determining set) are equal.

Proof: The first follows from Lemma 5.1, and implies that Φ and Ψ are projectively similar, i.e.,

$$w_j = c_j Q v_j = Q(c_j v_j), \quad \forall j,$$

if and only if $\Psi = (w_j)$ and $\Phi' = (c_j v_j)$ are similar, for some choice of unit scalars (c_j) , i.e.,

$$P_\Psi = P_{(c_j v_j)} = C^* P_\Phi C. \quad (5.18)$$

Here the last equality follows by a simple calculation. Since Φ and Ψ are similar to $(P_\Phi e_j)$ and $(P_\Psi e_j)$, which have Gramian matrices P_Φ and P_Ψ , it follows from (1.2) that (5.18) is equivalent to $(P_\Phi e_j)$ and $(P_\Psi e_j)$ being projectively unitary equivalent, and by Theorem 2.2, this is equivalent to their m -products, i.e., the canonical m -products of (v_j) and (w_j) being equal. \square

For the case of projective similarity, one can calculate the c_j and Q in $w_j = c_j Q v_j$ explicitly, as we now illustrate.

Example 5.3 Suppose that $\Phi = (v_j)$ spans a 2-dimensional space, i.e.,

$$\alpha v_1 + \beta v_2 + \gamma v_3 = 0, \quad |\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1.$$

Then $\text{dep}(\Phi) = \text{span}\{u\}$, $u = (\alpha, \beta, \gamma)$, so that

$$P_\Phi = I - uu^* = \begin{pmatrix} 1 - |\alpha|^2 & -\alpha\bar{\beta} & -\alpha\bar{\gamma} \\ -\bar{\alpha}\beta & 1 - |\beta|^2 & -\beta\bar{\gamma} \\ -\bar{\alpha}\gamma & -\bar{\beta}\gamma & 1 - |\gamma|^2 \end{pmatrix}.$$

The canonical 2-products are uniquely determined by $|a|, |b|, |c|$, e.g.,

$$\Delta_C(v_1, v_1) = (1 - |\alpha|^2)^2, \quad \Delta_C(v_1, v_2) = |-\bar{\alpha}\beta|^2 = |\alpha|^2 |\beta|^2,$$

as is the canonical 3-product corresponding to the unique 3-cycle

$$\Delta_C(v_1, v_2, v_3) = (-\bar{\alpha}\beta)(-\bar{\beta}\gamma)(-\alpha\bar{\gamma}) = -|\alpha|^2 |\beta|^2 |\gamma|^2.$$

Thus if $\Psi = (w_j)$ is given by

$$\tilde{\alpha} w_1 + \tilde{\beta} w_2 + \tilde{\gamma} w_3 = 0, \quad |\tilde{\alpha}|^2 + |\tilde{\beta}|^2 + |\tilde{\gamma}|^2 = 1,$$

then

1. Ψ is similar to Φ if and only if $P_\Psi = P_\Phi$, i.e.,

$$\tilde{\alpha}\bar{\tilde{\beta}} = \alpha\bar{\beta}, \quad \tilde{\alpha}\bar{\tilde{\gamma}} = \alpha\bar{\gamma}, \quad \tilde{\beta}\bar{\tilde{\gamma}} = \beta\bar{\gamma}.$$

2. Ψ is projectively similar to Φ if and only if their canonical m -products are equal, i.e.,

$$|\tilde{\alpha}| = |\alpha|, \quad |\tilde{\beta}| = |\beta|, \quad |\tilde{\gamma}| = |\gamma|.$$

When Ψ and Φ are projectively similar, i.e., $w_j = c_j Q v_j$ (the canonical m -products are equal), one has $P_\Psi = C^* P_\Phi C$. Here, suppose $\alpha, \beta, \gamma \neq 0$, then we have

$$\bar{c}_1 c_2 \alpha \bar{\beta} = \tilde{\alpha} \bar{\tilde{\beta}}, \quad \bar{c}_1 c_3 \alpha \bar{\gamma} = \tilde{\alpha} \bar{\tilde{\gamma}}, \quad \bar{c}_2 c_3 \beta \bar{\gamma} = \tilde{\beta} \bar{\tilde{\gamma}} \implies c_2 = \frac{\tilde{\alpha} \beta}{\alpha \bar{\tilde{\beta}}} c_1, \quad c_3 = \frac{\tilde{\alpha} \gamma}{\alpha \bar{\tilde{\gamma}}} c_1.$$

where Q is defined by

$$Q v_1 := \bar{c}_1 w_1, \quad Q v_2 := \bar{c}_2 w_2 = \frac{\alpha \tilde{\beta}}{\tilde{\alpha} \beta} \bar{c}_1 w_2.$$

6 Projectively equivalent harmonic frames

Orthogonal bases can be generalised as follows (cf. [1], [22]). We say, a sequence of n vectors (v_j) is a **tight frame** for a d -dimensional inner product space \mathcal{H} if for some $A > 0$

$$f = \frac{1}{A} \sum_j \langle f, v_j \rangle v_j, \quad \forall f \in \mathcal{H},$$

Examples of tight frames of more than d vectors include SICs, MUBs, and harmonic frames.

Let G be a finite abelian group of order n , with irreducible characters $\xi \in \hat{G}$. Here \hat{G} is known as the character group (which is isomorphic to G). Let $J \subset \hat{G}$, with $|J| = d$, then any tight frame which is unitarily equivalent to the equal-norm tight frame for $\mathbb{C}^J \approx \mathbb{C}^d$ given by

$$\Phi_J = (\xi|_J)_{\xi \in \hat{G}}$$

is called a **harmonic frame**, and is said to be **cyclic** if G is a cyclic group. This is the class of tight frames which are the orbit of a group of unitary transformations on \mathbb{C}^d , which is isomorphic to G (see [23],[24]). The harmonic frames were studied up to unitary equivalence in [25], [24]. We now recount some of the basic details.

Let G be a fixed finite abelian group. Subsets J and K of G are **multiplicatively equivalent** if there is an automorphism $\sigma : G \rightarrow G$ for which $K = \sigma J$. In this case,

$$\hat{\sigma} : \hat{G} \rightarrow \hat{G} : \chi \mapsto \chi \circ \sigma^{-1}$$

is an automorphism of \hat{G} , and

$$\langle \xi|_J, \eta|_J \rangle = \langle \hat{\sigma}\xi|_K, \hat{\sigma}\eta|_K \rangle,$$

i.e., Φ_J and Φ_K are unitarily equivalent after reindexing by the automorphism $\hat{\sigma}$.

The **translations** of G are the bijections

$$\tau_b : G \rightarrow G : j \mapsto j - b, \quad b \in G,$$

and we say K is a **translate** of J if $K = J - b$, i.e., $K = \tau_b J$. We define the **affine group** of G to be the group of bijections $\pi : G \rightarrow G$ generated by the translations and automorphisms, i.e., the $|G| \cdot |\text{Aut}(G)|$ maps of the form

$$\pi(g) = \sigma(g) - b, \quad \sigma \in \text{Aut}(G), \quad b \in G.$$

If $K = \pi J$, for some π in the affine group, we say J and K are **affinely equivalent**.

Lemma 6.1 *If J and K are subsets of a finite abelian group G , which are translates of each other, then the harmonic frames Φ_J and Φ_K are projectively unitarily equivalent.*

Proof: Suppose $K = J - b$. Since $\Phi_J = (\xi|_J)_{\xi \in \hat{G}}$, we need to show

$$\xi|_K = c_\xi U(\xi|_J), \quad \xi \in \hat{G},$$

where $U : \mathbb{C}^J \rightarrow \mathbb{C}^K$ is unitary. Let $U_b : \mathbb{C}^J \rightarrow \mathbb{C}^K$ be the unitary map

$$(U_b v)(k) := v(k + b), \quad k \in K.$$

Since ξ is a character, we have

$$(U_b \xi|_J)(k) = \xi|_J(k + b) = \xi(k + b) = \xi(k)\xi(b) = \xi|_K(k)\xi(b),$$

and so we can take $U = U_b$ and $c_\xi = 1/\xi(b)$. \square

The converse: that projective unitary equivalence implies J and K are translates of each other appears to be true.

Theorem 6.2 *Suppose J and K are subsets of a finite abelian group G . Then*

1. *If J and K are translates, then Φ_J and Φ_K are projectively unitarily equivalent.*
2. *If J and K are multiplicatively equivalent, then Φ_J and Φ_K are unitarily equivalent after reindexing by an automorphism.*
3. *If J and K are affinely equivalent, then Φ_J and Φ_K are projectively unitarily equivalent after reindexing by an automorphism.*

Proof: The first part is Lemma 6.1, the second is given in [24] (Theorem 3.5), and third follows by combining the first two. 6.1. \square

Example 6.3 *Let $p > 2$ be a prime. Then all harmonic frames of p vectors in \mathbb{C}^2 are projectively unitarily equivalent up to reindexing (to p equally spaced vectors in \mathbb{R}^d). This follows since there is a unique affine map, taking a sequence of two distinct elements of \mathbb{Z}_p to any other. In particular, allowing for reindexing, the two harmonic frames of three vectors in \mathbb{C}^2 which are unitarily inequivalent (one is real, one is complex) are projectively unitarily equivalent.*

The conditions of 1,2,3 of Theorem 6.2 say that J and K are in the same orbit under action of the *group of translations*, the *automorphism group*, and the *affine group* on the subsets of G , respectively. Using this, we were able to calculate the various equivalences using the computer algebra package MAGMA [26]. The results of these calculations for the cyclic harmonic frames are summarised in Fig. 5. These indicate that the number of projective unitary equivalence classes is much smaller than the number of unitary equivalence classes (up to any reindexing). There are just a few cases where the number of equivalence classes is smaller than that predicted by the group theoretic calculations, because there is a reindexing which is *not* an automorphism which makes harmonic frames equivalent. This was previously observed in the case of unitary equivalence [24]. In these cases the larger group theoretic estimate is given in the row below in Fig. 5.

$d = 2$			$d = 3$			$d = 4$		
n	uni	proj	n	uni	proj	n	uni	proj
2	1	1	3	1	1	5	2	1
3	2	1	4	3	1	6	9	3
4	3	2	5	3	1	7	7	2
5	3	1	6	11	3	8	21	6
6	6	3	7	7	2		23	5
7	4	1	8	16	4	9	23	4
8	7	3		17			24	
9	6	2	9	15	3	10	53	9
10	9	3	10	29	4		54	
11	6	1	11	17	2	11	34	4
12	13	5	12	56	9	12	138	21
13	7	1		57			141	
14	12	3	13	25	3			
15	13	3						

$d = 5$			$d = 6$			$d = 7$		
n	uni	proj	n	uni	proj	n	uni	proj
5	1	1	6	1	1	7	1	1
6	4	1	7	2	1	8	4	1
7	4	1	8	11	3	9	8	2
8	19	4	9	16	3	10	32	4
	20		10	55	9	11	34	4
9	23	4		56		12	228	25
	24		11	48	6		234	
10	67	9						
11	48	6						

Figure 5: The number of unitary and projective unitary equivalence classes (up to reindexing) of cyclic harmonic frames of n vectors in \mathbb{C}^d , $d = 2, \dots, 7$. When the group theoretic estimate of Theorem 6.2 is larger (because there are reindexings which are not automorphisms) it is given in the row below.

References

- [1] P. G. Casazza and G. Kutyniok (Eds.), “Finite Frames”, Springer-Verlag, New York, 2013.
- [2] J. M. Renes, R. Blume–Kohout, A. J. Scott, and C. M. Caves, Symmetric informationally complete quantum measurements, *J. Math. Phys.* **45** (2004), 2171–2180.

- [3] A. J. Scott and M. Grassl, SIC-POVMs: A new computer study, arXiv:0910.5784v2 [quant-ph], 2009.
- [4] D. M. Appleby, Symmetric informationally complete-positive operator valued measures and the extended Clifford group, *J. Math. Phys.* **46** (2005), 1–27.
- [5] I.D. Ivonovic, Geometrical description of quantal state determination, *Journal of Physics A: Mathematical and General*, **14(12)** (1981) 3241.
- [6] W.K. Wootters, B.D. Fields, Optimal state-determination by mutually unbiased measurements, *Annals of Physics*, **191(2)** (1989) 363–381.
- [7] W.M. Kantor, MUBs inequivalence and affine planes, *Journal of Mathematical Physics*, **53** (2012) 032204.
- [8] V. K. Goyal, J. Kovačević, and J. A. Kelner, Quantized Frame Expansions with Erasures, *Appl. Comput. Harmonic Anal.* **10** (2001), 203–233.
- [9] P. G. Casazza, and J. Kovačević, Equal-norm tight frames with erasures, **18** (2003), 387–430.
- [10] R. B. Holmes and V. I. Paulsen, Optimal frames for erasures, *Linear Algebra Appl.* **377** (2004), 31–51.
- [11] P. W. H. Lemmens, and J. J. Seidel, Equiangular lines, *J. Algebra* **24** (1973), 494–512.
- [12] I. Halperin, On the Gram matrix, *Canad. Math. Bull.* **5** (1962), 265–280.
- [13] C. Godsil and G. Royle, “Algebraic graph theory”, Springer-Verlag, New York, 2001.
- [14] A.R. Calderbank, P.J. Cameron, W.M. Kantor and J.J. Seidel, Z_4 -Kerdock codes, orthogonal spreads, and extremal Euclidean line-sets, *Proceedings of the London Mathematical Society*, **75(2)** (1997) 436–480.
- [15] C. Godsil and A. Roy, Equiangular lines, mutually unbiased bases, and spin models, *European Journal of Combinatorics*, **30(1)** (2009) 246–262.
- [16] I. Bengtsson, Three ways to look at mutually unbiased bases, arXiv:0610216 [quant-ph], 2006.
- [17] D. M. Appleby, S. T. Flammia, and C. A. Fuchs, The Lie Algebraic Significance of Symmetric Informationally Complete Measurements, arXiv:1001.0004 [quant-ph], 2009.
- [18] N. Mukunda, A. S. Chaturvedi, and R. Simon, Bargmann invariants and off-diagonal geometric phases for multi-level quantum systems – a unitary group approach, arXiv:quant-ph/0107006, 2001.
- [19] A. Abdollahi, H. Najafi, Frame graph, Preprint, 2013.

- [20] T. Chien and S. Waldron, Computing the projective symmetry group of a finite frame, Preprint.
- [21] S. Waldron, Frames for vector spaces and affine spaces, *Linear Algebra Appl.* **435** (2011), 77–94.
- [22] S. Waldron, “An introduction to finite tight frames”, Springer-Verlag, New York, 2015.
- [23] R. Vale and S. Waldron, Tight frames and their symmetries, *Constr. Approx.* **21** (2005), 83–112.
- [24] T. Chien and S. Waldron, A classification of the harmonic frames up to unitary equivalence, *Appl. Comput. Harmon. Anal.* **30** (2011), 307–318.
- [25] N. Hay and S. Waldron, On computing all harmonic frames of n vectors in \mathbb{C}^d , *Appl. Comput. Harmonic Anal.* **21** (2006), 168–181.
- [26] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system. I. The user language, *J. Symbolic Comput.* **24** (1997), 235–265.